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ON THE GEODESICS AND S-CURVATURE OF A HOMOGENEOUS FINSLER SPACE WITH SQUARE-ROOT (α, β)-METRIC

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Abstract. In this paper, we consider the square-root (α, β)-metric F which satisfies $F(\alpha, \beta) = \sqrt{\alpha(\alpha + \beta)}$. We prove the existence of invariant vector fields on a homogeneous Finsler space with square-root metric. Then we obtain the explicit formula for the S-curvature and mean Berwald curvature of homogeneous Finsler space with square-root metric. We study geodesics and geodesic vectors for homogeneous square-root (α, β)-metric.
Keywords: homogeneous Finsler space, square-root metric, S-curvature, invariant vector field, geodesic vector, mean Berwald curvature.

1. Introduction

An important family of Finsler metrics is the family of (α, β)-metric. These metrics are introduced by Matsumoto[11]. An (α, β)-metric is a Finsler metric of the form $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$ where $\alpha = \sqrt{\tilde{a}_{ij}(x)y^i y^j}$ is induced by a Riemannian metric $\tilde{a} = \tilde{a}_{ij}dx^i \otimes dx^j$ on a connected smooth n-dimensional manifold M and $\beta = b_i(x)y^i$ is a 1-form on M. The class of p-power (α, β)-metrics on a manifold M is in the following form

$$F = \alpha \left(1 + \frac{\beta}{\alpha}\right)^p,$$

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where $p \neq 0$ is a real constant. If $p = 1$, then we get the Randers metric $F = \alpha + \beta$. This metric was first recognized as kind of Finsler metric in 1957 by Ingarden, who first named them Randers metric [9]. If $p = -1$, then we have the Matsumoto metric $F = \frac{\alpha^2}{(\alpha+\beta)}$. Matsumoto metric is an important metric in Finsler geometry.

In the case of $p = 1\backslash 2$, we get

$$F = \sqrt{\alpha(\alpha + \beta)},$$

which is called a square-root metric. In this paper, we study square-root metrics. We study the existence of invariant vector fields on homogeneous Finsler spaces with square-root metrics. Invariant vector fields on homogeneous Finsler spaces has been studied by some authors in recent years (see [10, 13, 15]). Further, we give an explicit formula for S-curvature of square-root (α, β)-metric.

2. Preliminaries

In this section, we recall some known facts about Finsler spaces, for details see [2]. Let M be a smooth n- dimensional C^∞ manifold and TM be its tangent bundle. A Finsler metric on a manifold M is a non-negative function $F : TM \to R$ with the following properties [2]:

1) F is smooth on the slit tangent bundle $TM^0 := TM\backslash\{0\}$.

2) $F(x, \lambda y) = \lambda F(x, y)$ for any $x \in M$, $y \in T_xM$ and $\lambda > 0$.

3) The following bilinear symmetric form $g_y : T_xM \times T_xM \longrightarrow R$ is positive definite

$$g_y(u, v) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t} F^2(x, y + su + tv)|_{s=t=0}.$$

Definition 2.1. Let $\alpha = \sqrt{\tilde{a}_{ij}(x)y^i y^j}$ be a norm iduced by a Riemannian metric $\tilde{a}$ and $\beta(x, y) = b_i(x)y^i$ be a 1−form on an n−dimensional manifold M. Let

$$\|\beta(x)\|_\alpha := \sqrt{\tilde{a}^{ij}(x)b_i(x)b_j(x)}.$$

Now, let the function F is defined as follows

$$F := \alpha\phi(s) \quad , \quad s = \frac{\beta}{\alpha}, \tag{2.1}$$

where $\phi = \phi(s)$ is a positive C^∞ function on $(-b_0, b_0)$ satisfying

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0 \quad , \quad |s| \leq b < b_0.$$

Then by lemma 1.1.2 of [5], F is a Finsler metric if $\|\beta(x)\|_\alpha < b_0$ for any $x \in M$. A Finsler metric in the form (2.1) is called an (α, β)−metric [1, 5].

Definition 2.2. A Finsler space having the Finsler function:

$$F = \sqrt{\alpha(x,y)(\alpha(x,y)+\beta(x,y))}, \tag{2.2}$$

is called a square-root space with $\phi(s) = \sqrt{1+s}$.

Before defining homogeneous Finsler spaces, we discuss here some basic concepts required.

Definition 2.3. Let G be a smooth manifold having the structure of an abstract group. G is called a Lie group, if the maps $i : G \to G$ and $\mu : G \times G \to G$ defined as $i(g) = g^{-1}$, and $\mu(g,h) = gh$, respectively, are smooth.
Let G be a Lie group and M, a smooth manifold. Then a smooth map $f : G \times M \to M$ satisfying

$$f(g_2, f(g_1, x)) = f(g_2 g_1, x), \quad \forall g_1, g_2 \in G, \quad x \in M,$$

$$f(e, x) = x, \quad \forall x \in M,$$

is called a smooth action of G on M.

Definition 2.4. Let M be a smooth manifold and G, a Lie group. If G acts smoothly on M, then G is called a Lie transformation group of M.

The following Theorem gives us a differentiable structure on the coset space of a Lie group.

Theorem 2.1. *Let G be a Lie group and H, its closed subgroup. Then there exists a unique differentiable structure on the left coset space G/H with the induced topology that turns G/H into a smooth manifold such that G is a Lie transformation group of G/H.*

Definition 2.5. Let (M,F) be a connected Finsler space and $I(M,F$ the group of isometries of (M,F). If the action of $I(M,F)$ is transitive on M, then (M,F) is said to be a homogeneous Finsler space.

Let G be a Lie group acting transitively on a smooth manifold M. Then for $a \in M$, the isotropy subgroup G_a of G is a closed subgroup and by Theorem 2.1, G is a Lie transformation group of G/G_a. Further, G/G_a is diffeomorphic to M.

BochnerMontgomery in [3] proved that a locally compact group of differentiable transformations of a manifold is a Lie transformation group. Therefore we have the following Theorem.

Theorem 2.2. *[6] Let (M,F) be a Finsler space. Then $G = I(M,F)$, the group of isometries of M is a Lie transformation group of M. Let $a \in M$ and $I_a(M,F)$ be the isotropy subgroup of $I(M,F)$ at a. Then $I_a(M,F)$ is compact.*

Let (M, F) be a homogeneous Finsler space, i.e. $G = I(M, F)$ acts transitively on M. For $a \in M$, let $H = I_a(M, F)$ be a closed isotropy subgroup of G which is compact. Then H is a Lie group itself being a closed subgroup of G. Write M as the quotient space G/H.

Definition 2.6. [12] Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of the Lie groups G and H respectively. Then the direct sum decomposition of $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{h} + \mathfrak{n}$, where $\mathfrak{n}$ is a subspace of $\mathfrak{g}$ such that $Ad(h)(\mathfrak{n}) \subset \mathfrak{n}, \forall h \in H$, is called a reductive decomposition of $\mathfrak{g}$, and if such decomposition exists, then $(G/H, F)$ is called reductive homogeneous space.

Therefore, we can write any homogeneous Finsler space as a coset space of a connected Lie group with an invariant Finsler metric. Here, the Finsler metric F is viewed as G invariant Finsler metric on M.

Definition 2.7. A one-parameter subgroup of a Lie group G is a homomorphism $\xi : R \to G$, such that $\xi(0) = e$, where e is the identity of G.

Recall [6] the following result which gives us the existence of one-parameter subgroup of a Lie group.

Theorem 2.3. *Let G be a Lie group having Lie algebra $\mathfrak{g}$. Then for any $Y \in \mathfrak{g}$, there exists a unique locally one-parameter subgroup ξ such that $\dot{\xi}(0) = Y_e$, where e is the identity element of G.*

Definition 2.8. Let G be a Lie group with identity element e and $\mathfrak{g}$ its Lie algebra. The exponential map $exp : \mathfrak{g} \to G$ is defined by

$$exp(tY) = \xi(t), \quad \forall t \in R,$$

where $\xi : R \to G$ is unique one-parameter subgroup of G with $\dot{\xi}(0) = Y_e$.

In the case of reductive homogeneous manifold, we can identify the tangent space $T_H(G/H)$ of G/H at the origin $eH = H$ with $\mathfrak{n}$ through the map

$$Y \to \frac{d}{dt} exp(tX)H|_{t=0}, \quad Y \in \mathfrak{n},$$

since M is identified with G/H and Lie algebra of any Lie group G is viewed as T_eG.

3. Invariant Vector Field

The Riemannian metric $\tilde{a}$ induces an inner product on any cotangent space T_x^*M such that $\langle dx^i(x), dx^j(x)\rangle = \tilde{a}^{ij}(x)$. The induced inner product on T_x^*M induces a linear isomorphism between T_x^*M and T_xM. Then the 1-form β corresponds to a vector field $\tilde{X}$ on M such that

$$\tilde{a}(y, \tilde{X}(x)) = \beta(x, y).$$

Also we have $\|\beta(x)\|_\alpha = \|\tilde{X}(x)\|_\alpha$. Therefore we can write $(\alpha, \beta)-$metrics as follows:

$$F(x, y) = \alpha(x, y)\phi\Big(\frac{\tilde{a}(\tilde{X}(x), y)}{\alpha(x, y)}\Big),$$

where for any $x \in M$, $\sqrt{\tilde{a}(\tilde{X}(x), \tilde{X}(x))} = \|\tilde{X}(x)\|_\alpha < b_0$.

So for square-root metric, we can write

$$F(x, y) = \sqrt{\tilde{a}(y_x, y_x) + \sqrt{\tilde{a}(y_x, y_x)}\tilde{a}(X_x, y_x)}. \tag{3.1}$$

Lemma 3.1. *Let (M, F) be a Finsler space with square-root metric $F = \sqrt{\alpha(\alpha + \beta)}$. Let $I(M, F)$ be the group of isometries of (M, F) and $I(M, \tilde{a})$ be that of Riemannian space $(M, \tilde{a})$. Then $I(M, F)$ is a closed subgroup of $I(M, \tilde{a})$.*

Proof. Let $x \in M$ and $\xi : (M, F) \to (M, F)$ be an isometry. Therefore, we have

$$F(x, Y) = F(\xi(x), d\xi_x(Y)), \quad \forall Y \in T_xM.$$

So we have

$$\begin{aligned} &\sqrt{\tilde{a}(Y, Y) + \sqrt{\tilde{a}(Y, Y)}\tilde{a}(X_x, Y)} \\ = \ &\sqrt{\tilde{a}(d\xi_x(Y), d\xi_x(Y)) + \sqrt{\tilde{a}(d\xi_x(Y), d\xi_x(Y))}\tilde{a}(X_{\xi(x)}, d\xi_x(Y))}. \end{aligned}$$

After simplification, we get

$$\begin{aligned} &\tilde{a}(Y, Y) + \sqrt{\tilde{a}(Y, Y)}\tilde{a}(X_x, Y) \\ = \ &\tilde{a}(d\xi_x(Y), d\xi_x(Y)) + \sqrt{\tilde{a}(d\xi_x(Y), d\xi_x(Y))}\tilde{a}(X_{\xi(x)}, d\xi_x(Y)). \end{aligned} \tag{3.2}$$

Replacing Y by $-Y$ in 3.2 implies that

$$\begin{aligned} &\tilde{a}(Y, Y) - \sqrt{\tilde{a}(Y, Y)}\tilde{a}(X_x, Y) \\ = \ &\tilde{a}(d\xi_x(Y), d\xi_x(Y)) - \sqrt{\tilde{a}(d\xi_x(Y), d\xi_x(Y))}\tilde{a}(X_{\xi(x)}, d\xi_x(Y)). \end{aligned} \tag{3.3}$$

Adding equations 3.2 and 3.3, we get

$$\tilde{a}(Y, Y) = \tilde{a}(d\xi_x(Y), d\xi_x(Y)). \tag{3.4}$$

Subtracting equation 3.3 from equation 3.2 and use equation 3.4, we get

$$\tilde{a}(X_x, Y) = \tilde{a}(X_{\xi(x)}, d\xi_x(Y)).$$

Therefore, ξ is an isometry with respect to the Riemannian metric $\tilde{a}$ and $d\xi_x(X_x) = X_{\xi(x)}$. Thus, $I(M, F)$ is a closed subgroup of $I(M, \tilde{a})$. □

From Lemma 3.1, we conclude that if (M, F) is a homogeneous Finsler space with square-root metric $F = \sqrt{\alpha(\alpha+\beta)}$, then the Riemannian space (M, α) is homogeneous. Further, M can be written as a coset space G/H, where $G = I(M, F)$ is a Lie transformation group of M and H, the compact isotropy subgroup $I_a(M, F)$ of $I(M, F)$ at some point $a \in M$ [8]. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of the Lie groups G and H, respectively. If $\mathfrak{g}$ can be written as a direct sum of subspaces $\mathfrak{h}$ and $\mathfrak{n}$ of $\mathfrak{g}$ such that $Ad(h)\mathfrak{n} \subset \mathfrak{n}$, $\forall h \in H$, then from Definition 2.6, $(G/H, F)$ is a reductive homogeneous space.

Therefore, homogeneous Finsler space with square-root metric can be written as a coset space of a connected Lie group with square metric. Here, the square-root metric $F = \sqrt{\alpha(\alpha+\beta)}$ is viewed as G invariant Finsler metric on M.

Theorem 3.1. *Let $F = \sqrt{\alpha(\alpha+\beta)}$ be a G-invariant square-root metric on G/H, X the vector field corresponding to 1-form β. Then α is a G-invariant Riemannian metric and the vector field X is also G-invariant.*

Proof. Let F be G-invariant metric on G/H, we have

$$F(y) = F(Ad(h)y), \quad \forall h \in H, \quad Y \in \mathfrak{n}.$$

By 3.1, we get

$$\begin{aligned} & \sqrt{\tilde{a}(Y,Y) + \sqrt{\tilde{a}(Y,Y)}\tilde{a}(X,Y)} \\ = & \sqrt{\tilde{a}(Ad(h)Y, Ad(h)Y) + \sqrt{\tilde{a}(Ad(h)Y, Ad(h)Y)}\tilde{a}(X, Ad(h)Y)}. \end{aligned}$$

After simplification, we get

$$\begin{aligned} & \tilde{a}(Y,Y) + \sqrt{\tilde{a}(Y,Y)}\tilde{a}(X,Y) \\ = & \tilde{a}(Ad(h)Y, Ad(h)Y) + \sqrt{\tilde{a}(Ad(h)Y, Ad(h)Y)}\tilde{a}(X, Ad(h)Y). \end{aligned} \tag{3.5}$$

Replacing Y by $-Y$ in 3.5 implies that

$$\begin{aligned} & \tilde{a}(Y,Y) - \sqrt{\tilde{a}(Y,Y)}\tilde{a}(X,Y) \\ = & \tilde{a}(Ad(h)Y, Ad(h)Y) - \sqrt{\tilde{a}(Ad(h)Y, Ad(h)Y)}\tilde{a}(X, Ad(h)Y). \end{aligned} \tag{3.6}$$

Adding equations 3.5 and 3.6, we get

$$\tilde{a}(Y,Y) = \tilde{a}(Ad(h)Y, Ad(h)Y). \tag{3.7}$$

Subtracting equation 3.6 from equation 3.5 and use equation 3.7, we get

$$\tilde{a}(X,Y) = \tilde{a}(X, Ad(h)Y).$$

Therefore, α is a G-invariant Riemannian metric and

$$Ad(h)X = X,$$

which proves that X is also G-invariant. □

The following Theorem gives us a complete description of invariant vector fields.

Theorem 3.2. *[7] There exists a bijection between the set of invariant vector fields on G/H and the subspace*

$$V = \{Y \in \mathfrak{n} : \quad Ad(h)Y = Y, \quad \forall h \in H\}.$$

4. S-Curvature of Homogeneous Finsler Space with Square-root Metric

S-curvature was introduced by Shen in [16]. It is a quantity to measure the rate of change of the volume form of a Finsler space along geodesics. Let V be an n-dimensional real vector space and F a Minkowski norm on V. For a basis $\{b_i\}$ of V, let

$$\sigma_F = \frac{Vol(B^n)}{Vol\{(y^i) \in R^n | \quad F(y^i b_i) < 1\}},$$

where Vol means the volume of a subset in the standard Euclidean space R^n and B^n is the open ball of radius 1. This quantity is generally dependent on the choice of the basis $\{b_i\}$. But it is easily seen that

$$\tau(y) = \ln \frac{\sqrt{det(g_{ij}(y))}}{\sigma_F}, \quad y \in V - \{0\},$$

is independent of the choice of basis. We call $\tau = \tau(y)$ the distortion of (V, F).

Now let (M, F) be a Finsler space. Let $\tau(x, y)$ be the distortion of the Minkowski norm F_x on $T_x(M)$ and σ the geodesic with $\sigma(0) = x$ and $\dot{\sigma}(0) = y$. Then the quantity

$$S(x, y) = \frac{d}{dt}[\tau(\sigma(t), \dot{\sigma}(t))]|_{t=0},$$

is called the S-curvature of the Finsler space (M, F).

The formula for S-curvature of an (α, β)-metric, in local coordinate system, introduced by Cheng and Shen [4], is as follows:

$$S = \Big(2\psi - \frac{f^{'}(b)}{bf(b)}\Big)(r_0 + s_0) - \frac{\Phi}{2\alpha\Delta^2}(r_{00} - 2\alpha Q s_0), \tag{4.1}$$

where

$$\begin{aligned} Q &= \frac{\phi^{'}}{\phi - s\phi^{'}}, \\ \Delta &= 1 + sQ + (b^2 - s^2)Q^{'}, \end{aligned}$$

$$
\begin{aligned}
\psi &= \frac{Q^{'}}{2\Delta},\\
\Phi &= (sQ^{'} - Q)(n\Delta + 1 + sQ) - (b^2 - s^2)(1 + sQ)Q^{''},\\
r_{ij} &= \frac{1}{2}(b_{i|j} + b_{j|i}), \quad r_j = b^i r_{ij}, \quad r_0 = r_i y^i, \quad r_{00} = r_{ij} y^i y^j,\\
s_{ij} &= \frac{1}{2}(b_{i|j} - b_{j|i}), \quad s_j = b^i s_{ij}, \quad s_0 = s_i y^i.
\end{aligned}
$$

Definition 4.1. Let (M, F) be an n-dimensional Finsler space. If there exists a smooth function $c(x)$ on M and a closed 1-form ω such that

$$S(x, y) = (n + 1)(c(x)F(y) + \omega(y)), \quad x \in M, \quad y \in T_x(M),$$

then (M, F) is said to have almost isotropic S-curvature. In addition, if ω is zero, then (M, F) is said to have isotropic S-curvature. Also, if ω is zero and $c(x)$ is constant, then we say, (M, F) has constant S-curvature.

With above notations, let us recall from [14] the following Theorem.

Theorem 4.1. *Let $F = \alpha\varphi(s)$ be a G-invariant (α, β)-metric on the reductive homogeneous Finsler space G/H with a decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{h} + \mathfrak{n}$. Then the S-curvature is given by*

$$S(H, y) = \frac{\Phi}{2\alpha\Delta^2}\big(\langle [v, y]_{\mathfrak{n}}, y\rangle + \alpha Q\langle [v, y]_{\mathfrak{n}}, v\rangle\big), \tag{4.2}$$

where $v \in \mathfrak{n}$ corresponds to the 1-form β and $\mathfrak{n}$ is identified with the tangent space $T_H(G/H)$ of G/H at the origin H.

Now, we establish a formula for S-curvature of homogeneous Finsler spaces with square-root metric.

Theorem 4.2. *Let G/H be reductive homogeneous Finsler space with a decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{h} + \mathfrak{n}$, and $F = \sqrt{\alpha(\alpha + \beta)}$ be a G-invariant square-root metric on G/H. Then the S-curvature is given by*

$$
\begin{aligned}
S(H, y) &= \left[\frac{6ns^3 + 6(3n + 2)s^2 + 4(3n + b^2 + 5)s + 2(4(n + 1) - b^2(n - 2))}{-2(3s^2 + 6s - (b + 2)(b - 2))^2}\right]\\
&\quad \times \left(\frac{1}{\alpha}\langle [v, y]_{\mathfrak{n}}, y\rangle + \frac{1}{s + 2}\langle [v, y]_{\mathfrak{n}}, v\rangle\right), \qquad (4.3)
\end{aligned}
$$

where $v \in \mathfrak{n}$ corresponds to the 1-form β and $\mathfrak{n}$ is identified with the tangent space $T_H(G/H)$ of G/H at the origin H.

Proof. For square-root metric $F = \alpha\varphi(s)$, where $\varphi(s) = \sqrt{1 + s}$, the entities written in 4.1 take the values as follows:

$$Q = \frac{\phi^{'}}{\phi - s\phi^{'}} = \frac{1}{s + 2}, \quad Q^{'} = \frac{-1}{(s + 2)^2}, \quad Q^{''} = \frac{2}{(s + 2)^3},$$

$$\begin{aligned}
\Delta &= 1 + sQ + (b^2 - s^2)Q^{'} \\
&= 1 + \frac{s}{s+2} + (b^2 - s^2)\Big(\frac{-1}{(s+2)^2}\Big) \\
&= \frac{3s^2 + 6s - (b+2)(b-2)}{(s+2)^2}, \\
\Phi &= (sQ^{'} - Q)(n\Delta + 1 + sQ) + (s^2 - b^2)(1 + sQ)Q^{''} \\
&= \Big(\frac{-s}{(s+2)^2} - \frac{1}{s+2}\Big)\Big(1 + \frac{3ns^2 + 6ns - n(b-2)(b+2)}{(s+2)^2}\Big) \\
&\quad +(s^2 - b^2)(1 + \frac{s}{s+2})(\frac{2}{(s+2)^3}) \\
&= \frac{6ns^3 + 6(3n+2)s^2 + 4(3n + b^2 + 5)s + 2\big(4(n+1) - b^2(n-2)\big)}{-(s+2)^4}.
\end{aligned}$$

After substituting these values in 4.2, we get formula 4.3 for S-curvature of homogeneous Finsler space with square-root metric. □

Theorem 4.3. *Let G/H be reductive homogeneous Finsler space with a decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{h} + \mathfrak{n}$, and $F = \sqrt{(\alpha(\alpha+\beta)}$ be a G-invariant square-root metric on G/H. Then $(G/H, F)$ has isotropic S-curvature if and only if it has vanishing S-curvature.*

Proof. For necessary part, suppose G/H has isotropic S-curvature, then

$$S(x, y) = (n+1)c(x)F(y), \quad x \in G/H, \quad y \in T_x(G/H).$$

Taking $x = H$ and $y = v$ in 4.3, we get $c(H) = 0$. Consequently $S(H, y) = 0$, $\forall y \in TH(G/H)$. Since F is a homogeneous metric, we have $S = 0$ everywhere.

For the converse part, let G/H has vanishing S-curvature.then

$$0 = (n+1)(c(x)F(y) + \omega(y)), \quad x \in M, \quad y \in T_x(M).$$

Then we have, $c(x)F(y) + \omega(y) = 0$ and $\omega(y) = 0$. This proof the Theorem. □

5. Homogeneous Geodesics

Definition 5.1. A Finsler space (M, F) is called a homogeneous Finsler space if the group of isometries of (M, F), $I(M, L)$ acts transitively on M.

We recall that, Any homogeneous Finsler manifold $M = G/H$ is a reductive homogeneous space.

Definition 5.2. Let $(G/H, F)$ be a homogeneous Finsler space and e be the identity of G. A non-zero vector $X \in \mathfrak{g}$ is called a geodesic vector if the curve $exp(tX).eH$ is a geodesic of $(G/H, F)$.

In [10], the author proved the following result that gives a criterion for a non-zero vector to be a geodesic vector in a homogeneous Finsler space.

Lemma 5.1. *A non-zero vector $Y \in \mathfrak{g}$ is a geodesic vector if and only if*

$$g_{Y_\mathfrak{n}} = (Y_\mathfrak{n}, [Y, Z]_\mathfrak{n}) = 0, \quad \forall Z \in \mathfrak{g}.$$

Next, we deduce necessary and sufficient condition for a nonzero vector in a homogeneous Finsler space with square-root (α, β)-metric to be a geodesic vector.

Theorem 5.1. *Let $(G/H, F)$ be a homogeneous Finsler space with*

$$F(x, y) = \sqrt{\tilde{a}(y_x, y_x) + \sqrt{\tilde{a}(y_x, y_x)}\tilde{a}(X_x, y_x)}.$$

defined by the Riemannian metric $\tilde{a}$ and the vector field X. Then, X is a geodesic vector of $(G/H, \tilde{a})$ if and only if X is a geodesic vector of $(G/H, F)$.

Proof. We know that

$$g_y(u, v) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t}F^2(x, y + su + tv)|_{s=t=0}.$$

After some calculations, we get

$$\begin{aligned} g_y(u, v) &= \tilde{a}(u, v) - \frac{1}{2}\frac{\tilde{a}(y, v)\tilde{a}(u, y)\tilde{a}(X, y)}{\tilde{a}(y, y)^{\frac{3}{2}}} \\ &+ \frac{1}{2}\frac{\tilde{a}(u, v)\tilde{a}(X, y) + \tilde{a}(X, v)\tilde{a}(u, y) + \tilde{a}(y, v)\tilde{a}(y, u)}{\tilde{a}(y, y)^{\frac{1}{2}}}. \end{aligned} \tag{5.1}$$

So for all $Z \in \mathfrak{n}$, we have

$$g_{X_\mathfrak{n}}(X_\mathfrak{n}, [X, Z]_\mathfrak{n}) = \tilde{a}(X_\mathfrak{n}, [X, Z]_\mathfrak{n})\left[1 + \sqrt{\tilde{a}(X, X)}\right].$$

Thus, $g_{X_\mathfrak{n}}(X_\mathfrak{n}, [X, Z]_\mathfrak{n}) = 0$ if and only if

$$\tilde{a}(X_\mathfrak{n}, [X, Z]_\mathfrak{n}) = 0.$$

This completes the proof. □

Theorem 5.2. *Let $(G/H, F)$ be a homogeneous Finsler space with*

$$F(x, y) = \sqrt{\tilde{a}(y_x, y_x) + \sqrt{\tilde{a}(y_x, y_x)}\tilde{a}(X_x, y_x)}.$$

defined by the Riemannian metric $\tilde{a}$ and the vector field X. Let $y \in \mathfrak{g} - \{0\}$ be a vector which $\tilde{a}(X, [y, z]_\mathfrak{n}) = 0$, for all $z \in \mathfrak{n}$. Then, y is a geodesic vector of $(G/H, F)$ if and only if y is a geodesic vector of $(G/H, \tilde{a})$.

Proof. By using the relation 5.1 and some computations, we have

$$g_{y_{\mathfrak{n}}}(y_{\mathfrak{n}}, [y,z]_{\mathfrak{n}}) = \tilde{a}(y_{\mathfrak{n}}, [y,z]_{\mathfrak{n}}) \left[1 + \frac{1}{2}\frac{\tilde{a}(X,y)}{\sqrt{\tilde{a}(y,y)}}\right] + \frac{1}{2}\tilde{a}(X, [y,z]_{\mathfrak{n}})\sqrt{\tilde{a}(y,y)}.$$

This completes the proof. □

6. Mean Berwald Curvature

Let $E_{ij} = \frac{1}{2}\frac{\partial^2}{\partial y^i \partial y^j}(\frac{\partial G^m}{\partial y^m})(x,y)$, where G^m are spray coefficients. Then $\Xi := E_{ij}dx^i \otimes dx^j$ is a tensor on $TM\backslash\{0\}$, which called E tensor. E tensor can also be viewed as a family of symmetric forms defined as

$$E_y : T_xM \times T_xM \to \mathbb{R},$$
$$E_y(u,v) = E_{ij}(x,y)u^i v^j,$$

where $u = u^i \frac{\partial}{\partial x^i}|_x, v = v^i \frac{\partial}{\partial x^i}|_x \in T_xM$.Then the collection $\{E_y : y \in TM\backslash\{0\}\}$ is called E-curvature or Mean Berwald curvature.
In this section, we calculate the mean Berwald curvature of a homogeneous Finsler space with square-root metric. We need the following:
At the origin, we have $a_{ij} = \delta^i_j$, therefore,

$$y_i = a_{ij}y^j = \delta^i_j y^j = y^i,$$

$$\alpha_{y^i} = \frac{y_i}{\alpha}, \quad \beta_{y^i} = b_i,$$

$$s_{y^i} = \frac{\partial}{\partial y^i}(\frac{\beta}{\alpha}) = \frac{b_i\alpha - sy_i}{\alpha^2},$$

$$s_{y^iy^j} = \frac{\partial}{\partial y^j}\left(\frac{b_i\alpha - sy_i}{\alpha^2}\right) = \frac{-(b_iy_j + b_jy_i)\alpha + 3sy_iy_j - \alpha^2 s\delta^i_j}{\alpha^4}.$$

Now let in 4.3 we set:

$$\left[\frac{6ns^3 + 6(3n+2)s^2 + 4(3n+b^2+5)s + 2(4(n+1) - b^2(n-2))}{-2(3s^2 + 6s - (b+2)(b-2))^2}\right] = A.$$

Then we have

$$
\begin{aligned}
\frac{\partial A}{\partial y^j} &= \left[\frac{8+2b^4+60s+90s^2+36s^3+3n(8+4s+6s^2+12s^3+3s^4)}{(4-b^2+6s+3s^2)^3}\right.\\
&\quad \left. + \frac{b^2(26+48s+18s^2+n(-6+6s+9s^2))}{(4-b^2+6s+3s^2)^3}\right] s_{y^j},
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial^2 A}{\partial y^i \partial y^j} &= \frac{\partial}{\partial y^i}\left[\frac{8+2b^4+60s+90s^2+36s^3+3n(8+4s+6s^2+12s^3+3s^4)}{(4-b^2+6s+3s^2)^3}\right.\\
&\quad \left. + \frac{b^2(26+48s+18s^2+n(-6+6s+9s^2))}{(4-b^2+6s+3s^2)^3}\right] s_{y^j}\\
&+ \left[\frac{8+2b^4+60s+90s^2+36s^3+3n(8+4s+6s^2+12s^3+3s^4)}{(4-b^2+6s+3s^2)^3}\right.\\
&\quad \left. + \frac{b^2(26+48s+18s^2+n(-6+6s+9s^2))}{(4-b^2+6s+3s^2)^3}\right] s_{y^i y^j}\\
&= \left[\frac{6(-9ns^5-45ns^4-54s^4-36b^2s^3-24b^2ns^3-12ns^3-180s^3+24ns^2}{(4-b^2+6s+3s^2)^4}\right.\\
&\quad + \frac{-156b^2s^2-42b^2ns^2-168s^2+12b^2ns-12b^4s-180b^2s-3b^4ns-72ns}{(4-b^2+6s+3s^2)^4}\\
&\quad \left. + \frac{-24s+20b^2n+16-14b^2-b^4n-56b^2-64n}{(4-b^2+6s+3s^2)^4}\right] s_{y^i} s_{y^j}\\
&+ \left[\frac{8+2b^4+60s+90s^2+36s^3+3n(8+4s+6s^2+12s^3+3s^4)}{(4-b^2+6s+3s^2)^3}\right.\\
&\quad \left. + \frac{b^2(26+48s+18s^2+n(-6+6s+9s^2))}{(4-b^2+6s+3s^2)^3}\right] s_{y^j y^j}.
\end{aligned}
$$

Theorem 6.1. *Let G/H be a reductive homogeneous Finsler space with a decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{h} + \mathfrak{n}$, and $F = \sqrt{\alpha(\alpha+\beta)}$ be a G-invariant square-root metric on G/H. Then the mean Berwald curvature of the homogeneous Finsler space with suare-root metric is given by*

$$
\begin{aligned}
E_{ij}(H,y) &= \frac{1}{2}\left(\frac{1}{\alpha}\frac{\partial^2 A}{\partial y^i \partial y^j} - \frac{y_i}{\alpha^3}\frac{\partial A}{\partial y^j} - \frac{y_j}{\alpha^3}\frac{\partial A}{\partial y^i} - \frac{A}{\alpha^3}\delta_i^j + \frac{3A}{\alpha^5} y_i y_j\right)\langle [v,y\rangle_{\mathfrak{n}}, y\rangle\\
&+ \frac{1}{2}\left(\frac{1}{\alpha}\frac{\partial A}{\partial y^j} - \frac{A y_j}{\alpha^3}\right)(\langle [v,v_i]_{\mathfrak{n}}, y\rangle + \langle [v,y]_{\mathfrak{n}}, v_i\rangle)
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \frac{1}{2}\left(\frac{1}{\alpha}\frac{\partial A}{\partial y^i} - \frac{Ay_i}{\alpha^3}\right)(\langle[v,v_j]_{\mathfrak{n}},y\rangle + \langle[v,y]_{\mathfrak{n}},v_j\rangle)\\
&+ \quad \frac{A}{2\alpha}(\langle[v,v_j]_{\mathfrak{n}},v_i\rangle + \langle[v,v_i]_{\mathfrak{n}},v_j\rangle)\\
&+ \quad \frac{1}{2}\left(\frac{1}{s+2}\frac{\partial^2 A}{\partial y^i\partial y^j} - \frac{1}{(s+2)^2}s_{y^i}\frac{\partial A}{\partial y^j} - \frac{1}{(s+2)^2}s_{y^j}\frac{\partial A}{\partial y^i}\right)\\
&+ \quad \frac{1}{2}\left(\frac{2A}{(s+2)^3}s_{y^i}s_{y^j} - \frac{A}{(s+2)^2}s_{y^iy^j}\right)\langle[v,y]_{\mathfrak{n}},v\rangle\\
&+ \quad \frac{1}{2}\left(\frac{1}{s+2}\frac{\partial A}{\partial y^j} - \frac{A}{(s+2)^2}s_{y^j}\right)\langle[v,v_i]_{\mathfrak{n}},v\rangle\\
&+ \quad \frac{1}{2}\left(\frac{1}{s+2}\frac{\partial A}{\partial y^i} - \frac{A}{(s+2)^2}s_{y^i}\right)\langle[v,v_j]_{\mathfrak{n}},v\rangle. \qquad (6.1)
\end{aligned}
$$

where $v \in \mathfrak{n}$ corresponds to the 1-form β and $\mathfrak{n}$ is identified with the tangent space $T_H(G/H)$ of G/H at the origin H.

Proof. From 4.3, we can write S-curvature at the origin as follows:

$$S(H,y) = \phi + \psi,$$

where

$$\phi = \frac{A}{\alpha}\langle[v,y]_{\mathfrak{n}},y\rangle \quad \textit{and} \quad \psi = \frac{A}{s+2}\langle[v,y]_{\mathfrak{n}},v\rangle.$$

Therefore, mean Berwald curvature is

$$E_{ij} = \frac{1}{2}\frac{\partial^2 S}{\partial y^i\partial y^j} = \frac{1}{2}\left(\frac{\partial^2\phi}{\partial y^i\partial y^j} + \frac{\partial^2\psi}{\partial y^i\partial y^j}\right), \qquad (6.2)$$

where $\frac{\partial^2\phi}{\partial y^i\partial y^j}$ and $\frac{\partial^2\psi}{\partial y^i\partial y^j}$ are calculated as follows:

$$
\begin{aligned}
\frac{\partial\phi}{\partial y^j} &= \frac{\partial}{\partial y^j}\left(\frac{A}{\alpha}\langle[v,y]_{\mathfrak{n}},y\rangle\right)\\
&= \left(\frac{1}{\alpha}\frac{\partial A}{\partial y^j} - \frac{A}{\alpha^2}\frac{y_j}{\alpha}\right)\langle[v,y]_{\mathfrak{n}},y]\rangle + \frac{A}{\alpha}(\langle[v,v_j]_{\mathfrak{n}},y\rangle + \langle[v,y]_{\mathfrak{n}},v_j\rangle),
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2\phi}{\partial y^i\partial y^j} &= \frac{\partial}{\partial y^i}\left[\left(\frac{1}{\alpha}\frac{\partial A}{\partial y^j} - \frac{Ay_j}{\alpha^3}\right)\langle[v,y]_{\mathfrak{n}},y\rangle + \frac{A}{\alpha}(\langle[v,v_j]_{\mathfrak{n}},y]\rangle + \langle[v,y]_{\mathfrak{n}},v_j\rangle)\right]\\
&= \left(\frac{1}{\alpha}\frac{\partial^2 A}{\partial y^i\partial y^j} - \frac{y_i}{\alpha^3}\frac{\partial A}{\partial y^j} - \frac{y_j}{\alpha^3}\frac{\partial A}{\partial y^i} - \frac{A}{\alpha^3}\delta^j_i + \frac{3A}{\alpha^5}y_iy_j\right)\langle[v,y\rangle_{\mathfrak{n}},y\rangle\\
&+ \left(\frac{1}{\alpha}\frac{\partial A}{\partial y^j} - \frac{Ay_j}{\alpha^3}\right)(\langle[v,v_i]_{\mathfrak{n}},y\rangle + \langle[v,y]_{\mathfrak{n}},v_i\rangle)
\end{aligned}
$$

$$
\begin{aligned}
&+ \left(\frac{1}{\alpha}\frac{\partial A}{\partial y^i} - \frac{Ay_i}{\alpha^3}\right)(\langle [v, v_j]_{\mathfrak{n}}, y\rangle + \langle [v, y]_{\mathfrak{n}}, v_j\rangle) \\
&+ \frac{A}{\alpha}(\langle [v, v_j]_{\mathfrak{n}}, v_i\rangle + \langle [v, v_i]_{\mathfrak{n}}, v_j\rangle),
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial \psi}{\partial y^j} &= \frac{\partial}{\partial y^j}\left(\frac{A}{s+2}\langle [v, y]_{\mathfrak{n}}, v\rangle\right) \\
&= \left[\frac{1}{s+2}\frac{\partial A}{\partial y^j} - \frac{A}{(s+2)^2}s_{y^j}\right]\langle [v, y]_{\mathfrak{n}}, v\rangle + \frac{A}{s+2}\langle [v, v_j]_{\mathfrak{n}}, v\rangle,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 \psi}{\partial y^i \partial y^j} &= \frac{\partial}{\partial y^i}\left[\left(\frac{1}{s+2}\frac{\partial A}{\partial y^j} - \frac{A}{(s+2)^2}s_{y^j}\right)\langle [v, y]_{\mathfrak{n}}, v\rangle + \frac{A}{s+2}\langle [v, v_j]_{\mathfrak{n}}, v\rangle\right] \\
&= \left(\frac{1}{s+2}\frac{\partial^2 A}{\partial y^i \partial y^j} - \frac{1}{(s+2)^2}s_{y^i}\frac{\partial A}{\partial y^j} - \frac{1}{(s+2)^2}s_{y^j}\frac{\partial A}{\partial y^i}\right) \\
&+ \left(\frac{2A}{(s+2)^3}s_{y^i}s_{y^j} - \frac{A}{(s+2)^2}s_{y^i y^j}\right)\langle [v, y]_{\mathfrak{n}}, v\rangle \\
&+ \left(\frac{1}{s+2}\frac{\partial A}{\partial y^j} - \frac{A}{(s+2)^2}s_{y^j}\right)\langle [v, v_i]_{\mathfrak{n}}, v\rangle \\
&+ \left(\frac{1}{s+2}\frac{\partial A}{\partial y^i} - \frac{A}{(s+2)^2}s_{y^i}\right)\langle [v, v_j]_{\mathfrak{n}}, v\rangle.
\end{aligned}
$$

Substituting all above values in 6.2, we get formula 6.1. □

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