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BANACH AND CARISTI TYPE THEOREMS IN B-METRIC-LIKE SPACES

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Abstract. The aim of this paper is to introduce new contractive mappings in the setting of b-metric-like spaces. We establish a new fixed point result by extending the famous Caristi fixed point result from the framework of b-metric spaces. An example illustrates and supports this result.
Keywords: b-metric-like space, Caristi fixed point theorem.

1. Introduction and Preliminaries

Many authors significantly enhanced and expanded upon the idea of metric spaces, along with other various concepts.

Metric type spaces (or b-metric spaces) is one of the important generalization of metric spaces. This concept was introduced by Bakhtin 1989 [4] and Czerwik 1993 [8].

Recently, M. A. Alghamdi et al. [1] have introduced the notion of b-metric-like space that generalized the notions of partial b-metric space and metric-like space. Moreover, many authors proved some fixed point theorems for single-valued and multi-valued mappings in b-metric-like space. For more details, we refer the reader to ([2], [6], [7] and [11]).

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Motivated by the results and notions mentioned above, in this paper we present a new type fixed point theorems for this class of mappings defined on b-metric like spaces. Our main result is essentially inspired by G. H. Joonaghany et al [14].

We shall now present some definitions and basic notions of generalized metric spaces.

Definition 1.1. [4] Let X be a (nonempty) set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0, +\infty)$ is a b-metric on X if for all $u, v, w \in X$, the following conditions hold:

$d1)$ $d(u, v) = 0$ if and only if $u = v$,

$d2)$ $d(u, v) = d(v, u)$,

$d3)$ $d(u, w) \leq s[d(u, v) + d(v, w)]$.

The triplet (X, d, s) is called a b-metric space. Clearly, every metric space is a b-metric space with $s = 1$, but the converse is not true in general. In fact, the class of b-metric spaces is larger than the class of metric spaces.

Definition 1.2. [13] Let X be a nonempty set. A mapping $\sigma : X \times X \to [0, +\infty)$ is said to metric-like if the following conditions hold for all $u, v, w \in X$:

$\sigma 1)$ $\sigma(u, v) = 0$ implies $u = v$;

$\sigma 2)$ $\sigma(u, v) = \sigma(v, u)$;

$\sigma 3)$ $\sigma(u, w) \leq \sigma(u, v) + \sigma(v, w)$.

In this case, the pair (X, σ) is called a metric-like space.

Definition 1.3. [1] A b-metric-like on a nonempty set X is a function $\sigma_b : X \times X \to [0, +\infty)$ such that for all $u, v, w \in X$ and a constant $s \geq 1$, the following three conditions hold true:

$(\sigma_b 1)$ $\sigma_b(u, v) = 0$ implies $u = v$,

$(\sigma_b 2)$ $\sigma_b(u, v) = \sigma_b(v, u)$,

$(\sigma_b 3)$ $\sigma_b(u, w) \leq s(\sigma_b(u, v) + \sigma_b(v, w))$.

The pair (X, σ_b) is then called a b-metric-like space.

Example 1.1. Let $X = \{0, 1, 2\}$ and let

$$\sigma_b(u, v) = \begin{cases} 2, & u = v = 0 \\ \frac{1}{2}, & otherwise. \end{cases}$$

Then (X, σ_b) is a b-metric-like space with the constant $s = 2$.

For more such examples and details see ([3], [9], [12]).

Definition 1.4. [1] **a**) A sequence $\{x_n\}$ in the b-metric-like space (X, σ_b) converges to a point $x \in X$ if and only if

$$\sigma_b(x, x) = \lim_{n\to+\infty} \sigma_b(x, x_n).$$

b) A sequence $\{x_n\}$ in the b-metric-like space (X, σ_b) is called a Cauchy sequence if there exists (and is finite) $\lim_{m,n\to+\infty} \sigma_b(x_m, x_n)$.

c) A b-metric-like space is called complete if for every Cauchy sequence $\{x_n\}$ in X if there exists a point $x \in X$ such that

$$\lim_{n\to+\infty} \sigma_b(x, x_n) = \sigma_b(x, x) = \lim_{m,n\to+\infty} \sigma_b(x_m, x_n).$$

Lemma 1.1. [1] *Let $\{x_n\}$ be a sequence in a b-metric-like space (X, σ_b, s) such that*

$$\sigma_b(x_n, x_{n+1}) \leq \gamma\sigma_b(x_{n-1}, x_n)$$

for some γ, $0 < \gamma < 1/s$, and each $n \in \mathbb{N}$. Then $\lim_{m,n\to+\infty} \sigma_b(x_m, x_n) = 0$.

Remark 1.1. It is worth to notice that the previous lemma holds in the context of b-metric-like spaces for each $\gamma \in [0, 1)$.

Remark 1.2. In each metric (resp. b-metric) space (X, d) if the sequence $\{x_n\}$ converges to say some x then this limit is unique.

While, in partial metric (resp. metric like, partial b-metric, b-metric like) space this is not true always (see [10]).

Also in each of these 4 type spaces if the sequence x_n is a Cauchy such that $d(x_n, x_m)$ tends to 0 as n, m tend to infinity then its limit is unique if it exists.

2. Main Results

In the following theorem, we extend recent results in b-metric-like spaces.

Theorem 2.1. *Let $(X, \sigma_b, s \geq 1)$ be a complete b-metric-like space and $T : X \to X$ be a map. Suppose that there exists a function $\varphi : X \to [0, +\infty)$ such that for all $x, y \in X$,*

$\sigma_b(x, Tx) > 0$ *implies*

$$\sigma_b(Tx, Ty) \leq (\varphi(x) - \varphi(Tx))\, M(x, y), \tag{2.1}$$

where

$$M(x, y) = \max\left\{\sigma_b(x, y), \sigma_b(x, Tx), \sigma_b(y, Ty), \frac{\sigma_b(x, Ty) + \sigma_b(y, Tx) - \sigma_b(y, y)}{2s}\right\}.$$

Then, T has at least one fixed point in X.

Proof. Let $x_0 \in X$. Define a sequence $\{x_n\}$ in X as $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. We first assume that $\sigma_b(x_0, Tx_0) > 0$.If $x_{p-1} = x_p$ for some $p \in \mathbb{N}$, then x_{p-1} is a fixed point of T and the proof of theorem is finished. Suppose further that $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$.

For any $y \in X$ with $M(x_0, y) > 0$ we have

$$\varphi(x_0) - \varphi(Tx_0) \geq 0.$$

By the same manner, for all $n \in \mathbb{N}$, we obtain

$$\varphi(x_n) - \varphi(Tx_n) \geq 0,$$

which means that

$$\varphi(x_{n-1}) \geq \varphi(x_n).$$

Thus the sequence $\{\varphi(x_n)\}$ is non-negative and non-increasing. Hence, it converges to some $r \geq 0$. So

$$\lim_{n\to+\infty} (\varphi(x_{n-1}) - \varphi(x_n)) = 0. \tag{2.2}$$

So, there exists n_0 from $\mathbb{N}$ such that for each $n \geq n_0$

$$\varphi(x_{n-1}) - \varphi(x_n) < 1. \tag{2.3}$$

On the other hand, for each $n \in \mathbb{N}$, denote $a_n = \sigma_b(x_n, x_{n+1})$. From (2.1), we consider

$$\begin{aligned} a_{n+1} &= \sigma_b(x_{n+1}, x_{n+2}) = \sigma_b(Tx_n, Tx_{n+1}) && (2.4)\\ &\leq (\varphi(x_n) - \varphi(Tx_n))M(x_{n+1}, x_n) && (2.5)\\ &= (\varphi(x_n) - \varphi(x_{n+1}))M(x_{n+1}, x_n). && (2.6) \end{aligned}$$

Using (2.3) and (2.6), we find

$$a_{n+1} < M(x_{n+1}, x_n), \text{for each n} \geq n_0 \tag{2.7}$$

where

$$\begin{aligned} M(x_n, x_{n+1}) &= \max\{\sigma_b(x_n, x_{n+1}), \sigma_b(x_n, Tx_n), \sigma_b(x_{n+1}, Tx_{n+1}),\\ &\quad \frac{\sigma_b(x_n, Tx_{n+1}) + \sigma_b(x_{n+1}, Tx_n) - \sigma_b(x_{n+1}, x_{n+1})}{2s}\}\\ &= \max\{\sigma_b(x_n, x_{n+1}), \sigma_b(x_n, x_{n+1}), \sigma_b(x_{n+1}, x_{n+2}),\\ &\quad \frac{\sigma_b(x_n, x_{n+2}) + \sigma_b(x_{n+1}, x_{n+1}) - \sigma_b(x_{n+1}, x_{n+1})}{2s}\}\\ &= \max\left\{\sigma_b(x_n, x_{n+1}), \sigma_b(x_{n+1}, x_{n+2}), \frac{\sigma_b(x_n, x_{n+2})}{2s}\right\} \end{aligned}$$

$$
\begin{aligned}
&\leq \max\{\sigma_b(x_n, x_{n+1}), \sigma_b(x_{n+1}, x_{n+2}), \\
&\quad \frac{\sigma_b(x_n, x_{n+1}) + \sigma_b(x_{n+1}, x_{n+2})}{2}\} \\
&\leq \max\{\sigma_b(x_n, x_{n+1}), \sigma_b(x_{n+1}, x_{n+2})\} \\
&= \max\{a_n, a_{n+1}\}.
\end{aligned}
$$

If

$$\max\{a_n, a_{n+1}\} = a_{n+1}.$$

Then, for $n \geq n_0$ we get

$$a_n \leq a_{n+1}.$$

It is a contradiction with (2.7). Hence

$$a_{n+1} < a_n.$$

Accordingly with (2.7), we have

$$a_{n+1} < M(x_{n+1}, x_n) < a_n, \text{foreach} n \geq n_0 \tag{2.8}$$

By (2.6) and (2.7), it follows that

$$a_{n+1} \leq (\varphi(x_n) - \varphi(x_{n+1}))a_n, \text{foreach} n \geq n_0 \tag{2.9}$$

On account of (2.2), for given γ from $(0, 1)$, since $\frac{a_{n+1}}{a_n}$ tends to 0 as n tends to infinity, there exists n_1 from $\mathbb{N}$ such that

$$\frac{a_{n+1}}{a_n} \leq \gamma \text{whenever} n \geq n_1.$$

It yields that

$$\sigma_b(x_{n+2}, x_{n+1}) \leq \gamma\sigma_b(x_n, x_{n+1}), \text{foreach} n \geq n_1.$$

Now, using Lemma 1.1 we have $\lim_{m,n\to+\infty} \sigma_b(x_m, x_n) = 0$.

As, $\lim_{m,n\to+\infty} \sigma_b(x_m, x_n)$ exists (and is finite), so $\{x_n\}$ is a Cauchy sequence. Since $(X, \sigma_b, s \geq 1)$ is a complete b-metric-like space, the sequence $\{x_n\}$ in X converges to a unique $z \in X$ such that

$$\lim_{n\to+\infty} \sigma_b(z, x_n) = \sigma_b(z, z) = \lim_{m,n\to+\infty} \sigma_b(x_m, x_n) = 0.$$

We claim that z is the fixed point of T. Employing assumption (2.1) of the Theorem 2.1, we find that

$$\begin{aligned}\sigma_b(z,Tz) &\leq s[\sigma_b(z,x_{n+1})+\sigma_b(x_{n+1},Tz)]\\ &= s\sigma_b(z,x_{n+1})+s\sigma_b(x_{n+1},Tz)\\ &\leq s\sigma_b(z,x_{n+1})+s(\varphi(x_n)-\varphi(x_{n+1}))M(x_n,z),\end{aligned}$$

where

$$\begin{aligned}M\left(x_n,z\right) &= \max\{\sigma_b\left(x_n,z\right),\sigma_b\left(x_n,Tz\right),\sigma_b\left(z,Tz\right),\\ &\qquad \frac{\sigma_b\left(x_n,Tz\right)+\sigma_b\left(z,Tx_n\right)-\sigma_b\left(z,z\right)}{2s}\}\\ &\leq \max\{\sigma_b\left(x_n,z\right),s\sigma_b\left(x_n,z\right)+s\sigma_b\left(z,Tz\right),\sigma_b\left(z,Tz\right),\\ &\qquad \frac{s\sigma_b\left(x_n,z\right)+s\sigma_b\left(z,Tz\right)-\sigma_b\left(z,z\right)}{2s}\}\\ &= \max\{\sigma_b\left(x_n,z\right),s\sigma_b\left(x_n,z\right)+s\sigma_b\left(z,Tz\right),\sigma_b\left(z,Tz\right),\\ &\qquad \frac{s\sigma_b\left(x_n,z\right)+s\sigma_b\left(z,Tz\right)-\sigma_b\left(z,z\right)}{2s}\}\\ &\to s\sigma_b\left(z,Tz\right) \quad \text{as} \quad n\to+\infty.\end{aligned}$$

Therefore, for all $n\in\mathbb{N}$

$$\sigma_b(z,Tz)\leq s\sigma_b(z,x_{n+1})+s(\varphi(x_n)-\varphi(x_{n+1}))M(x_n,z).$$

Now, taking n to infinity, we get $\sigma_b(z,Tz)=0$, that is, $Tz=z$, and the proof of Theorem 2.1 is completed. □

Remark 2.1. - Taking (X,σ) a complete metric-like space in Theorem 2.1, one can easily obtain the result of [14].

- Taking (X,σ) a complete metric-like space and $M(x,y)=d(x,y)$ in Theorem 2.1 one can easily obtain the result of [15].

Now we present an example to support our Theorem 2.1.

Example 2.1. Let $X=\{0,1,2\}$ endowed with the following b-metric-like σ_b such as

$\sigma_b(0,0)=0$, $\sigma_b(1,1)=3$, $\sigma_b(2,2)=1$, $\sigma_b(0,1)=8$, $\sigma_b(0,2)=1$, $\sigma_b(1,2)=4$, $\sigma_b(a,b)=\sigma_b(b,a)$, for all $a,b\in X$.

It is clear that $(X,\sigma_b,s=\frac{8}{5})$ is a complete b-metric-like space.

Let $T:X\to X$ be defined by

$$Tx=\begin{cases}0 & if \quad x=0\\ 1 & if \quad x=2\\ 2 & if \quad x=1\end{cases}.$$

Also, define $\varphi:X\to[0,+\infty)$ as

$$\varphi\left(x\right)=\begin{cases}2 & if \quad x=2\\ 4 & if \quad x=1\\ 0 & if \quad x=0\end{cases}.$$

Now, we show that (2.1) in Theorem is true.
Thus for all $x \in X$ such that $\sigma_b(x, Tx) > 0$, (in this example, $x \neq 0$), we have

$$\begin{array}{rcl} \sigma_b(T1, T2) & \leq & (\varphi(1) - \varphi(T(1)))M(2,1), \\ \sigma_b(T2, T1) & \leq & (\varphi(2) - \varphi(T(2)))M(2,1), \\ \sigma_b(T1, T0) & \leq & (\varphi(1) - \varphi(T(1)))M(1,0), \\ \sigma_b(T2, T0) & \leq & (\varphi(2) - \varphi(T(2)))M(2,0). \end{array}$$

So, for all $x, y \in X$,

$$\sigma_b(Tx, Ty) \leq (\varphi(x) - \varphi(Tx))\, M(x, y).$$

Where

$$M(x,y) = \max\left\{\sigma_b(x,y), \sigma_b(x,Tx), \sigma_b(y,Ty), \frac{\sigma_b(x,Ty) + \sigma_b(y,Tx) - \sigma_b(y,y)}{2s}\right\}.$$

Therefore, all the conditions of Theorem 2.1 are satisfied, and we see that 0 is a fixed point of T.

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