FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 39, No 2 (2024), 317–326
https://doi.org/10.22190/FUMI231123022J
**Original Scientific Paper**

# ON PROJECTIVELY FLAT GENERALIZED BERWALD $(\alpha, \beta)$-METRICS

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**Abstract.** Every Berwald metric is a special generalized Berwald metric. In this paper, we study the class of projectively flat generalized Berwald $(\alpha, \beta)$-metrics of isotropic S-curvature. We find some conditions under which this class of Finsler metrics reduces to the class of Berwald metrics.
**Keywords**: Berwald $(\alpha, \beta)$-metric, Finsler metric, isotropic S-curvature.

## 1. Introduction

The geodesics curves of an arbitrary Finsler metric $F = F(x, y)$ on a manifold $M$ are characterized by the following system of differential equations

$$\ddot{c}^i + 2G^i(\dot{c}) = 0,$$

where the local functions $G^i = G^i(x, y)$ are called the spray coefficients of $F$. Two Finsler metrics $F$ and $\bar{F}$ on a manifold $M$ are called projectively related if any geodesic of the first is also geodesic for the second and the other way around. In this case, there is a scalar function $P = P(x, y)$ defined on the slit tangent bundle $TM_0 = TM - \{0\}$ such that

$$G^i = \bar{G}^i + Py^i.$$

Received November 23, 2023, accepted: February 11, 2024
Communicated by Uday Chand De
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2020 *Mathematics Subject Classification.* Primary 53B40; Secondary 53C60

Here, $G^i$ and $\bar{G}^i$ denote the geodesic spray coefficients of $F$ and $\bar{F}$, respectively [6]. The problem of projectively related Finsler metrics is quite old in geometry and its origin is formulated in Hilberts Fourth Problem: to determine the metrics on an open subset in $\mathbb{R}^n$, whose geodesics are straight lines [2]. Projectively flat Finsler metrics on a convex domain in $\mathbb{R}^n$ are regular solutions to Hilbert's Fourth Problem. A Finsler metric $F$ on an open subset $U \subset \mathbb{R}^n$ is called projectively flat if all geodesics are straight in $U$. In this case, $F$ and the Euclidean metric on $U$ are projectively related.

In order to find projectively flat Finsler metrics, one can search in the class of generalized Berwald metrics. A Finsler metric $F = F(x, y)$ on a manifold $M$ is called a generalized Berwald metric if there exists a covariant derivative $D$ on $M$ such that the parallel translations induced by $D$ preserve the Finsler function $F$ [1][12]. In this case, $F$ is called a generalized Berwald metric on $M$. If the covariant derivative $D$ is also torsion-free, then $F$ reduces to a Berwald metric. In this case, the spray coefficients of $F$ is quadratic in direction $y$. By definition, the class of Berwald metrics belongs to the class of generalized Berwald metrics.

The class of generalized Berwald metrics is very large to search, and finding projectively flat Finsler metrics in this class is very complex. Thus, one can focus on a meaningful subclasses of these Finsler metrics, maybe the class of generalized Berwald $(\alpha, \beta)$-metrics. An $(\alpha, \beta)$-metric is a Finsler metric on a manifold $M$ defined by $F := \alpha\phi(s)$, where $s = \beta/\alpha$, $\phi = \phi(s)$ is a $C^\infty$ function on the $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}y^iy^j}$ is a positive-definite Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on $M$.

It is interesting to find some conditions under which a projectively flat generalized Berwald $(\alpha, \beta)$-metric reduces to a Berwald metric. To find the mentioned condition, for an $(\alpha, \beta)$-metric $F := \alpha\phi(s)$, let us put

$$Q := \frac{\phi'}{\phi - s\phi'}, \qquad \Psi := \frac{Q'}{2[1 + sQ + (b^2 - s^2)Q']}.$$

Define

$$(1.1) \quad \Lambda := b^ib^jb^kb^l\Big[\alpha\beta Q\Big]_{y^iy^jy^ky^l} \text{and} \quad \Upsilon := b^jb^jb^kb^lb^m\Big[\Psi\Big]_{y^iy^jy^ky^ly^m}.$$

Then, we will prove the following result.

**Theorem 1.1.** *Let $F = \alpha\phi(\beta/\alpha)$ be a projectively flat $(\alpha, \beta)$-metric on a manifold $M$. Suppose that $\phi$ satisfies $\phi'(0) \neq 0$, $\Lambda \neq 0$ and $\Upsilon \neq 0$. Then $F$ is a generalized Berwald metric of isotropic S-curvature if and only if it is a Berwald metric. In this case, $F$ is a locally Minkowskian metric.*

We remark that the S-curvature is constructed by Zhongmin Shen for given comparison theorems on Finsler manifolds [11]. A natural problem is to study and characterize Finsler metrics of vanishing S-curvature. An $n$-dimensional Finsler metric is said to have isotropic S-curvature if $\mathbf{S} = (n + 1)cF$, for some scalar function $c = c(x)$ on $M$.

## 2. Preliminary

Let $M$ be an $n$-dimensional $C^\infty$ manifold, $TM = \bigcup_{x\in M} T_xM$ the tangent space and $TM_0 := TM - \{0\}$ the slit tangent space of $M$. A Finsler structure on manifold $M$ is a function $F : TM \to [0, \infty)$ with the following properties:
(i) $F$ is $C^\infty$ on $TM_0$;
(ii) $F$ is positively 1-homogeneous on the fibers of tangent bundle $TM$, i.e., $F(x, \lambda y) = \lambda F(x, y)$, $\forall \lambda > 0$;
(iii) The quadratic form $\mathbf{g}_y : T_xM \times T_xM \to \mathbb{R}$ is positive-definite on $T_xM$

$$\mathbf{g}_y(u, v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\Big[F^2(y + su + tv)\Big]_{s=t=0}, \quad u, v \in T_xM.$$

Then, the pair $(M, F)$ is called a Finsler manifold.

Given a Finsler manifold $(M, F)$, then a global vector field $\mathbf{G}$ is induced by $F$ on $TM_0$, which in a standard coordinate $(x^i, y^i)$ for $TM_0$ is given by

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i},$$

where

$$G^i := \frac{1}{4} g^{il}\Big[\frac{\partial^2 F^2}{\partial x^k \partial y^l} y^k - \frac{\partial F^2}{\partial x^l}\Big], \quad y \in T_xM. \tag{2.1}$$

$\mathbf{G}$ is called the spray associated to $(M, F)$.

For a tangent vector $y \in T_xM_0$, define $\mathbf{B}_y : T_xM \times T_xM \times T_xM \to T_xM$ by $\mathbf{B}_y(u, v, w) := B^i{}_{jkl}(y)u^j v^k w^l \partial/\partial x^i|_x$ where

$$B^i{}_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}.$$

$\mathbf{B}$ is called the Berwald curvature. Then $F$ is called a Berwald metric if $\mathbf{B} = \mathbf{0}$ [10].

For a Finsler metric $F$ on an $n$-dimensional manifold $M$, the Busemann-Hausdorff volume form $dV_F = \sigma_F(x)dx^1 \cdots dx^n$ is defined by

$$\sigma_F(x) := \frac{\mathrm{Vol}(\mathbb{B}^n(1))}{\mathrm{Vol}\big[(y^i) \in R^n \mid F\big(y^i \frac{\partial}{\partial x^i}|_x\big) < 1\big]}.$$

Let $G^i$ denote the geodesic coefficients of $F$ in the same local coordinate system. Then for $\mathbf{y} = y^i \frac{\partial}{\partial x^i}|_x \in T_xM$, the S-curvature is defined by

$$\mathbf{S}(\mathbf{y}) := \frac{\partial G^i}{\partial y^i}(x, y) - y^i \frac{\partial}{\partial x^i}\big[\ln \sigma_F(x)\big].$$

This quantity was first introduced by Shen for a volume comparison theorem [10]. A Finsler metric $F$ on an $n$-dimensional manifold $M$ has isotropic S-curvature if

$$\mathbf{S} = (n+1)cF,$$

where $c = c(x)$ is a scalar function on $M$. Also, $F$ has vanishing S-curvature if $\mathbf{S} = 0$.

It is known that a Finsler metric $F(x, y)$ on $\mathcal{U}$ is projective if and only if its geodesic coefficients $G^i$ are in the form

$$G^i(x, y) = P(x, y)y^i,$$

where $P : T\mathcal{U} = \mathcal{U} \times \mathbb{R}^n \to \mathbb{R}$ is positively homogeneous with degree one, $P(x, \lambda y) = \lambda P(x, y)$, $\lambda > 0$. We call $P(x, y)$ the *projective factor* of $F(x, y)$.

For a non-zero vector $y \in T_xM_0$, the Riemann curvature is a family of linear transformation $\mathbf{R}_y : T_xM \to T_xM$ which is defined by $\mathbf{R}_y(u) := R^i_k(y)u^k\partial/\partial x^i$, where

$$R^i_k(y) = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k}y^j + 2G^j\frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

The family $\mathbf{R} := \{\mathbf{R}_y\}_{y \in TM_0}$ is called the Riemann curvature.

For a flag $P := \mathrm{span}\{y, u\} \subset T_xM$ with flagpole $y$, the flag curvature $\mathbf{K} = \mathbf{K}(P, y)$ is defined by

$$\mathbf{K}(x, y, P) := \frac{\mathbf{g}_y\big(u, \mathbf{R}_y(u)\big)}{\mathbf{g}_y(y, y)\mathbf{g}_y(u, u) - \mathbf{g}_y(y, u)^2}. \tag{2.2}$$

The flag curvature $\mathbf{K}(x, y, P)$ is a function of tangent planes $P = \mathrm{span}\{y, v\} \subset T_xM$. $F$ is of scalar flag curvature if $\mathbf{K} = \mathbf{K}(x, y)$ is independent of flag $P$.

## 3. Proof of Theorem 1.1

An $(\alpha, \beta)$-metric is a Finsler metric on a manifold $M$ defined by $F := \alpha\phi(s)$, where $s = \beta/\alpha$, $\phi = \phi(s)$ is a $C^\infty$ function on the $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on $M$. For an $(\alpha, \beta)$-metric, let us define $b_{i|j}$ by $b_{i|j}\theta^j := db_i - b_j\theta_i^j$, where $\theta^i := dx^i$ and $\theta_i^j := \Gamma^j_{ik}dx^k$ denote the Levi-Civita connection form of $\alpha$. Let us define

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}),$$
$$r_{i0} := r_{ij}y^j, \quad r_{00} := r_{ij}y^iy^j, \quad r_j := b^ir_{ij},$$
$$s_{i0} := s_{ij}y^j, \quad s_j := b^is_{ij}, \quad r_0 := r_jy^j, \quad s_0 := s_jy^j.$$

Let $\phi = \phi(s)$ be a positive $C^\infty$ function on $(-b_0, b_0)$. For a number $b \in [0, b_0)$, put

$$\Delta := 1 + sQ + (b^2 - s^2)Q',$$
$$\Phi := -(n\Delta + 1 + sQ)(Q - sQ') - (b^2 - s^2)(1 + sQ)Q''.$$

In [4], Cheng-Shen characterized $(\alpha, \beta)$-metrics with isotropic S-curvature on a manifold $M$ of dimension $n \geq 3$. Soon, they found that their result holds for the class of $(\alpha, \beta)$-metrics with constant length one-forms, only. Here, we modify their result as follows.

**Lemma 3.1.** *Let $F = \alpha\phi(\beta/\alpha)$ be an non-Randers type $(\alpha, \beta)$-metric on an manifold $M$ of dimension $n \geq 3$. Suppose that $\beta$ has constant length with respect to $\alpha$. Then, $F$ is of isotropic S-curvature $\mathbf{S} = (n+1)cF$, if and only if one of the following holds*

*(i) $\beta$ satisfies*

(3.1) $$r_{ij} = \epsilon\Big\{b^2 a_{ij} - b_i b_j\Big\}, \qquad s_j = 0,$$

*where $\epsilon = \epsilon(x)$ is a scalar function, and $\phi = \phi(s)$ satisfies*

(3.2) $$\Phi = -2(n+1)k\frac{\phi\Delta^2}{b^2 - s^2},$$

*where $k$ is a constant. In this case, $\mathbf{S} = (n+1)k\epsilon F$.*

*(ii) $\beta$ satisfies*

(3.3) $$r_{ij} = 0, \qquad s_j = 0.$$

*In this case, $\boldsymbol{S} = 0$.*

In [18], the following is proved.

**Lemma 3.2.** ([18]) *An $(\alpha, \beta)$-metric satisfying $\phi'(0) \neq 0$ is a generalized Berwald manifold if and only if $\beta$ has constant length with respect to $\alpha$.*

By Lemmas 3.1 and 3.2, we get the following.

**Lemma 3.3.** *Let $F = \alpha\phi(\beta/\alpha)$ be an non-Randers type generalized Berwald $(\alpha, \beta)$-metric on a manifold $M$ of dimension $n \geq 3$ such that $\phi'(0) \neq 0$. Then, $F$ is of isotropic S-curvature $\mathbf{S} = (n+1)cF$, if and only if one of the following holds:*

*(i) $\beta$ satisfies*

$$r_{ij} = \epsilon\Big\{b^2 a_{ij} - b_i b_j\Big\}, \qquad s_j = 0, \tag{3.4}$$

*where $\epsilon = \epsilon(x)$ is a scalar function, and $\phi = \phi(s)$ satisfies*

$$\Phi = -2(n+1)k\frac{\phi\Delta^2}{b^2 - s^2}, \tag{3.5}$$

*where $k$ is a constant. In this case, $\mathbf{S} = (n+1)k\epsilon F$.*

*(ii) $\beta$ satisfies*

$$r_{ij} = 0, \qquad s_j = 0. \tag{3.6}$$

*In this case, $\boldsymbol{S} = 0$.*

To prove Theorem 1.1, we need the following.

**Proposition 3.1.** *Let $F = \alpha\phi(\beta/\alpha)$ be a non-Randers type $(\alpha, \beta)$-metric on a manifold $M$ of dimension $n \geq 3$ such that $\Lambda \neq 0$. Then $F$ is a generalized Berwald metric with vanishing S-curvature $\mathbf{S} = 0$ if and only if it is a Berwald metric.*

*Proof.* Let $G^i = G^i(x, y)$ and $G^i_\alpha = G^i_\alpha(x, y)$ denote the spray coefficients of $F$ and $\alpha$ respectively in the same coordinate system. By (2.1), we have

$$G^i = G^i_\alpha + Py^i + Q^i, \tag{3.7}$$

where

$$P := \alpha^{-1}\Theta(r_{00} - 2Q\alpha s_0),$$
$$Q^i := \alpha Q {s^i}_0 + \Psi(r_{00} - 2Q\alpha s_0)b^i.$$

In [3], Cheng proved that every regular $(\alpha, \beta)$-metric with isotropic S-curvature has vanishing S-curvature (see Theorem 2.4). In this case, by Lemma 3.3, we have $r_{00} = s_0 = 0$. Then (3.7) reduces to following

$$G^i = G^i_\alpha + \alpha Q {s^i}_0. \tag{3.8}$$

$F$ is a projectively flat Finsler metric which is equal to following

$$G^i = Py^i, \tag{3.9}$$

where $P = P(x, y)$ is a local scalar function satisfying $P(x, \lambda y) = \lambda P(x, y)$. By (3.8) and (3.9), we have

$$Py^i = G^i_\alpha + \alpha Q {s^i}_0. \tag{3.10}$$

Multiplying (3.10) with $b_i$ and $y_i$, respectively, imply that

$$P\beta = b_i G^i_\alpha, \tag{3.11}$$
$$P\alpha^2 = y_i G^i_\alpha. \tag{3.12}$$

Contracting (3.10) with $\beta$ yields

$$P\beta y^i = \beta G^i_\alpha + \alpha\beta Q {s^i}_0. \tag{3.13}$$

By (3.11) and (3.13) it follows that

$$(b_r G^r_\alpha) y^i - \beta G^i_\alpha = \alpha\beta Q {s^i}_0. \tag{3.14}$$

The following holds

$$\big[(b_r G^r_\alpha) y^i - \beta G^i_\alpha\big]_{y^j y^k y^l y^m} = 0. \tag{3.15}$$

(3.14) and (3.15) give us

$$[\alpha\beta Q s^i_0]_{y^j y^k y^l y^m} = 0. \tag{3.16}$$

We have

$$\begin{aligned}\big[\alpha\beta Q s^i_0\big]_{y^j y^k y^l y^m} = {} & \big[\alpha\beta Q\big]_{y^j y^k y^l} {s^i}_m + \big[\alpha\beta Q\big]_{y^j y^k y^m} {s^i}_l + \big[\alpha\beta Q\big]_{y^j y^l y^m} {s^i}_k \\ & + \big[\alpha\beta Q\big]_{y^l y^k y^m} {s^i}_j + \big[\alpha\beta Q\big]_{y^j y^k y^l y^m} {s^i}_0 = 0\end{aligned} \tag{3.17}$$

By part ($b$) of Lemma 3.3, we have $s^k = b^m s^k_m = 0$. Then multiplying (3.17) with $b^j b^k b^l b^m$ and considering (3.16) imply that

$$b^j b^k b^l b^m \big[\alpha\beta Q\big]_{y^j y^k y^l y^m} {s^i}_0 = 0 \tag{3.18}$$

By assumption, we get

$${s^i}_j = 0. \tag{3.19}$$

Putting (3.19) in (3.8) gives us $G^i = G^i_\alpha$. It implies that $F$ is a Berwald metric. □

Now, we are ready to prove Theorem 1.1.

**Proof of Theorem 1.1:** The proof divided to three main cases as follows:

**Case (i).** ***F* is not a Randers metric and** $dim(M) \geq 3$**:** Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be a generalized Berwald non-Randers type $(\alpha,\beta)$-metric on a manifold $M$ of dimension $dim(M) \geq 3$. Suppose that $F$ has isotropic S-curvature,

$\mathbf{S} = (n+1)cF$, where $c = c(x)$ is a scalar function on $M$. In this case, by Lemma 3.3 we have

$$s_0 = 0, \tag{3.20}$$

$$r_{00} = c(b^2 - s^2)\alpha^2. \tag{3.21}$$

Since $F$ is a projectively flat metric, then there exists a local scalar function $P = P(x,y)$ satisfies $P(x,\lambda y) = \lambda P(x,y)$. By (3.7) and (3.20), it follows that

$$Py^i = G^i_\alpha + \alpha Q {s^i}_0 + r_{00}\Big[\Theta\frac{y^i}{\alpha} + \Psi b^i\Big]. \tag{3.22}$$

Multiplying (3.22) with $b_i$ and $y_i$, respectively, imply that

$$P\beta = b_i G^i_\alpha + r_{00}\Big[\Theta\frac{\beta}{\alpha} + \Psi b^2\Big], \tag{3.23}$$

$$P\alpha^2 = y_i G^i_\alpha + r_{00}\Big[\Theta\alpha + \Psi\beta\Big]. \tag{3.24}$$

(3.23) $\times \alpha^2 -$ (3.24) $\times \beta$ yields

$$\Psi r_{00}(b^2\alpha^2 - \beta^2) = (y_i G^i_\alpha)\beta - (b_i G^i_\alpha)\alpha^2. \tag{3.25}$$

By (3.21) and (3.25), we get

$$c\Psi(b^2\alpha^2 - \beta^2)^2 = (y_i G^i_\alpha)\beta - (b_i G^i_\alpha)\alpha^2. \tag{3.26}$$

Since

$$\Big[(y_i G^i_\alpha)\beta - (b_i G^i_\alpha)\alpha^2\Big]_{y^j y^k y^l y^m y^p} = 0$$

then

$$\Big[c\Psi(b^2\alpha^2 - \beta^2)^2\Big]_{y^j y^k y^l y^m y^p} = 0.$$

It is easy to see that the following holds

$$b^t\Big[(b^2\alpha^2 - \beta^2)^2\Big]_{y^t} = 0.$$

Then

$$\begin{aligned} b^j b^k b^l b^m b^p \Big[c\Psi(b^2\alpha^2 - \beta^2)^2\Big]_{y^j y^k y^l y^m y^p} &= c b^j b^k b^l b^m b^p \Big[\Psi\Big]_{y^j y^k y^l y^m y^p}(b^2\alpha^2 - \beta^2)^2 \\ &= 0 \end{aligned} \tag{3.27}$$

According to the assumption, (3.27) implies that $c = 0$. Then $r_{00} = 0$ and by (3.20) we get $s_0 = 0$. By Lemma 3.3, $F$ has vanishing S-curvature. Then by Proposition 3.1, we conclude that $F$ is a Berwald metric. Since $F$ is projectively flat metric then it is of scalar flag curvature $\mathbf{K} = \mathbf{K}(x,y)$. $F$ is not Randers-type and then is not Riemannian. Then $\mathbf{K} = 0$, and $F$ is a locally Minkowsian metric.

**Case (ii).** **$F$ is not a Randers metric and $dim(M) = 2$:** Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be a two-dimensional generalized Berwald non-Randers type $(\alpha, \beta)$-metric on a manifold $M$. Suppose that $F$ has isotropic S-curvature. By Theorem 2.4 of [3], every regular $(\alpha, \beta)$-metric with isotropic S-curvature has vanishing S-curvature. In [13], it is proved that such metric reduces to a locally Minkowskian metric. This completes the proof.

**Case (iii).** **$F$ is a Randers metric:** A Randers metric $F = \alpha + \beta$ is locally projectively flat if and only if $\alpha$ is locally projectively flat and $\beta$ is closed, i.e., $s_{ij} = 0$ (see [10]). On the other hand, in [18], it is proved that $F$ is a generalized Berwald manifold if and only if $\beta$ is of constant Riemannian length, namely $r_i + s_i = 0$. These imply

$$s_{ij} = 0, \quad r_i = 0. \tag{3.28}$$

In [4], it is proved that $F = \alpha + \beta$ has isotropic S-curvature $\mathbf{S} = (n + 1)cF$ if and only if

$$e_{00} = 2c(\alpha^2 - \beta^2), \tag{3.29}$$

where $c = c(x)$ is a scalar function on $M$, $e_{00} = e_{ij}y^iy^j$ and $e_{ij} = r_{ij} + b_is_j + b_js_i$. By (3.28) and (3.29), we get

$$r_{ij} = 2c(a_{ij} - b_ib_j). \tag{3.30}$$

Multiplying (3.30) with $b^i$ yields

$$r_j = 2c(1 - b^2)b_j. \tag{3.31}$$

Since $b < 1$ then by (3.28) and (3.31) we get $b_j = 0$ or $c = 0$. If $b_j = 0$ then $F$ is Riemannian. If $c = 0$ then by (3.30) implies that $r_{ij} = 0$. By considering (3.28), $\beta$ is parallel with respect to $\alpha$ and $F$ reduces to a Berwald metric. This completes the proof. □

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