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ON AN OVER-RING $C(X)_{\Delta}$ OF $C(X)$

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Abstract. Our aim in this paper is to introduce a ring of functions defined on a topological space X having a special property. By $C(X)_{\Delta}$ we denote the set of all real-valued functions defined on the topological space X, the discontinuity set of elements of which are members of $\Delta \subseteq \mathcal{P}(X)$, where Δ satisfies the following properties: (i) for each $x \in X, \{x\} \in \Delta, (ii)$ for $A, B \in \mathcal{P}(X)$ with $A \subseteq B, B \in \Delta$ implies that $A \in \Delta$ and (iii) for $A, B \in \Delta, A \cup B \in \Delta$. This $C(X)_{\Delta}$ is an over-ring of $C(X)$, moreover, $C(X) \subseteq C(X)_{F} \subseteq C(X)_{\Delta} \subseteq \mathbb{R}^X$. The ring $C(X)_{\Delta}$ is also almost regular. We study the Δ-completely separated sets and C_{Δ}-embedded subsets of X. Complete characterizations of fixed maximal ideals are then done and algebraic properties of $C(X)_{\Delta}$ have been studied. In [6], the authors have introduced $\mathcal{FP}$-spaces, for which the ring $C(X)_{F}$ is regular. Here we have generalized the notion of $\mathcal{FP}$-spaces in the context of $C(X)_{\Delta}$, so that the ring in question becomes regular. As a result, ΔP-spaces have been introduced, it has been proved that every P-space is a ΔP-space and examples are given in support of the fact that there exist ΔP-spaces which are not P-spaces.
Keywords: $C(X)_{\Delta}$, $C^{*}(X)_{\Delta}$, Δ-completely separated sets, Z_{Δ}-ideals, Z_{Δ}-filters, ΔP-spaces.

1. Introduction

Unless otherwise mentioned, all topological spaces are assumed to be T_1. Let $\mathbb{R}^X$ be the ring of all real-valued functions defined on a nonempty topological space

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X with pointwise addition and multiplication. We here note that all subrings of $\mathbb{R}^X$ are reduced (see [8]), in the sense that they have no non-zero nilpotent elements. Also recall that the ring $T'(X)$ [1] of all $f \in \mathbb{R}^X$, where for each f there is an open dense subset D of X such that $f|_D$ is continuous on D, is a (Von Neumann) regular ring, where a ring R is called regular if for any $a \in R$, there exists $b \in R$ such that $a = aba$. In this sequel, we also want to mention about the ring $T(X)$ [1] of all $f \in \mathbb{R}^X$ such that $f|_D \in C(D)$, for a dense subspace D of X. Also the collection of all continuous members of $\mathbb{R}^X$ is denoted by $C(X)$, and the collection of all bounded members of $C(X)$ is denoted by $C^*(X)$. In this connection, we refer to the reader [7], where these two rings have been studied extensively. If f is a function from a topological space (X, τ) to the real line $\mathbb{R}$ which is not necessarily continuous, it is well known that the set $D_f = \{x \in X : f$ is discontinuous at x w.r.t the topology $\tau\}$ is an F_σ-subset of X. The proof of this fact is followed by some simple modification in the arguments to prove that for a function $f : \mathbb{R} \to \mathbb{R}$, the set of all points of discontinuity of f is an F_σ-set (see [11]). Gharebaghi, Ghirati and Taherifar in [6] first introduced and studied the ring $C(X)_F$ of all real-valued functions on X which are discontinuous on some finite subset of X, i.e. all those members $f \in \mathbb{R}^X$ for which D_f is a finite subset of X. After that this ring has been further studied by M. R. Ahmadi Zand and Z. Khosravi in [2]. Very recently, the authors in [3] investigated the family $\mathcal{M}_0(X, \mu)$ of all those functions f of $\mathcal{M}(X, \mathcal{A})$ ($\equiv$ the ring of all real-valued measurable functions defined over a measurable space $(X, \mathcal{A})$), for which $\mu(D_f) = 0$. Fortunately, using the properties of the measure μ, it can be checked that $\mathcal{M}_0(X, \mu)$ is a commutative lattice ordered ring with unity if the relevant operations are defined pointwise on X. In this connection, one can go through [4], where the authors have studied the ring of functions which are discontinuous on a countable set. Regarding the rings $C(X)_F$, $T(X)$ and $\mathcal{M}_0(X, \mu)$, the most common features are that the discontinuity set D_f, for any f in all these rings are closed under finite unions and forming subsets. These particular properties motivate us to consider a subcollection $\mathcal{D} \subseteq \mathcal{P}(X)$ closed under forming subsets and finite unions. [These urge us to consider a collection $C(X)_D$ of all those members f of $\mathbb{R}^X$ for which $D_f \in \mathcal{D}$. This $C(X)_D$ also happens to be a commutative ring with unity if the relevant operations are defined pointwise on X. Note that, if $\mathcal{D} =$ the collection of all finite subsets of X (resp., set of all nowhere dense subsets of X), then $C(X)_D$ reduces to $C(X)_F$ (resp., $T(X)$) and if $\mathcal{D} =$ the collection of all sets having measure zero in a complete measure space, then $C(X)_D = \mathcal{M}_0(X, \mu)$]. We now impose another condition on $\mathcal{D}$ mainly, $\mathcal{D}$ is closed under containing singletons, i.e. for any $x \in X$, $\{x\} \in \mathcal{D}$. So, in this paper our key element is a subcollection $\Delta \subseteq \mathcal{P}(X)$ with the following properties:

1) For each $x \in X$, $\{x\} \in \Delta$.
2) For $A, B \in \mathcal{P}(X)$ with $A \subseteq B$, $B \in \Delta$ implies that $A \in \Delta$.
3) For $A, B \in \Delta$, $A \cup B \in \Delta$.

As mentioned before, $C(X)_\Delta$ becomes a commutative ring with unity. Now, the benefits of switching to Δ from $\mathcal{D}$ yield the following results.

1) X is discrete if and only if $C(X) = C(X)_{\Delta}$.
2) X is connected if and only if $\bar{\mathbf{0}}$ and $\bar{\mathbf{1}}$ are the only idempotent elements of $C(X)$ (where for any $r \in \mathbb{R}$, $\bar{\mathbf{r}}$ denotes the constant function $f(x) = r$, for all $x \in X$), whereas in the case of $C(X)_{\Delta}$, $\chi_{\{x\}}$ becomes an idempotent element, for each $x \in X$, irrespective of the connectedness of X.
3) Any element of $C(X)_{\Delta}$ is either a unit or a zero-divisor.
4) Also while studying ideals and z-filters, a necessary and sufficient condition for a proper ideal as well as a maximal ideal to be fixed can be solved.

Let us now briefly explain the organization of the paper. Section 2 starts with the definition of the rings $C(X)_{\Delta}$ and $C^{*}(X)_{\Delta}$. It is shown that unlike the ring $C(X)$, the equality $C(X)_{\Delta} = C^{*}(X)_{\Delta}$ is only a sufficient condition for the pseudocompactness of X but not necessary. We define the zero sets $Z_{\Delta}(f)$, for a function $f \in C(X)_{\Delta}$. Examples are given in support of the fact that $Z_{\Delta}(f)$ is not necessarily closed as well as not G_{δ}, like the case of the ring $C(X)$. In fact, it is shown that for any $f \in C(X)_{\Delta}$, $Z_{\Delta}(f)$ can be written as a disjoint union of a G_{δ}-subset of X and a member of Δ. It is proved that $C(X)_{\Delta}$ is an almost regular ring. This section ends with some dissimilarities between $C(X)$ and $C(X)_{\Delta}$.

In section 3, we introduce the notion of Δ-completely separated sets and characterize them in terms of zero sets of $C(X)_{\Delta}$. It has been shown that Δ-complete separation is a generalization of both $\mathcal{F}$-complete separation and that of complete separation of subsets of X. Next we introduce C_{Δ}-embedded and C^{*}_{Δ}-embedded subsets of X. A necesarry and sufficient condition is obtained for a C^{*}_{Δ}-embedded subset to be C_{Δ}-embedded. Also it is established that if a discrete zero set is C^{*}_{Δ}-embedded, then all its subsets are also zero sets.

In section 4, we introduce the notions of ideals of $C(X)_{\Delta}$ and Z_{Δ}-filters on X. Naturally it is shown that there is a one-to-one correspondence between the set of all maximal ideals of $C(X)_{\Delta}$ and the set of all Z_{Δ}-ultrafilters of X. After the introduction of Z_{Δ}-ideals it is shown that every Z_{Δ}-ideal is a radical ideal. That the sum of two Z_{Δ}-ideals is a Z_{Δ}-ideal is established, as a consequence of which we have that, if $\{I_{\alpha}\}_{\alpha\in\Lambda}$ be a collection of Z_{Δ}-ideals in $C(X)_{\Delta}$, then either $\sum_{\alpha\in\Lambda} I_{\alpha} = C(X)_{\Delta}$ or $\sum_{\alpha\in\Lambda} I_{\alpha}$ is a Z_{Δ}-ideal.

In section 5, the complete list of fixed maximal ideals of $C(X)_{\Delta}$ and $C(X)^{*}_{\Delta}$ are given in terms of M_{p}^{Δ} and $M_{p}^{\Delta^{*}}$ respectively. Here with the help of M_{p}^{Δ}, we give another description of Z_{Δ}-ideals. Finally a finite space is characterized as one in which every proper ideal of $C(X)_{\Delta}$ is fixed and also every maximal ideal of $C(X)_{\Delta}$ is fixed.

Section 6 is devoted to the study of residue class rings of $C(X)_{\Delta}$ modulo ideals. It is shown that every Z_{Δ}-ideal is absolutely convex, and for every maximal ideal M in $C(X)_{\Delta}$, the quotient ring $C(X)_{\Delta}/M$ is a lattice ordered ring. Also for a Z_{Δ}-ideal I in $C(X)_{\Delta}$ which is prime, the lattice ordered ring $C(X)_{\Delta}/I$ is totally ordered. It is proved that every hyper-real residue class field $C(X)_{\Delta}/M$ is non-archimedean and each maximal ideal M in $C^*(X)_{\Delta}$ is real. Lastly it is established that $f \in C(X)_{\Delta}$ is unbounded on X if and only if there exists a maximal ideal M in $C(X)_{\Delta}$ such that $|M(f)|$ is infinitely large in $C(X)_{\Delta}/M$.

Section 7 deals with some algebraric aspects of $C(X)_{\Delta}$. Relations between the rings $C(X)$, $C(X)_{\Delta}$ and $T'(X)$ have been investigated.

Section 8 studies ΔP-spaces. It has been shown that every P-space is a ΔP-space. Examples are provided in support of the fact that the converse is not true in general.

Throughout the paper $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{N}$ respectively denote the set of reals, the set of rationals and the set of natural numbers.

2. The rings $C(X)_{\Delta}$ and $C^*(X)_{\Delta}$

In this section our main interest is to explore the properties of the ring $C(X)_{\Delta}$. We then introduce a subring $C^*(X)_{\Delta}$ of $C(X)_{\Delta}$ and also discuss about the zero sets for functions in $C(X)_{\Delta}$.

Definition 2.1. For a topological space X and a subcollection Δ of $\mathcal{P}(X)$ ($\equiv$ the power set of X), where Δ is closed under forming subsets, finite unions and containing all singletons, we define,

$$C(X)_{\Delta} = \{f \in \mathbb{R}^X : \text{the set of points of discontinuities of } f \text{ is a member of } \Delta\}.$$

It can be easily observed that $C(X)_{\Delta}$ is a commutative ring with unity (with respect to pointwise addition and multiplication) containing $C(X)$, in addition, $C(X)_{\Delta}$ is a super-ring or an over-ring of $C(X)_F \supseteq C(X)$, i.e. $C(X) \subseteq C(X)_F \subseteq C(X)_{\Delta}$.

We note that $C(X)_{\Delta}$ is a sublattice of $\mathbb{R}^X$, in fact, $(C(X)_{\Delta}, +, ., \vee, \wedge)$ is a lattice-ordered ring if for any $f, g \in C(X)_{\Delta}$, one defines $(f \vee g)(x) = f(x) \vee g(x)$ and $(f \wedge g)(x) = f(x) \wedge g(x)$, $x \in X$. Also $f \vee g = \frac{f+g+|f-g|}{2} \in C(X)_{\Delta}$, for all $f, g \in C(X)_{\Delta}$. For $f \in C(X)_{\Delta}$ and $f > 0$, we note that there exists $h \in C(X)_{\Delta}$ such that $f = h^2$. Also, whenever $f \in C(X)_{\Delta}$ and f^r is defined where $r \in \mathbb{R}$, then $f^r \in C(X)_{\Delta}$.

Definition 2.2. We next define,

$$C^*(X)_{\Delta} = \{f \in C(X)_{\Delta} : \ f \text{ is bounded}\}$$

which is obviously closed under the algebraic and order operations as discussed above. Hence $C^*(X)_{\Delta}$ is a subring as well as a sublattice of $C(X)_{\Delta}$.

Remark 2.1. We see that unlike the ring $C(X)$, the equality $C(X)_\vartriangle = C^*(X)_\vartriangle$ is only a sufficient condition for the pseudocompactness of X but not necessary, as it follows from the next example.

Example 2.1. Consider $X = [0,1]$ equipped with the subspace topology of the usual topology of reals and take $\Delta = \{A \subseteq X : A \text{ is nowhere dense in } X\}$. Take the function $f : [0,1] \to \mathbb{R}$ defined by,

$$f(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 1, & x = 0. \end{cases}$$

Clearly $f \in C(X)_\vartriangle$, but $f \notin C^*(X)_\vartriangle$. But here X is pseudocompact.

Definition 2.3. For $f \in C(X)_\vartriangle$, the set $f^{-1}(0) = \{x \in X : f(x) = 0\}$ will be called the zero set of f, to be denoted by $Z_\vartriangle(f)$.

We will use the notation $Z_\vartriangle(C(X)_\vartriangle)$ (or, $Z_\vartriangle(X)$) for the collection $\{Z_\vartriangle(f) : f \in C(X)_\vartriangle\}$ of all zero sets in X.

Some elementary properties of the zero sets of functions of $C(X)_\vartriangle$ are listed below, which are trivial to check as in the classical setting of $C(X)$ (see, 1.10, 1.11 of [7]).

Theorem 2.1. *For $f, g \in C(X)_\vartriangle$ and $r \in \mathbb{R}$, the following holds.*

i) $Z_\vartriangle(f) = Z_\vartriangle(|f|) = Z_\vartriangle(f^n)$, *for all* $n \in \mathbb{N}$.
ii) $Z_\vartriangle(\bar{\mathbf{0}}) = X$ *and* $Z_\vartriangle(\bar{\mathbf{1}}) = \varnothing$.
iii) $Z_\vartriangle(fg) = Z_\vartriangle(f) \cup Z_\vartriangle(g)$.
iv) $Z_\vartriangle(f^2 + g^2) = Z_\vartriangle(f) \cap Z_\vartriangle(g)$.
v) $\{x \in X : f(x) \geq r\}$ *and* $\{x \in X : f(x) \leq r\}$ *are zero sets in* X.
vi) Also for a given $f \in C(X)_\vartriangle$, *the function* $h = |f| \wedge \bar{\mathbf{1}} \in C(X)_\vartriangle$, *so that* $Z_\vartriangle(f) = Z_\vartriangle(h)$ *and hence we can conclude that* $C(X)_\vartriangle$ *and* $C^*(X)_\vartriangle$ *produce the same zero sets.*

Remark 2.2. Unlike $C(X)$, $Z_\vartriangle(f)$ is not necessarily closed as is seen below.

Example 2.2. Consider $X = [0,1]$ with the subspace topology of the usual topology of reals and $\Delta = \{A \subseteq X : A \text{ is nowhere dense in } X\}$. Take the function $f : X \to \mathbb{R}$ defined by, for any $n \in \mathbb{N}$,

$$f(x) = \begin{cases} 1, & x \neq \frac{1}{n} \\ 0, & x = \frac{1}{n}. \end{cases}$$

Then the set of points of discontinuities of f is $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\} \in \Delta$, so that $f \in C(X)_\vartriangle$, but $Z_\vartriangle(f) = \{\frac{1}{n} : n \in \mathbb{N}\}$ which is not closed in X.

Remark 2.3. $Z_\Delta(f)$ need not be a G_δ-set as in the case of $C(X)$ as is seen below.

Example 2.3. Consider $X = \mathbb{R}$ with the cofinite topology. Then no finite set in $\mathbb{R}$ is a G_δ-set. Take the function $f : \mathbb{R} \to \mathbb{R}$ defined by,

$$f(x) = \begin{cases} 1, & x \in \mathbb{R} \setminus \{0\} \\ 0, & x = 0. \end{cases}$$

Then $f \in C(X)_\Delta$ for any subcollection $\Delta \subseteq \mathcal{P}(X)$ and $Z_\Delta(f) = \{0\}$, which is not a G_δ-set.

The following theorem gives the nature of a zero set for a function in $C(X)_\Delta$.

Theorem 2.2. *For any $f \in C(X)_\Delta$, $Z_\Delta(f)$ can be written as a disjoint union of a G_δ-subset of X and a member of Δ.*

Proof. Write $Z_\Delta(f) = P \cup Q$, where $P = Z_\Delta(f) \cap (X \setminus D_f)$ and $Q = Z_\Delta(f) \cap D_f$. As $D_f \in \Delta, Q \in \Delta$. Now the function $h = f|_{X \setminus D_f}$ is a continuous function. Hence $P = Z(h)$ is a G_δ-subset of $X \setminus D_f$ (where $Z(h)$ as usual denotes the zero set for the continuous function h in $X \setminus D_f$). Also D_f being an F_σ-subset of X, P is a G_δ-set in X. Hence the proof. □

Theorem 2.3. *For an arbitrary topological space X (i.e. X does not have any separation axioms), whenever $f \in C(X)_\Delta$ and $Z_\Delta(f) \subseteq X \setminus D_f$, $Z_\Delta(f)$ becomes a G_δ-set in X.*

Proof. From Theorem 2.2, we have $Z_\Delta(f) = P \cup Q$, where P is a G_δ-set in X and $Q = Z_\Delta(f) \cap D_f$ is a member of Δ. Now if $Z_\Delta(f) \subseteq X \setminus D_f$, then $Q = \varnothing$, so that $Z_\Delta(f) = P$, a G_δ-set in X. Hence the proof. □

The following example shows that the converse of Theorem 2.3 is not true in general.

Example 2.4. Let $X = [0,1]$ with the subspace topology of the usual topology of reals and $\Delta = \{A \subseteq [0,1] : A \text{ is countable}\}$. Take the function $f : X \to \mathbb{R}$ defined by,

$$f(x) = \begin{cases} 1, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Then $f \in C(X)_\Delta$ and $Z_\Delta(f) = \{0\}$ is a G_δ-set but $Z_\Delta(f) \nsubseteq X \setminus D_f$.

Remark 2.4. In [2], in the discussion after Theorem 2.1, the authors have mentioned that if X is a T_1 space, $f \in C(X)_F$ and $\mathcal{Z}(f) \subseteq X \setminus D_f$, then $\mathcal{Z}(f)$ is G_δ. But from the above theorem, we can say that if we consider $\Delta =$ the set of all finite subsets of X there, then the same is true without assuming any separation axioms (in particular, T_1-ness) of X.

Theorem 2.4. *For a topological space X and a subcollection $\Delta \subseteq \mathcal{P}(X)$, the following statements hold.*

i) $C(X)_\vartriangle$ is a reduced ring.
ii) $f \in C(X)_\vartriangle$ is a unit if and only if $Z_\vartriangle(f) = \varnothing$.
iii) Any element of $C(X)_\vartriangle$ is either a zero-divisor or a unit.
iv) For $f, g \in C(X)_\vartriangle$, if $|f| < |g|^r$ for some real number $r > 1$, then f is a multiple of g. In particular, if $|f| < |g|$ and $r \in \mathbb{R}$ with $r > 1$ be such that f^r is defined, then f^r is a multiple of g.

Proof. *i)* It is trivial.
ii) Let $f \in C(X)_\vartriangle$ be a unit. Then there exists $g \in C(X)_\vartriangle$ such that $f.g = \bar{\mathbf{1}}$, so that $Z_\vartriangle(f) = \varnothing$. Conversely, if $Z_\vartriangle(f) = \varnothing$, then the function $g = \frac{1}{f} \in C(X)_\vartriangle$ is the required inverse of f, so that f becomes a unit in $C(X)_\vartriangle$.
iii) Let $f \in C(X)_\vartriangle$ be not a unit. Then $Z_\vartriangle(f) \neq \varnothing$. Choose $p \in Z_\vartriangle(f)$ and define a function $g : X \to \mathbb{R}$ by $g(p) = 0$ and $g(X \setminus \{p\}) = \{1\}$. Then $g \in C(X)_\vartriangle$ and $X \setminus Z_\vartriangle(f) \subseteq Z_\vartriangle(g)$, which implies that $fg = 0$, i.e. f is a zero-divisor of $C(X)_\vartriangle$.
iv) Let $|f| < |g|^r$ for some real number $r > 1$, where $f, g \in C(X)_\vartriangle$. Clearly $Z_\vartriangle(g) \subseteq Z_\vartriangle(f)$. Take $D = D_f \cup D_g$. Then $D \in \Delta$ and f, g are continuous on $X \setminus D$. Define a function $h : X \to \mathbb{R}$ by

$$h(x) = \begin{cases} \frac{f(x)}{g(x)}, & x \in X \setminus Z_\vartriangle(g) \\ \\ 0, & x \in Z_\vartriangle(g). \end{cases}$$

We now show that h is continuous on the set $X \setminus D$. Let $x \in (X \setminus D) \setminus Z_\vartriangle(g)$. Since f and g are continuous at x and $g(x) \neq 0$, so $\frac{f}{g}$ is continuous at x, i.e. h is continuous at x.
Now $|f| < |g|^r$ implies that $\frac{|f(x)|}{|g(x)|} < |g(x)|^{r-1}$, for all $x \in X \setminus Z_\vartriangle(g)$ which gives that $|h(x)| < |g(x)|^{r-1}$, for all $x \in X \setminus Z_\vartriangle(g)$. Again, $x \in Z_\vartriangle(g)$ implies that $g(x) = 0$, so that $h(x) = 0$. Hence $|h| \leq |g|^{r-1}$, for all $x \in X$.
Let $x \in (X \setminus D) \cap Z_\vartriangle(g)$. Then $h(x) = 0 \in (-\epsilon, \epsilon)$. Also we have $g(x) = 0$ and g is continuous at x, so there exists a neighbourhood U of x such that $g(U) \subseteq (-\epsilon^{\frac{1}{r-1}}, \epsilon^{\frac{1}{r-1}})$ which implies that $|g(x)| < \epsilon^{\frac{1}{r-1}}$, for all $x \in U$. Thus $|g(x)|^{r-1} < \epsilon$, for all $x \in U$ which implies that $|h(x)| < \epsilon$, for all $x \in U$. Hence h is continuous on $X \setminus D$ so that $h \in C(X)_\vartriangle$ and $f = gh$.
The second part follows from the first part. □

Remark 2.5. In $C(X)_F$, we have seen that $C(X)_F = C^*(X)_F$ if and only if for any finite subset F of X, $X \setminus F$ is pseudocompact ([6], Lemma 2.4). That means if we consider $\Delta =$ the set of all finite subsets of X, then $C(X)_\vartriangle = C^*(X)_\vartriangle$ if and only if for any $F \in \Delta$, $X \setminus F$ is pseudocompact. But for any arbitrary Δ, it is not necessarily true as is seen below.

Example 2.5. Let $X = \mathbb{N}$ be endowed with the cofinite topology. Consider $\Delta = \{P : P$ is a countable subset of $\mathbb{N}\}$. Then $\mathbb{R}^{\mathbb{N}} = C(\mathbb{N})_\vartriangle \neq C^*(\mathbb{N})_\vartriangle$. Now the function f defined by

$f(n) = n$, for all $n \in \mathbb{N}$, is a member of $C(\mathbb{N})_{\triangle}$, but $f \notin C^{*}(\mathbb{N})_{\triangle}$. But for any countable set F, $X \setminus F$ is always pseudocompact.

Remark 2.6. In view of Theorem 2.4, we can conclude that $C(X)_{\triangle}$ is an almost regular ring.

Next we give an example to show that the result analogous to Theorem 2.4 ii) is not true if we replace $C(X)_{\triangle}$ by $C^{*}(X)_{\triangle}$.

Example 2.6. In the view of Example 2.1, the function $\frac{1}{f} = h$ has an empty zero set. This function $h \in C^{*}(X)_{\triangle}$, whereas $\frac{1}{h} = f \notin C^{*}(X)_{\triangle}$.

The nature of the units of $C^{*}(X)_{\triangle}$ is given by the following theorem.

Theorem 2.5. *A function $f \in C^{*}(X)_{\triangle}$ is a unit in $C^{*}(X)_{\triangle}$ if and only if f is bounded away from zero, i.e. there exists $r > 0$ such that $|f(x)| \geq r$, for all $x \in X$.*

Proof. Just take into account that whenever for some $f \in C^{*}(X)_{\triangle}$, $Z_{\triangle}(f) = \varnothing$, then $D_f = D_{\frac{1}{f}}$. □

Remark 2.7. We next provide two dissimilarities between $C(X)$ and $C(X)_{\triangle}$.

Example 2.7. $C(X)_{\triangle}$ is not closed under uniform limits: Consider $X = [0,1]$ with the subspace topology of the usual topology of $\mathbb{R}$ and Δ = set of all finite subsets of $[0,1]$. Enummerate $[0,1] \cap \mathbb{Q}$ as, $[0,1] \cap \mathbb{Q} = \{x_1, x_2, ..., x_n, ...\}, n \in \mathbb{N}$. Now define a sequence of functions $\{f_n\}$ on X by,

$$f_n(x) = \begin{cases} \frac{1}{i}, & x = x_i, 1 \leq i \leq n \\ 0, & \textit{otherwise.} \end{cases}$$

Clearly each $f_n \in C(X)_{\triangle}$ and this sequence of functions converges uniformly to the function f given by,

$$f(x) = \begin{cases} \frac{1}{n}, & x = x_n \\ 0, & \textit{otherwise.} \end{cases}$$

But $f \notin C(X)_{\triangle}$, as f is discontinuous on $\mathbb{Q}$. Hence $C(X)_{\triangle}$ is not closed under uniform limits.

Example 2.8. $Z_{\triangle}(C(X)_{\triangle})$ is not closed under countable intersections: Let $X = [0,1]$ with the subspace topology of the usual topology of $\mathbb{R}$ and Δ = set of all finite subsets of $[0,1]$. Consider $[0,1] \cap \mathbb{Q} = \{x_1, x_2, ..., x_n, ...\}, n \in \mathbb{N}$. Now define a sequence of functions $\{f_n\}$ on X by,

$$f_n(x) = \begin{cases} 1, & x = x_1, x_2, ..., x_n \\ 0, & \textit{otherwise.} \end{cases}$$

Clearly each $f_n \in C(X)_\Delta$, $n \in \mathbb{N}$.

Now, $\bigcap_{n=1}^{\infty} Z_\Delta(f_n) = \bigcap_{n=1}^{\infty}([0,1] \setminus \{x_1, x_2, ..., x_n\}) = [0,1] \setminus \bigcup_{n=1}^{\infty}\{x_1, x_2, ..., x_n\} = [0,1] \bigcap \mathbb{Q}^c$.

Now we show that there does not exist any $f \in C(X)_\Delta$ such that $Z_\Delta(f) = [0,1] \bigcap \mathbb{Q}^c$.

If possible, let there exist $f \in C(X)_\Delta$ with $Z_\Delta(f) = [0,1] \bigcap \mathbb{Q}^c$. Choose $c \in [0,1] \bigcap \mathbb{Q}$, then $f(c) \neq 0$. Without loss of generality, let $f(c) > 0$. Choose $\epsilon > 0$ such that $f(c) - \epsilon > 0$. If f is continuous at c, then there exists an open set $G \subseteq [0,1]$ containing c such that $|f(x) - f(c)| < \epsilon$, for all $x \in G$ which implies that $f(x) > f(c) - \epsilon > 0$, for all $x \in G$, i.e. $f(x) > 0$, for all $x \in G$, which contradicts the fact that $[0,1] \bigcap \mathbb{Q}^c$ is dense in $[0,1]$. Hence f is not continuous at any rational number, so that $f \notin C(X)_\Delta$.

Remark 2.8. From the definition of Δ it can be easily observed that if the set of all non-isolated points of X is a member of Δ, then $C(X)_\Delta = \mathbb{R}^X = C(Y)$, where $X = Y$ is equipped with the discrete topology. So in this case we can say that $C(X)_\Delta$ is a C-ring..

3. Δ-completely separated and C_Δ-embedded subsets of X

Recall that two subsets A and B of a topological space X are said to be completely separated in X ([7], Theorem 1.15) if there exists a function $f \in C^*(X)$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$, with $\bar{\mathbf{0}} \leq f \leq \bar{\mathbf{1}}$.

Analogously we define the following.

Definition 3.1. Two subsets A and B of X are said to be Δ-completely separated in X, if there exists a function f in $C^*(X)_\Delta$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

In $C(X)$, it is true that two sets A and B are completely separated if and only if their respective closures $\overline{A}$ and $\overline{B}$ are also completely separated. But we here notice that $\overline{A}$ and $\overline{B}$ are Δ-completely separated in X implies that A and B are Δ-completely separated. That the converse is not true in general, is seen by the following example.

Example 3.1. Take $X = [0,1]$ with the subspace topology of the usual topology of reals, $A = [0,1), B = \{1\}$. Then A and B are Δ-completely separated by the function $f : X \to \mathbb{R}$ defined by,

$$f(x) = \begin{cases} 1, & 0 \leq x < 1 \\ 2, & x = 1, \end{cases}$$

where $f \in C^*(X)_\Delta$, for any arbitrary subcollection $\Delta \subseteq \mathcal{P}(X)$, but $\overline{A}$, $\overline{B}$ are not Δ-completely separated, as $\overline{A} \cap \overline{B} \neq \varnothing$.

Also in this connection we want to mention the notion of $\mathcal{F}$-completely separated sets (see [6]), where any two completely separated sets are $\mathcal{F}$-completely separated but not the converse.

Remark 3.1. Any two $\mathcal{F}$-completely separated sets are Δ-completely separated but not conversely as is seen by the following example.

Example 3.2. Consider $X = [0, 1]$ with the subspace topology of the usual topology of reals, $\Delta = \{A \subseteq X : A$ is nowhere dense in $X\}$ and $K =$ Cantor set. Define $f : X \to \mathbb{R}$ by,

$$f(x) = \begin{cases} 1, & x \in K \\ 0, & x \notin K, \end{cases}$$

i.e. $f = \chi_K$. Then $D_f = K \in \Delta$, so that $f \in C(X)_\Delta$. Now, the sets K and $X \setminus K$ are Δ-completely separated but not $\mathcal{F}$-completely separated, as K is uncountable.

The next result is the counterpart of ([7], Theorem 1.15) and can be proved in a similar manner.

Theorem 3.1. *Two subsets A, B of a space X are Δ-completely separated if and only if they are contained in disjoint members of $Z_\Delta(X)$.*

Corollary 3.1. *If A and A' are Δ-completely separated, then there exist zero sets Z and H in $Z_\Delta(X)$ such that*

$$A \subseteq X \setminus Z \subseteq H \subseteq X \setminus A'.$$

Theorem 3.2. *If two disjoint subsets A and B of X are Δ-completely separated, then there is a member D of Δ such that $A \setminus D$ and $B \setminus D$ are completely separated in $X \setminus D$.*

Proof. Assume that A, B are Δ-completely separated. Then by Theorem 3.1, there exist two disjoint zero sets $Z_\Delta(f_1)$ and $Z_\Delta(f_2)$ in $Z_\Delta(X)$ such that $A \subseteq Z_\Delta(f_1)$ and $B \subseteq Z_\Delta(f_2)$. Let D_{f_1} and D_{f_2} be the sets of points of discontinuities of f_1 and f_2 respectively. Then $f_1 \in C(X \setminus D_{f_1})$, $f_2 \in C(X \setminus D_{f_2})$. Consider $D = D_{f_1} \cup D_{f_2}$. Then $D \in \Delta$ and $f_1, f_2 \in C(X \setminus D)$. Also, $A \setminus D \subseteq Z_\Delta(f_1) \setminus D$, $B \setminus D \subseteq Z_\Delta(f_2) \setminus D$, where $Z_\Delta(f_1) \setminus D$ and $Z_\Delta(f_2) \setminus D$ are disjoint zero-sets in $X \setminus D$. By ([7], Theorem 1.15), $A \setminus D$ and $B \setminus D$ are completely separated in $X \setminus D$. $\square$

Remark 3.2. The converse of the above theorem holds good if D is closed. For let, $A \setminus D$ and $B \setminus D$ be completely separated in $X \setminus D$, where $D \in \Delta$ and D is closed. Then there exists $f \in C^*(X \setminus D)$ with $f(A \setminus D) = \{0\}$ and $f(B \setminus D) = \{1\}$. Now consider the function $g : X \to \mathbb{R}$ defined as follows:

$$g(x) = \begin{cases} f(x), & x \in X \setminus D \\ 0, & x \in D \cap A \\ 1, & x \in D \cap B. \end{cases}$$

Since D is closed, $g \in C^*(X)_\Delta$ with $g(A) = \{0\}$ and $g(B) = \{1\}$. Hence A and B are Δ-completely separated in X.

Next, we introduce the analogues of C-embedding and C^*-embedding in our settings, called C_Δ-embedding and C^*_Δ-embedding to deal with the problem of extension of functions belonging to such rings.

Definition 3.2. A subset Y of a topological space X is said to be C_Δ-embedded in X, if each $f \in C(Y)_{\Delta_Y}$ has an extension to a $g \in C(X)_\Delta$, i.e. there exists $g \in C(X)_\Delta$ such that $g|_Y = f$, where $\Delta \subseteq \mathcal{P}(X)$ and $\Delta_Y = \Delta|_{\mathcal{P}(Y)}$.

Likewise, Y is said to be C^*_Δ-embedded in X, if each $f \in C^*(Y)_\Delta$ has an extension to a $g \in C^*(X)_\Delta$.

Remark 3.3. It is noteworthy to mention here that any C_Δ-embebbed subset is C^*_Δ-embebbed also.

Example 3.3. Consider $X = \mathbb{R}^2$ with the Euclidean topology, $\Delta = \{A \subseteq \mathbb{R}^2 : A$ is nowhere dense in $\mathbb{R}^2\}$, $S = \mathbb{R}^2 \setminus \mathbb{R} \times \{0\}$ and a function $f : S \to \mathbb{R}$ defined by,

$$f(x, y) = \frac{1}{y}, (x, y) \in \mathbb{R}^2 \setminus \mathbb{R} \times \{0\}.$$

As $f \in C(S)$, clearly $f \in C(S)_\Delta$. But there does not exist any $g \in C(\mathbb{R}^2)_F$ such that $g|_S = f$. Hence S is not C_F-embedded (see [2], Definition 2.15) and hence not C-embedded in X. Now, consider the function $g : X \to \mathbb{R}$ defined by $g(X \setminus S) = f$ and $g(S) = 0$. Then S is C_Δ-embedded but not C_F-embedded and hence not C-embedded.

In view of the above example we observe that if S is a closed subset of a topological space X with $X \setminus S \in \Delta$, then S is both C^*_Δ-embedded and C_Δ-embedded.

As a converse of Remark 3.3, we have the following.

Theorem 3.3. *A C^*_Δ-embedded subset is C_Δ-embedded if and only if it is Δ-completely separated from every zero set disjoint from it.*

Proof. First, let S be C^*_Δ-embedded in X and $h \in C(X)_\Delta$ be such that $Z_\Delta(h) \cap S = \varnothing$. Define a function $f : S \to \mathbb{R}$ by $f(s) = \frac{1}{h(s)}, s \in S$. Then $f \in C(S)_\Delta$. By the given condition, there exists $g \in C(X)_\Delta$ such that $g|_S = f$. Hence $gh \in C(X)_\Delta$. Also $gh(S) = \{1\}$ and $gh(Z_\Delta(h)) = \{0\}$, so that $Z_\Delta(h)$ and S are Δ-completely separated in X.

Conversely, let $f \in C(S)_\Delta$. As $\arctan \circ f \in C^*(S)_\Delta$, there exists $g \in C(X)_\Delta$ such that $g|_S = \arctan \circ f$. Now, the set $Z = \{x \in X : |g(x)| \geq \frac{\pi}{2}\}$ is a member of $Z_\Delta(X)$ with $Z \cap S = \varnothing$. So by hypothesis, there exists $h \in C^*(X)_\Delta$ such that $h(S) = \{1\}$ and $h(Z) = \{0\}$. We see that $g \cdot h \in C(X)_\Delta$ and for all $x \in X$, $|(g \cdot h)(x)| < \frac{\pi}{2}$. Hence, $\tan(g \cdot h) \in C(X)_\Delta$ and for all $s \in S$, $\tan(g \cdot h)(s) = f(s)$. So S is C_Δ-embedded. □

Corollary 3.2. *For any topological space X, a zero set $Z \in Z_\Delta(X)$ is C^*_Δ-embedded if and only if it is C_Δ-embedded.*

Example 3.4. (i) If a discrete zero set is C^*_Δ-embedded, then all of its subsets are zero sets: for if $Z \in Z_\Delta(X)$ be a discrete, C^*_Δ-embedded subset of X, then for any $Y \subseteq Z$, Y is also discrete. Define a function $f : Z \to \mathbb{R}$ by,

$$f(x) = \begin{cases} 1, & x \notin Y \\ 0, & x \in Y. \end{cases}$$

Then $f \in C(Z)_\Delta$. As Z is C^*_Δ-embedded, there exists $h \in C^*(X)_\Delta$ such that $h|_Z = f$. Also, as Z is a zero set, $Z = Z_\Delta(g)$, for some $g \in C^*(X)_\Delta$. Now, consider the function $k \in C^*(X)_\Delta$ by $k = g^2 + h^2$. Certainly, $Z_\Delta(k) = Z \cap Z_\Delta(h) = Y$, so that Y becomes a zero set in X.

(ii) If for every $f \in C^*(X)_\Delta$, $f(X)$ is compact, then X becomes pseudocompact. But the converse is not true. Consider $X = [0,1]$ with the subspace topology of the usual topology of reals, $\Delta = \{A \subseteq [0,1] : A$ is nowhere dense in $X\}$ and a function $f : X \to \mathbb{R}$ defined by, for $n \in \mathbb{N}$,

$$f(x) = \begin{cases} \frac{1}{n}, & x = \frac{1}{n} \\ 1, & x \neq \frac{1}{n}. \end{cases}$$

Then $D_f = \{0\} \cup \{\frac{1}{n} : n \geq 2\} \in \Delta$ and $f \in C^*(X)_\Delta$. But $f(X) = \{\frac{1}{n} : n \in \mathbb{N}\}$, which is not compact.

4. Ideals of $C(X)_\Delta$ and Z_Δ-filters on X

Throughout our discussion, an ideal I, unmodified in any of the two rings $C(X)_\Delta$ and $C^*(X)_\Delta$ will always mean a proper ideal.

Definition 4.1. A nonempty subcollection $\mathcal{F}$ of $Z_\Delta(X)$ is called a Z_Δ-filter on X if it satisfies the following conditions:

(i) $\varnothing \notin \mathcal{F}$.
(ii) $Z_1, Z_2 \in \mathcal{F}$ implies that $Z_1 \cap Z_2 \in \mathcal{F}$.
(iii) If $Z \in \mathcal{F}, Z' \in Z_\Delta(X)$ with $Z \subset Z'$, then $Z' \in \mathcal{F}$.

A Z_Δ-filter on X which is not properly contained in any Z_Δ-filter on X is called a Z_Δ-ultrafilter on X.

Applying Zorn's lemma one can show that a Z_Δ-filter on X can be extended to a Z_Δ-ultrafilter on X.

There is a nice interplay between ideals (maximal ideals) in $C(X)_\Delta$ and the Z_Δ-filters (resp., Z_Δ-ultrafilters) on X. This fact is observed in the following theorem.

Theorem 4.1. *For the ring $C(X)_\Delta$, the following hold.*

i) If I is an ideal in $C(X)_\triangle$, then $Z_\triangle(I) = \{Z_\triangle(f) : f \in I\}$ is a $Z_\triangle$-filter on X. Dually, if $\mathcal{F}$ is a $Z_\triangle$-filter on X, then $Z_\triangle^{-1}(\mathcal{F})$ is an ideal in $C(X)_\triangle$.
ii) If M is a maximal ideal in $C(X)_\triangle$, then $Z_\triangle(M)$ is a $Z_\triangle$-ultrafilter on X. If $\mathcal{U}$ is a $Z_\triangle$-ultrafilter on X, then $Z_\triangle^{-1}(\mathcal{U})$ is a maximal ideal in $C(X)_\triangle$.
iii) The assignment : $M \to Z_\triangle(M)$ is a bijection from the set of all maximal ideals of $C(X)_\triangle$ to the set of all $Z_\triangle$-ultrafilters on X.

Proof. Can be done in same way as in Theorems 2.3 and 2.5 of [7]. □

Remark 4.1. The assignment : $I \to Z_\triangle(I)$ from the set of all ideals on $C(X)_\triangle$ to the set of all $Z_\triangle$-filters on X is a surjection but not an injection. In fact, for any ideal I in $C(X)_\triangle$, $Z_\triangle^{-1}Z_\triangle(I) \supseteq I$.

We therefore concentrate on those ideals of $C(X)_\triangle$ for which the above inclusion becomes an equality.

Definition 4.2. An ideal I of $C(X)_\triangle$ is called a $Z_\triangle$-ideal if $Z_\triangle^{-1}Z_\triangle(I) = I$. Equivalently, $Z_\triangle(f) = Z_\triangle(g)$, for $f \in I$ and $g \in C(X)_\triangle$ implies that $g \in I$.

Remark 4.2. It thus follows that
i) Every maximal ideal in $C(X)_\triangle$ is a $Z_\triangle$-ideal but not the converse (as shown below in Example 4.1).
ii) The mapping : $I \to Z_\triangle(I)$ is a bijection from the set of $Z_\triangle$-ideals onto the set of all $Z_\triangle$-filters.

Example 4.1. Consider $I = \{f \in C(X)_\triangle : f(p) = f(q) = 0\}$, for $p, q \in \mathbb{R}$ with $p \neq q$. Then I is a $Z_\triangle$-ideal in $C(X)_\triangle$. But I is not maximal, as $I \subset \{f \in C(X)_\triangle : f(p) = 0\}$. The ideal I is not a prime ideal also, as the function $(x-p)(x-q)$ belongs to I but neither the function $x - p$ nor the function $x - q$ belongs to I.

Remark 4.3. Clearly every $Z_\triangle$-ideal in $C(X)_\triangle$ is an intersection of prime ideals in $C(X)_\triangle$.

The next result establishes the relation between prime ideals and $Z_\triangle$-ideals to some extent.

Theorem 4.2. *Let I be a $Z_\triangle$-ideal in $C(X)_\triangle$. Then the following statements are equivalent:*

i) I is prime.
ii) I contains a prime ideal.
iii) For all $f, g \in C(X)_\triangle$, if $fg = 0$, then either $f \in I$ or $g \in I$.
iv) For each $f \in C(X)_\triangle$, there exists a zero set in $Z_\triangle(I)$ on which f does not change its sign.

Proof. Similar to the counterpart of Theorem 2.9 in [7]. □

Corollary 4.1. *Every prime ideal in $C(X)_\vartriangle$ is contained in a unique maximal ideal in $C(X)_\vartriangle$, i.e. $C(X)_\vartriangle$ is a Gelfand ring.*

Definition 4.3. A $Z_\vartriangle$-filter $\mathcal{F}$ on X is called a prime $Z_\vartriangle$-filter if whenever $A \cup B \in \mathcal{F}$, for some $A, B \in Z_\vartriangle(C(X)_\vartriangle)$, then either $A \in \mathcal{F}$ or $B \in \mathcal{F}$.

The next theorem is analogous to Theorem 2.12 of [7] and we therefore omit the proof.

Theorem 4.3. *For a space X, the following hold.*

i) If P is a prime ideal in $C(X)_\vartriangle$, then $Z_\vartriangle(P)$ is a prime $Z_\vartriangle$-filter.
ii) If $\mathcal{F}$ is a prime $Z_\vartriangle$-filter on X, then $Z_\vartriangle^{-1}(\mathcal{F})$ is a prime $Z_\vartriangle$-ideal.

Corollary 4.2. *For a space X, the following hold.*

i) Every prime $Z_\vartriangle$-filter is contained in a unique $Z_\vartriangle$-ultrafilter.
ii) Every $Z_\vartriangle$-ultrafilter is a prime $Z_\vartriangle$-filter.

It is known that in a commutative ring R with unity, the intersection of all prime ideals of R containing an ideal I is said to be the radical of I to be denoted by $\sqrt{I}$. For any ideal I, $\sqrt{I} = \{a \in R : a^n \in I, \text{ for some } n \in \mathbb{N}\}$ (see [7]) and also $I \subseteq \sqrt{I}$. Also I is called radical if $I = \sqrt{I}$.

Theorem 4.4. *Every $Z_\vartriangle$-ideal I in $C(X)_\vartriangle$ is a radical ideal.*

Proof. Only to use the definition of a $Z_\vartriangle$-ideal. □

It is well known that the sum of two z-ideals in $C(X)$ is a z-ideal, (see [7], Lemma 14.8 and [12]). This result can be modified in $C(X)_\vartriangle$ as follows.

Theorem 4.5. *The sum of two $Z_\vartriangle$-ideals in $C(X)_\vartriangle$ is a $Z_\vartriangle$-ideal.*

Proof. Let I, J be two $Z_\vartriangle$-ideals in $C(X)_\vartriangle$, $f \in I$, $g \in J$, $h \in C(X)_\vartriangle$ and $Z_\vartriangle(f + g) \subseteq Z_\vartriangle(h)$. First note that, $Z_\vartriangle(f) \cap Z_\vartriangle(g) \subseteq Z_\vartriangle(h)$ and there exists a subset $P \in \Delta$ such that $f, g, h \in C(X \setminus P)$. Define

$$k(x) = \begin{cases} 0, & x \in (Z_\vartriangle(f) \cap Z_\vartriangle(g)) \setminus P \\ \frac{hf^2}{f^2+g^2}, & x \in (X \setminus P) \setminus (Z_\vartriangle(f) \cap Z_\vartriangle(g)) \\ h(x), & x \in P \end{cases}$$

$$l(x) = \begin{cases} 0, & x \in (Z_\triangle(f) \cap Z_\triangle(g)) \setminus P \\ \frac{hg^2}{f^2+g^2}, & x \in (X \setminus P) \setminus (Z_\triangle(f) \cap Z_\triangle(g)) \\ 0, & x \in P. \end{cases}$$

We first prove that k is continuous on $X \setminus P$. So it requires only to prove that k is continuous at any point $x \in (Z_\triangle(f) \cap Z_\triangle(g)) \setminus P$. As $h(x) = 0$, for any given $\epsilon > 0$, there exists a neighbourhood U of x such that $h(U) \subseteq (-\epsilon, \epsilon)$. Also for any $x \in U$, $|k(x)| \leq |h(x)|$, which means that k is continuous on $X \setminus P$. Similarly l is continuous on $X \setminus P$. Then we have $l, k \in C(X)_\triangle$, $Z_\triangle(f) \subseteq Z_\triangle(k)$, $Z_\triangle(g) \subseteq Z_\triangle(l)$ and $h = l + k$. Since I, J are $Z_\triangle$-ideals, $k \in I$ and $l \in J$, hence $h \in I + J$. □

Corollary 4.3. *Let $\{I_\alpha\}_{\alpha\in\Lambda}$ be a collection of $Z_\triangle$-ideals in $C(X)_\triangle$. Then either $\sum_{\alpha\in\Lambda} I_\alpha = C(X)_\triangle$ or $\sum_{\alpha\in\Lambda} I_\alpha$ is a $Z_\triangle$-ideal.*

Lemma 4.1. *[10] If P is minimal in the class of prime ideals containing a z-ideal I, then P is a z-ideal.*

In view of the above result, we can have,

Corollary 4.4. *Let $\{P_\alpha\}_{\alpha\in\Lambda}$ be a collection of minimal prime ideals in $C(X)_\triangle$. Then either $\sum_{\alpha\in\Lambda} P_\alpha = C(X)_\triangle$ or $\sum_{\alpha\in\Lambda} P_\alpha$ is a prime ideal in $C(X)_\triangle$.*

The following result can be obtained in the same way as is done in ([12], Lemma 5.1).

Corollary 4.5. *The sum of a collection of semi prime ideals in $C(X)_\triangle$ is either a semiprime ideal or the entire ring $C(X)_\triangle$.*

5. Fixed and Free ideals in $C(X)_\triangle$

In this section, we introduce fixed and free ideals of $C(X)_\triangle$ and $C^*(X)_\triangle$ and completely characterize the fixed maximal ideals of $C(X)_\triangle$ and that of $C^*(X)_\triangle$.

Definition 5.1. A proper ideal I of $C(X)_\triangle$ (resp., $C^*(X)_\triangle$) is called fixed if $\cap Z_\triangle(I) \neq \varnothing$, where $\cap Z_\triangle(I) = \bigcap_{f\in I} Z_\triangle(f)$. If I is not fixed, then it is called free.

Let us now characterize the fixed maximal ideals of $C(X)_\triangle$ and those of $C^*(X)_\triangle$.

Theorem 5.1. *$\{M_p^{\triangle} : p \in X\}$ is a complete list of fixed maximal ideals of $C(X)_{\triangle}$, where $M_p^{\triangle} = \{f \in C(X)_{\triangle} : f(p) = 0\}$. Moreover, the ideals $M_p^{\triangle}$ are distinct for distinct p.*

Proof. First choose $p \in X$. The map $\psi : C(X)_{\triangle} \to \mathbb{R}$ defined by $\psi_p(f) = f(p)$ is a ring homomorphism. Also ψ_p is surjective and *ker* $\psi_p = \{f \in C(X)_{\triangle} : \psi_p(f) = 0\} = M_p^{\triangle}$ (say). Hence by the First Isomorphism theorem of rings, we have $C(X)_{\triangle}/M_p^{\triangle}$ is isomorphic to the field $\mathbb{R}$, so that $M_p^{\triangle}$ is a maximal ideal in $C(X)_{\triangle}$. Also, as $p \in \cap Z_{\triangle}[M_p^{\triangle}]$, $M_p^{\triangle}$ is a fixed ideal.

Now, $p \neq q$ implies that $\chi_{\{p\}} \neq \chi_{\{q\}}$, where $\chi_{\{p\}}, \chi_{\{q\}} \in C(X)_{\triangle}$ (since X is T_1). As $\chi_{\{p\}} \in M_q^{\triangle}$ but $\chi_{\{p\}} \notin M_p^{\triangle}$, it thus follows that for $p \neq q$, $M_p^{\triangle} \neq M_q^{\triangle}$. □

Similarly we have,

Theorem 5.2. *$\{M_p^{\triangle^*} : p \in X\}$ is a complete list of fixed maximal ideals of $C^*(X)_{\triangle}$, where $M_p^{\triangle^*} = \{f \in C^*(X)_{\triangle} : f(p) = 0\}$. Moreover, $p \neq q$ implies that $M_p^{\triangle^*} \neq M_q^{\triangle^*}$.*

From above it follows that the Jacobson radical of the ring $C(X)_{\triangle}$ and $C^*(X)_{\triangle}$ is zero. Also the interrelation between fixed ideals of $C(X)_{\triangle}$ and $C^*(X)_{\triangle}$ are as follows.

Corollary 5.1. *If I is a fixed maximal ideal of $C(X)_{\triangle}$, then $I \cap C^*(X)_{\triangle}$ is so in $C^*(X)_{\triangle}$. Also, if $I \cap C^*(X)_{\triangle}$ is a fixed ideal of $C^*(X)_{\triangle}$, for some ideal I of $C(X)_{\triangle}$, then I is a fixed ideal of $C(X)_{\triangle}$.*

We now give a result with the help of which we get another description of $Z_{\triangle}$-ideals.

Lemma 5.1. *For any $f \in C(X)_{\triangle}$, we have $M_f^{\triangle} = \{g \in C(X)_{\triangle} : Z_{\triangle}(f) \subseteq Z_{\triangle}(g)\}$, where $M_f^{\triangle}$ is the intersection of all maximal ideasl of $C(X)_{\triangle}$ containing f.*

Proof. The proof is same as that of Lemma 4.1 of [6]. □

The following is the counterpart of ([7], 4A).

Theorem 5.3. *A necessary and sufficient condition that an ideal I in $C(X)_{\triangle}$ be a $Z_{\triangle}$-ideal is that, for a given g, if there exists $f \in I$ such that $g \in M_f^{\triangle}$, then $g \in I$.*

Proof. Let I be a $Z_{\triangle}$-ideal and for a given g, there exists $f \in I$ such that $g \in M_f^{\triangle}$. Then $Z_{\triangle}(f) \subseteq Z_{\triangle}(g)$. Also $f \in I$ implies that $Z_{\triangle}(f) \in Z_{\triangle}(I)$, so that $Z_{\triangle}(g) \in Z_{\triangle}(I)$ (as $Z_{\triangle}(I)$ is a $Z_{\triangle}$-filter) which further implies that $g \in I$.

Conversely, let $Z_{\triangle}(g) \in Z_{\triangle}(I)$ imply that $Z_{\triangle}(g) = Z_{\triangle}(f)$, for some $f \in I$. So $g \in M_f^{\triangle}$. Thus by the given condition $g \in I$. Hence I is a $Z_{\triangle}$-ideal. □

Regarding the existence of free maximal ideals in $C(X)_\vartriangle$ and in $C(X)_\vartriangle$, we now establish the following.

Theorem 5.4. *For a space X, the following are equivalent:*

i) X is finite.
ii) Every proper ideal of $C(X)_\vartriangle$ is fixed.
iii) Every maximal ideal of $C(X)_\vartriangle$ is fixed.
iv) Every proper ideal of $C^*(X)_\vartriangle$ is fixed.
v) Every maximal ideal of $C^*(X)_\vartriangle$ is fixed.

Proof. $i) \Rightarrow ii)$: Let I be a proper ideal of $C(X)_\vartriangle$. Now $Z[I](\equiv \{Z(f) : f \in I\})$ is finite and also a $Z_\vartriangle$-filter. Hence I is fixed.
$ii) \Rightarrow iii)$: Obvious.
$iii) \Rightarrow i)$: If possible, let X be infinite. Let $S = \{\chi_{\{x\}} : x \in X\}$ and consider the ideal I generated by S in $C(X)_\vartriangle$. We claim that I is proper. If not, then there exists $x_1, x_2, ..., x_n$ and $f_1, f_2, ..., f_n \in C(X)_\vartriangle$ such that $\bar{\mathbf{1}} = f_1\chi_{\{x_1\}} + f_2\chi_{\{x_2\}} + ... + f_n\chi_{\{x_n\}}$. Then $\bigcap_{i=1}^{n} Z_\vartriangle[\chi_{\{x_i\}}] = \varnothing$. Hence $\bigcap_{i=1}^{n}(X \setminus \{x_i\}) = \varnothing$ which implies that X is finite, a contradiction. Let M be any maximal ideal of $C(X)_\vartriangle$ containing I. Then $\bigcap Z[M] \subseteq \bigcap Z[I] \subseteq \bigcap_{x\in X}(X \setminus \{x\}) = \varnothing$ which implies that M is a free ideal, a contradiction. Hence X is finite.
$i) \Rightarrow iv)$: Can be done as in $i) \Rightarrow ii)$.
$ii) \Rightarrow v)$: Obvious.
$v) \Rightarrow i)$: Obvious. □

In view of Example 4.7 of [7], since $C(X) = C(X)_\vartriangle$, for any discrete space X, we can conclude that

i) For any maximal ideal M of $C(X)_\vartriangle$, $M \cap C^*(X)_\vartriangle$ need not be a maximal ideal in $C^*(X)_\vartriangle$.
ii) All free maximal ideals in $C^*(X)_\vartriangle$ need not be of the form $M \cap C^*(X)_\vartriangle$, where M is a maximal ideal in $C(X)_\vartriangle$.

6. Residue class rings of $C(X)_\vartriangle$ modulo ideals

Let us recall that an ideal I in a partially ordered ring A is called convex if whenever $0 \leq x \leq y$ and $y \in I$, then $x \in I$. Equivalently, for all $a, b, c \in A$ with $a \leq b \leq c$ and $a, c \in I$ implies that $b \in I$.

If A is a lattice-ordered ring, then an ideal I of A is said to be absolutely convex if whenever $|x| \leq |y|$ and $y \in I$, then $x \in I$.

For an ideal I of $C(X)_\vartriangle$, we shall denote any member of the quotient ring $C(X)_\vartriangle/I$ by $I(f)$, for $f \in C(X)_\vartriangle$, i.e. $I(f) = f + I$.

Let us now recall the following.

Theorem 6.1. *[7]. Let I be an ideal in a partially ordered ring A. In order that A/I be a partially ordered ring, according to the definition:*

$$I(a) \geq 0 \text{ if there exists } x \in A \text{ such that } x \geq 0 \text{ and } a \equiv x \ (mod\ I),$$

it is necessary and sufficient that I is convex.

Theorem 6.2. *[7]. The following conditions on a convex ideal I in a lattice ordered ring A are equivalent:*

i) I is absolutely convex.
ii) $x \in I$ implies $|x| \in I$.
iii) $x, y \in I$ implies $x \vee y \in I$.
iv) $I(a \vee b) = I(a) \vee I(b)$, whence A/I is a lattice.
v) $I(a) \geq 0$ if and only if $a \equiv |a|$ (mod I).

Remark 6.1. $I(|a|) = |I(a)|$, $\forall\ a \in A$, when I is an absolutely convex ideal of A.

Theorem 6.3. *Every $Z_\vartriangle$-ideal in $C(X)_\vartriangle$ is absolutely convex.*

Proof. Let I be any $Z_\vartriangle$-ideal in $C(X)_\vartriangle$ and $|f| \leq |g|$, where $f \in C(X)_\vartriangle$ and $g \in I$. Then $Z_\vartriangle(f) \subseteq Z_\vartriangle(g)$. As $g \in I$, $Z_\vartriangle(g) \in Z_\vartriangle(I)$ which implies that $Z_\vartriangle(f) \in Z_\vartriangle(I)$. Now I being a $Z_\vartriangle$-ideal, it follows that $f \in I$. □

Corollary 6.1. *Every maximal ideal in $C(X)_\vartriangle$ is absolutely convex.*

Theorem 6.4. *For every maximal ideal M in $C(X)_\vartriangle$, the quotient ring $C(X)_\vartriangle/M$ is a lattice ordered ring.*

Proof. Obvious. □

Next we characterize the non-negative elements in the lattice-ordered ring $C(X)_\vartriangle/I$, for a $Z_\vartriangle$-ideal I.

Theorem 6.5. *For a $Z_\vartriangle$-ideal I and $f \in C(X)_\vartriangle$, $I(f) \geq 0$ if and only if there exists $Z \in Z_\vartriangle(I)$, such that $f \geq 0$ on Z.*

Proof. First let, $I(f) \geq 0$. By Theorem 6.2, $f \equiv |f|$ (mod I), i.e. $f - |f| \in I$. So, $Z_\vartriangle(f - |f|) \in Z_\vartriangle(I)$ and hence $f \geq 0$ on $Z_\vartriangle(f - |f|)$.

Conversely, let $f \geq 0$ on some $Z \in Z_\vartriangle(I)$. Then $f = |f|$ on Z, i.e. $Z \subseteq Z_\vartriangle(f-|f|)$ which implies that $Z_\vartriangle(f-|f|) \in Z_\vartriangle(I)$. Since I is a $Z_\vartriangle$-ideal, $f-|f| \in I$, i.e. $I(f) = I(|f|)$. As $I(|f|) \geq 0$, hence $I(f) \geq 0$. □

Theorem 6.6. *Let I be a $Z_{\vartriangle}$-ideal and $f \in C(X)_{\vartriangle}$. If there exists $Z \in Z_{\vartriangle}(I)$ such that $f(x) > 0$, for all $x \in Z$, then $I(f) > 0$. Converse is true if I is maximal.*

Proof. If f is positive on $Z \in Z_{\vartriangle}(I)$, then $Z_{\vartriangle}(f) \cap Z = \varnothing$, so that $Z_{\vartriangle}(f) \notin Z_{\vartriangle}(I)$. Hence $f \notin I$. So by the previous theorem $I(f) > 0$.

For the converse, if I is maximal, then there exists some zero set Z' of I such that $Z' \cap Z(f) = \varnothing$. Now $Z \cap Z' \in Z_{\vartriangle}(I)$, thus $f > 0$ on the zero set $Z \cap Z'$ of I. □

Remark 6.2. The converse part of the above theorem fails if I is not maximal: for let I be non-maximal. Then there exists a proper ideal J of $C(X)_{\vartriangle}$ such that $I \subset J$. Choose $f \in J \setminus I$. Then $I(f^2) > 0$. Now choose any $Z \in Z_{\vartriangle}(I)$. Then $Z \in Z_{\vartriangle}(J)$ also, so that $Z \cap Z(f^2) \neq \varnothing$. Now f is not strictly positive on the whole of Z.

We now characterize those ideals I in $C(X)_{\vartriangle}$ for which $C(X)_{\vartriangle}/I$ is a totally ordered ring.

Theorem 6.7. *For a $Z_{\vartriangle}$-ideal I in $C(X)_{\vartriangle}$, the lattice ordered ring $C(X)_{\vartriangle}/I$ is a totally ordered if I is prime.*

Proof. $C(X)_{\vartriangle}/I$ is totally ordered if and only if for any $f \in C(X)_{\vartriangle}$, $I(f) \geq 0$ or $I(-f) \geq 0$ if and only if for all $f \in C(X)_{\vartriangle}$, there exists $Z \in Z_{\vartriangle}(I)$ such that f does not change its sign of Z if and only if I is a prime ideal in view of Theorem 4.2. □

Corollary 6.2. *For every maximal ideal M in $C(X)_{\vartriangle}$, $C(X)_{\vartriangle}/M$ is a totally ordered ring.*

Theorem 6.8. *For a prime ideal P in $C(X)_{\vartriangle}$, the following are true.*

i) P is absolutely convex.
ii) The residue class ring $C(X)_{\vartriangle}/P$ is totally ordered.
iii) The mapping : $r \to P(\bar{\mathbf{r}})$ is an order-preserving monomorphism of the real field $\mathbb{R}$ into the residue class rings.

Proof. *i)* Let $0 \leq |f| \leq |g|$, for some $f \in C(X)_{\vartriangle}$ and $g \in P$. Then $f^2 = |f|^2 \leq |g|^2$. By Theorem 2.4, $f^2 = h \cdot g$, for some $h \in C(X)_{\vartriangle}$. Thus $f^2 \in P$ implies that $f \in P$ (as P is prime). Hence P is absolutely convex.
ii) Since P is prime, $C(X)_{\vartriangle}/P$ is a partially ordered ring. Now $(f-|f|)(f+|f|) = \bar{\mathbf{0}}$ which implies that either $f \equiv |f|$ (mod P), i.e. either $P(f) \geq 0$ or $P(-f) \geq 0$. Hence $C(X)_{\vartriangle}/P$ is totally ordered.
iii) Clearly the mapping: $r \to P(\bar{\mathbf{r}})$ is a monomorphism. We only need to show the order preserving property of the mapping. Choose $r, s \in \mathbb{R}$ with $r > s$. Then $r - s > 0$, so that $P(\bar{\mathbf{r}} - \bar{\mathbf{s}}) > \mathbf{0}$, i.e. $P(\bar{\mathbf{r}}) > \mathbf{P}(\bar{\mathbf{s}})$. □

For a maximal ideal M in $C(X)_{\vartriangle}$, $C(X)_{\vartriangle}/M$ can be considered as an extension of the real field $\mathbb{R}$, or in otherwords, $C(X)_{\vartriangle}/M$ contains a cannonical copy of $\mathbb{R}$.

Definition 6.1. If for a maximal ideal M, the canonical copy of $\mathbb{R}$ is the entire field $C(X)_{\vartriangle}/M$, (resp. $C^{*}(X)_{\vartriangle}/M$), then M is called a real ideal and $C(X)_{\vartriangle}/M$ is called real residue class field. If M is not real, then it is called hyper-real and $C(X)_{\vartriangle}/M$ is called a hyper-real residue class field

Definition 6.2. [7] A totally ordered field F is said to be archimedean if for every element a, there exists $n \in \mathbb{N}$ such that $n \geq a$. If F is not archimedean, then it is called non-archimedean. Thus, a non-archimedean field is characterized by the presence of infinitely large elements, i.e. there exists $a \in F$ such that $a > n$, $n \in \mathbb{N}$. Such elements are called infinitely large elements. The following is an important theorem in the context of archimedean field.

Theorem 6.9. *[7] A totally ordered field is archimedean if and only if it is isomorphic to a subfield of the ordered field of* $\mathbb{R}$.

Thus we get that the real residue class field $C(X)_{\vartriangle}/M$ is archimedean if M is a real maximal ideal of $C(X)_{\vartriangle}$.

Theorem 6.10. *Every hyper-real residue class field $C(X)_{\vartriangle}/M$ is non-archimedean.*

Proof. Since the identity is the only non-zero homomorphism on the ring $\mathbb{R}$ into itself, the proof follows. □

Corollary 6.3. *A maximal ideal in $C(X)_{\vartriangle}$ is hyper-real if and only if there exists $f \in C(X)_{\vartriangle}$ such that $M(f)$ is an infinitely large member of $C(X)_{\vartriangle}/M$.*

Theorem 6.11. *Each maximal ideal M in $C^{*}(X)_{\vartriangle}$ is real.*

Proof. In view of the above discussions, it sufficies to show that $C^{*}(X)_{\vartriangle}/M$ is archimedean. Choose $f \in C^{*}(X)_{\vartriangle}$. Then $|f(x)| \leq n$, for all $x \in X$ and for some $n \in \mathbb{N}$, i.e. $|M(f)| \leq M(\bar{\mathbf{n}})$. □

The following theorem relates to unbounded functions on X with infinitely large elements modulo maximal ideals.

Theorem 6.12. *For a given maximal ideal M in $C(X)_{\vartriangle}$ and $f \in C(X)_{\vartriangle}$, the following are equivalent:*

i) $|M(f)|$ *is infinitely large.*
ii) f *is unbounded on every zero set of* M.
iii) For each $n \in \mathbb{N}$, *the zero set* $Z_n = \{x \in X : |f(x)| \geq n\} \in Z_{\vartriangle}(M)$.

Proof. i) $\iff$ *ii)*: $|M(f)|$ is not infinitely large in $C(X)_{\vartriangle}/M$ if and only if there exists $n \in \mathbb{N}$ such that $|M(f)| = M(|f|) \leq M(\bar{\mathbf{n}})$ if and only if $|f| \leq \bar{\mathbf{n}}$ on some $Z \in Z_{\vartriangle}(M)$ if and only if f is bounded on some $Z \in Z_{\vartriangle}(M)$.

ii) $\iff$ *iii)*: Choose $n \in \mathbb{N}$. Since Z_n intersects each member in $Z_{\vartriangle}(M)$, $Z_n \in Z_{\vartriangle}(M)$, as because $Z_{\vartriangle}(M)$ is $Z_{\vartriangle}$-ultrafilter.
iii) $\iff$ *ii)*: Since for each $n \in \mathbb{N}$, $|f| \geq n$ on some zero set in $Z_{\vartriangle}(M)$, $|M(f)| \geq M(\bar{\mathbf{n}})$, for all $n \in \mathbb{N}$. This implies that $|M(f)|$ is an infinitely large element of $C(X)_{\vartriangle}/M$. □

Theorem 6.13. *$f \in C(X)_{\vartriangle}$ is unbounded on X if and only if there exists a maximal ideal M in $C(X)_{\vartriangle}$ such that $|M(f)|$ is infinitely large in $C(X)_{\vartriangle}/M$.*

Proof. One part follows from Theorem 6.12.

For the other part, let f be unbounded on X. Then each $Z_n = \{x \in X : |f| \geq n\} \neq \varnothing$, for $n \in \mathbb{N}$ and $\{Z_n : n \in \mathbb{N}\}$ has the finite intersection property. So there exists a $Z_{\vartriangle}$-ultrafilter $\mathcal{U}$ on X containing $\{Z_n : n \in \mathbb{N}\}$. Hence there exists a maximal ideal M in $C(X)_{\vartriangle}$ such that $\mathcal{U} = Z_{\vartriangle}(M)$ and so $Z_n \in Z_{\vartriangle}(M)$, for all $n \in \mathbb{N}$. Now by Theorem 6.12, it follows that $|M(f)|$ is infinitely large. □

Remark 6.3. In the case of $C(X)$, the pseudocompactness of X ensures that every maximal ideal of $C(X)$ is real. But in $C(X)_{\vartriangle}$, this may not hold. Consider $X = [0,1]$ with the subspace topology of the usual topology of reals, $\Delta = \{A \subseteq X : A$ is nowhere dense in $X\}$ and $f : X \to \mathbb{R}$ defined by,

$$f(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

As f is unbounded on X, by Theorem 6.12, there exists a maximal ideal M (say) such that $|M(f)|$ is infinitely large, so that M is not real.

7. Some algebraric aspects of $C(X)_{\vartriangle}$

Let us first recall that a ring S containing a reduced ring R is called a ring of quotients of R if and only if for each $0 \neq s \in S$, there exists $r \in R$ such that $0 \neq sr \in R$ (see [8]). Regarding rings of quotients of rings of functions one can go through [9, 5].

Theorem 7.1. *For a space X and a subcollection $\Delta \subseteq \mathcal{P}(X)$, the following are equivalent:*

i) $C(X) = C(X)_{\vartriangle}$.
ii) *X is a discrete space.*
iii) *$C(X)_{\vartriangle}$ is a ring of quotients of $C(X)$.*
iv) $C(X) = T'(X)$.

Proof. i) $\iff$ *ii)*: If X is discrete, then obviously $C(X) = C(X)_{\vartriangle}$. Next suppose that $C(X) = C(X)_{\vartriangle}$ and $x \in X$. As $\chi_{\{x\}} \in C(X)_{\vartriangle}$, $\chi_{\{x\}} \in C(X)$, so that X becomes discrete.

ii) $\Rightarrow$ *iii)*: Obvious.

iii) $\Rightarrow$ *iv)*: Choose $x_0 \in X$. Then $\chi_{\{x_0\}} \in C(X)_\Delta$. Hence there exists $f \in C(X)$ such that $0 \neq f(x)\chi_{\{x_0\}} \in C(X)$. Hence $f(x_0)\chi_{\{x_0\}} = f(x)\chi_{\{x_0\}} \in C(X)$, which implies that $\{x_0\}$ is an isolated point, so that X is discrete.

iv) $\Rightarrow$ *ii)*: If X is not discrete, then there exists a non-isolated point $x_0 \in X$. Now $\chi_{\{x_0\}} \in T'(X)$, but $\chi_{\{x_0\}} \notin C(X)$. Hence $T'(X) \neq C(X)$. $\square$

Theorem 7.2. *For a space X and a subcollection $\Delta \subseteq \mathcal{P}(X)$, $T'(X) \subseteq C(X)_\Delta$ if and only if every open dense subset D of X is of the form $X \setminus G$, for some $G \in \Delta$.*

Proof. First let $T'(X) \subseteq C(X)_\Delta$ and D be an open dense subset of X. Then $\chi_D \in T'(X)$ implies that $\chi_D \in C(X)_\Delta$. Hence the set of points of discontinuities of $\chi_D (\equiv G(\text{say})) = X \setminus D \in \Delta$, so that $D = X \setminus G$, where $G \in \Delta$.

Conversely, choose $f \in T'(X)$. Then there exists an open dense subset D of X such that f is continuous on D and by the given condition $D = X \setminus G$, for $G \in \Delta$. Hence the set D_f of points of discontinuities of f is a subset of $X \setminus D = G \in \Delta$, so that $D_f \in \Delta$. Thus $f \in C(X)_\Delta$, and hence $T'(X) \subseteq C(X)_\Delta$. $\square$

Remark 7.1. If X is T_1, we always have $C(X)_F \subseteq T'(X)$, but this inclusion is not true in case of $C(X)_\Delta$. Consider $X = \mathbb{R}$ with the usual topology of reals and $\Delta = \{A \subseteq X : A$ is countable$\}$. Define $f : X \to \mathbb{R}$ by,

$$f(x) = \begin{cases} \frac{1}{q}, & x = \frac{p}{q}, \text{ with g.c.d } (p,q) = 1 \\ \\ 0, & x = 0 \text{ or } x \text{ is an irrational.} \end{cases}$$

Then $f \in C(X)_\Delta$, but $f \notin T'(X)$. Hence $C(X)_\Delta \nsubseteq T'(X)$.

8. ΔP-space

Recall that a space X is called a P-space (resp., $\mathcal{F}P$ space) if $C(X)$ (resp., $C(X)_F$) is a regular ring, (see [7], $4J$ and [6]). We next introduce ΔP- spaces which is a generalization of the above types of spaces.

Definition 8.1. A space X is called a ΔP-space if $C(X)_\Delta$ is a regular ring.

Observe that any $\mathcal{F}P$ space is one kind of a ΔP-space if we consider $\Delta =$ the set of all finite subsets of X. Now we give an example of a ΔP-space which is not a $\mathcal{F}P$ space.

Example 8.1. Let $X = \mathbb{Q}$ and $\Delta =$ the set of all countable subsets of $\mathbb{Q}$. Then $C(X)_\Delta = \mathbb{R}^{\mathbb{Q}}$. So $\mathbb{Q}$ is a ΔP-space. But $\mathbb{Q}$ is not an $\mathcal{F}P$-space. To establish this, consider $f : \mathbb{Q} \to \mathbb{R}$ defined by,

$$f(x) = \begin{cases} 2(x - \overline{n-1}), & n-1 \leq x \leq \frac{2n-1}{2}, \\ \\ -2(x-n), & \frac{2n-1}{2} \leq x \leq n \\ \\ 1 & \textit{otherwise.} \end{cases}$$

Here the only point of discontinuity of f is $x = 0$. So $f \in C(\mathbb{Q})_F$ also. If $C(\mathbb{Q})_F$ be regular, then there exists $g \in C(\mathbb{Q})_F$ such that $f^2 g = f$ which implies that $g = \frac{1}{f}$, when $f(x) \neq 0, x \in \mathbb{Q}$. So we get,

$$g(x) = \begin{cases} \frac{1}{2(x-n-1)}, & n-1 < x < \frac{2n-1}{2}, \\ -\frac{1}{2(x-n)}, & \frac{2n-1}{2} < x < n \\ 1 & otherwise. \end{cases}$$

So whatever value we choose for $g(x)$, when $f(x) = 0$, g will be discontinuous at those points. Hence $g \notin C(\mathbb{Q})_F$. So $\mathbb{Q}$ is not an $\mathcal{F}P$ space, and hence not a P-space also.

Proposition 8.1. *Every P-space is a ΔP-space.*

Proof. Let X be a P-space and $f \in C(X)_\vartriangle$. Then $D_f \in \Delta$ and $X \setminus D_f$ is a G_δ-set in X. Also $X \setminus D_f$ is a P-space (as any subspace of a P-space is also a P-space), so that $X \setminus D_f$ is an open set in X. Now for $f \in C(X \setminus D_f)$, there exists $g \in C(X \setminus D_f)$ such that $f = f^2 g$. Now we define $g^* : X \to \mathbb{R}$ by,

$$g^*(x) = \begin{cases} g(x), & x \in X \setminus D_f \\ 0, & x \in D_f \cap Z_\vartriangle(f) \\ \frac{1}{f(x)}, & x =\in D_f \setminus Z_\vartriangle(f). \end{cases}$$

Then clearly $g^* \in C(X)_\vartriangle$. So $f = f^2 g^*$ and hence X is a ΔP-space. $\square$

It is known from literature that every zero set in $C(X)$ is clopen. The modification of this result in the setting of $C(X)_\vartriangle$ is furnished below.

Theorem 8.1. *If X is a ΔP-space, then for any $Z \in Z_\vartriangle(X)$, there exists $H \in \Delta$ such that $Z \setminus H$ is a clopen set in $X \setminus H$.*

Proof. Let $Z_\vartriangle(f) \in Z_\vartriangle(X)$, for $f \in C(X)_\vartriangle$. As X is a ΔP-space, there exists $g \in C(X)_\vartriangle$ such that $f^2 g = f$. Since $f, g \in C(X)_\vartriangle$, there exists $H \in C(X)_\vartriangle$ such that $f, g \in C(X \setminus H)$. So $f^2(x)g(x) = f(x)$, for all $x \in X \setminus H$ which implies that $Z_\vartriangle(f|_{X\setminus H}) \cup Z_\vartriangle((1-fg)|_{X\setminus H}) = X \setminus H$ and also $Z_\vartriangle(f|_{X\setminus H}) \cap Z_\vartriangle((1-fg)|_{X\setminus H}) = \varnothing$. So $Z_\vartriangle(f) \setminus H$ is clopen in $X \setminus H$. $\square$

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