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GEOMETRY AND CURVATURE ANALYSIS OF THE BERGER-TYPE CHEEGER-GROMOLL METRIC ON TANGENT BUNDLES OVER ANTI-PARAKÄHLER MANIFOLDS

Abderrahim Zagane[1], Aydin Gezer[2] and Saadia Chaoui[1]

[1]**Relizane University, Faculty of Science and Technology**
Department of Mathematics, P. O. Box 60, 4800 Relizane, Algeria
[2]**Ataturk University, Faculty of Science**
Department of Mathematics, P.O. Box 73, 25240 Erzurum, Turkiye

ORCID IDs: Abderrahim Zagane https://orcid.org/0000-0001-9339-3787
Aydin Gezer https://orcid.org/0000-0001-7505-0385
Saadia Chaoui https://orcid.org/0000-0001-9528-9131

Abstract. In this paper, we introduce a novel metric known as a Berger-type Cheeger-Gromoll metric on the tangent bundle TM over an anti-paraKähler manifold (M, φ, g). This metric is defined as a natural metric with respect to the base metric g on TM. We begin by exploring the properties of the Levi-Civita connection associated with this metric. Subsequently, we compute all the components of the Riemannian curvature tensor and provide an explicit expression for the sectional curvature and the scalar curvature. In the final part of our analysis, we delve into the geometry of the φ-unit tangent bundle, which is endowed with the Berger-type Cheeger-Gromoll metric. Within this context, we provide the Levi-Civita connection and detail all forms of the Riemannian curvature tensors associated with this metric.
Keywords: Tangent bundle, anti-paraKähler manifold, Riemannian curvature tensor, φ-unit tangent bundle, Berger-type Cheeger-Gromoll metric.

1. Introduction

The study of tangent bundles has been a subject of extensive research, with pivotal contributions made by Sasaki in his seminal paper [17]. Sasaki's work primarily

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Corresponding Author: A. Gezer
E-mail addresses: Zaganeabr2018@gmail.com (A. Zagane), aydingzr@gmail.com (A. Gezer), Ch.saadia2018@gmail.com (S. Chaoui)
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focused on the differential geometry of tangent bundles associated with Riemannian manifolds, leading to the introduction of the Sasaki metric as a fundamental Riemannian metric on these bundles. Following Sasaki's groundbreaking work, researchers have delved into exploring various geometric properties related to the Sasaki metric, as evidenced by references such as [6, 12, 14, 15, 20]. However, a significant limitation encountered in many of these studies was the inherent flatness of the base Riemannian manifold. This limitation prompted researchers to seek alternative deformations of the Sasaki metric to address this issue. In the direction, Musso and Tricerri introduced another Riemannian metric on the tangent bundle TM over a Riemannian manifold, which they referred to as the Cheeger-Gromoll metric. Although the metric was initially formulated by Cheeger and Gromoll in their paper [5], Musso and Tricerri provided a more accessible expression for it and played a significant role in its construction. In recognition of their contributions, they gave this metric its distinctive name [14]. Notably, Abbassi and Sarih, [1], introduced a family of natural metrics on both the tangent bundle and the unit tangent bundles. These metrics encompassed a wide range of cases, including the Sasaki metric, the Cheeger-Gromoll metric, the Kaluza-Klein type metric, and others (see [2, 11, 18]). Their work was inspired by prior contributions in the field, as indicated by [4, 7, 25]. Furthermore, Yampolsky introduced an innovative approach to deform the Sasaki metric on slashed and unit tangent bundles over Kählerian manifolds, incorporating the use of an almost complex structure J. This deformation resulted in the "Berger type deformed Sasaki metric," with a subsequent study of geodesics associated with this metric, as outlined in [19]. In the comprehensive work presented in [3], Altunbas, Simsek and Gezer extended the exploration of the Berger type deformed Sasaki metric to the tangent bundle over an anti-paraKähler manifold. They not only calculated all Riemannian curvature tensors for this metric but also provided significant geometric insights. Additionally, they introduced almost anti-paraHermitian structures on the tangent bundle and investigated the conditions under which these structures can be classified as anti-paraKähler or quasi-anti-paraKähler. It is important to note that the deformations of the Sasaki metric on the tangent bundle or cotangent bundle are not limited to the ones discussed here. This is evident from references such as [8, 9, 10, 13, 21, 22, 23, 24, 25], where researchers have likely explored additional deformation approaches. The field of differential geometry in this context has seen numerous contributions and remains an active and evolving area of research.

This research is primarily focused on a comprehensive exploration of the geometry associated with the Berger-type Cheeger-Gromoll metric, specifically applied to the tangent bundle TM over an anti-paraKähler manifold (M, φ, g). Our study unfolds in several key stages: We begin by thoroughly investigating the properties of the Levi-Civita connection corresponding to this metric. This investigation is elaborated in Theorem 3.1. We proceed to derive and present all relevant formulas for the Riemannian curvature tensors associated with this metric. This detailed analysis is encapsulated in Theorem 4.1 and Theorem 4.2. In addition to the Riemannian curvature tensors, we also establish formulas for the Ricci curvature, a fundamental curvature measure in Riemannian geometry. This aspect of our research is delineated in Theorem 4.3. We delve into characterizing the sectional curvature, offering insights into this curvature quantity in the context of the Berger-type Cheeger-Gromoll metric. The specifics of our findings are detailed in Theorem 4.4. Another crucial curvature measure, the scalar curvature, is comprehensively examined, and we provide explicit formulas for it. This component of our research is presented in Theorem 4.5. In the final section of our research, we extend our study to the geometry of the φ-unit tangent bundle. Within this framework, we present formulas for the Levi-Civita connection, as outlined in Theorem 5.1. Moreover, we provide a thorough exploration of the Riemannian curvature tensors pertinent to this specific context, which is detailed in Theorem 5.2. In summary, this research delves comprehensively into the geometry of the Berger-type Cheeger-Gromoll metric on the tangent bundle TM over an anti-paraKähler manifold. It encompasses a thorough examination of the Levi-Civita connection, Riemannian curvature tensor, Ricci curvature, sectional curvature, scalar curvature, as well as an exploration of the φ-unit tangent bundle with corresponding metric properties, thereby contributing to a deeper understanding of these mathematical structures.

2. Preliminaries

Consider TM as the tangent bundle over an m-dimensional Riemannian manifold (M, g), with the natural projection $\pi : TM \to M$. When we have a local chart $(u, x^i)_{i=\overline{1,m}}$ for M, it induces a local chart $(\pi^{-1}(u), x^i, u^i)_{i=\overline{1,m}}$ for TM. Let Γ_{ij}^k represent the Christoffel symbols of g, and let ∇ denote the Levi-Civita connection of g.

The Levi-Civita connection ∇ establishes a direct sum decomposition of the tangent bundle at any point $(x, u) \in TM$ into a vertical subspace:

$$V_{(x,u)}TM = Ker(d\pi_{(x,u)}) = \{\xi^i \frac{\partial}{\partial u^i}|_{(x,u)},\, \xi^i \in \mathbb{R}\},$$

and a horizontal subspace:

$$H_{(x,u)}TM = \{\xi^i \frac{\partial}{\partial x^i}|_{(x,u)} - \xi^i u^j \Gamma_{ij}^k \frac{\partial}{\partial u^k}|_{(x,u)},\, \xi^i \in \mathbb{R}\}.$$

Note that the mapping $\xi \to {}^H\xi = \xi^i \frac{\partial}{\partial x^i}|_{(x,u)} - \xi^i u^j \Gamma_{ij}^k \frac{\partial}{\partial u^k}|_{(x,u)}$ establishes an isomorphism between the vector spaces T_xM and $H_{(x,u)}TM$. Similarly, the mapping $\xi \to {}^V\xi = \xi^i \frac{\partial}{\partial u^i}|_{(x,u)}$ forms an isomorphism between the vector spaces T_xM and $V_{(x,u)}TM$. It is evident that any tangent vector $Z \in T_{(x,u)}TM$ can be expressed in the form $Z = {}^HX + {}^VY$, where X and Y are uniquely determined vectors belonging to T_xM.

Consider a local vector field $X = X^i \frac{\partial}{\partial x^i}$ on M. We define the vertical and horizontal lifts of X as follows:

$$\begin{aligned} {}^VX &= X^i \frac{\partial}{\partial u^i}, \\ {}^HX &= X^i \left(\frac{\partial}{\partial x^i} - u^j \Gamma_{ij}^k \frac{\partial}{\partial u^k}\right). \end{aligned} \tag{2.1}$$

It is worth noting that ${}^{H}(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial x^i} - u^j \Gamma_{ij}^k \frac{\partial}{\partial u^k}$ and ${}^{V}(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial u^i}$. Therefore, the set $({}^{H}(\frac{\partial}{\partial x^i}), {}^{V}(\frac{\partial}{\partial x^i}))$ for $i = 1$ to m forms a local adapted frame on TTM.

In particular, we have the vertical spray ${}^{V}u$ and the horizontal spray ${}^{H}u$ on TM defined by

$$ {}^{V}u = u^i \frac{\partial}{\partial u^i}, \quad {}^{H}u = u^i (\frac{\partial}{\partial x^i} - u^j \Gamma_{ij}^k \frac{\partial}{\partial u^k}). $$

${}^{V}u$ is also called the canonical or Liouville vector field on TM.

The bracket operation of vertical and horizontal vector fields is given by the formulas [6, 20]:

$$ \left\{ \begin{array}{l} \left[{}^{H}X, {}^{H}Y\right] = {}^{H}[X,Y] - {}^{V}(R(X,Y)u), \\ \left[{}^{H}X, {}^{V}Y\right] = {}^{V}(\nabla_X Y), \\ \left[{}^{V}X, {}^{V}Y\right] = 0 \end{array} \right. \tag{2.2} $$

for all vector fields X and Y on M.

3. The Berger-type Cheeger-Gromoll metric

An almost product structure φ on an m-dimensional manifold M is a $(1,1)$-tensor field on M satisfying $\varphi^2 = id_M$, where id_M represents the identity tensor field of type $(1,1)$ on M. Importantly, φ must not be equal to $\pm id_M$. The pair (M, φ) is then referred to as an almost product manifold. An almost para-complex manifold is essentially an almost product manifold (M, φ) for which the two eigenbundles TM^+ and TM^-, associated with the eigenvalues $+1$ and -1 of φ, respectively, have the same rank. It is essential to note that the dimension of an almost para-complex manifold is always even.

The integrability of an almost para-complex structure φ is determined by the vanishing of the Nijenhuis tensor:

$$ N_\varphi(X,Y) = [\varphi X, \varphi Y] - \varphi[X, \varphi Y] - \varphi[\varphi X, Y] + [X,Y], $$

which must vanish identically for all vector fields X and Y on M. Furthermore, an almost para-complex structure φ is integrable if and only if we can introduce a torsion-free linear connection ∇ such that $\nabla \varphi = 0$ [16].

A (pseudo-)Riemannian metric g is considered an anti-paraHermitian metric if it satisfies the condition:

$$ g(\varphi X, \varphi Y) = g(X,Y), $$

or equivalently (referred to as the purity condition or B-metric):

$$ g(\varphi X, Y) = g(X, \varphi Y) $$

for all vector fields X and Y on M [16].

When (M, φ) is an almost para-complex manifold equipped with an anti- para-Hermitian metric g, the triple (M, φ, g) is referred to as an almost anti-paraHermitian manifold or an almost B-manifold. Furthermore, (M, φ, g) is considered an anti-paraKähler manifold or B-manifold if φ is parallel with respect to the Levi-Civita connection ∇ of g, i.e., $\nabla\varphi = 0$ [16].

It is well-established that in an anti-paraKähler manifold (M, φ, g), the Riemannian curvature tensor has a specific property, where:

$$\begin{cases} R(\varphi Y, Z) & = R(Y, \varphi Z) = R(Y, Z)\varphi = \varphi R(Y, Z), \\ R(\varphi Y, \varphi Z) & = R(Y, Z), \end{cases}$$

for all vector fields Y and Z on M [16].

Definition 3.1. Consider a $2m$-dimensional almost anti-paraHermitian manifold (M, φ, g), with its tangent bundle denoted as TM. A Berger-type Cheeger-Gromoll metric on TM is defined as follows: For all vector fields X and Y on M

$$\begin{aligned} \tilde{g}({}^{H}X, {}^{H}Y) &= g(X, Y), \\ \tilde{g}({}^{V}X, {}^{H}Y) &= \tilde{g}({}^{H}X, {}^{V}Y) = 0, \\ \tilde{g}({}^{V}X, {}^{V}Y) &= \frac{1}{\alpha}(g(X, Y) + \delta^2 g(X, \varphi u) g(Y, \varphi u)), \end{aligned}$$

where $\alpha = 1 + \delta^2 g(u, u) = 1 + \delta^2 |u|^2$ and $|.|$ represents the norm with respect to the metric g.

Lemma 3.1. *[1] Consider a Riemannian manifold (M, g) with its tangent bundle denoted as TM. Additionally, let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function. In this context, we can establish the following relationships:*

$$\begin{array}{ll} 1) & {}^{H}X(f(r^2)) = 0, \\ 2) & {}^{V}X(f(r^2)) = 2f'(r^2) g(X, u), \\ 3) & {}^{H}X g(Y, u) = g(\nabla_X Y, u), \\ 4) & {}^{V}X g(Y, u) = g(X, Y) \end{array}$$

for any vector fields X, Y on M, where $r^2 = g(u, u)$.

Lemma 3.2. *Consider an anti-paraKähler manifold (M, φ, g) with its tangent bundle denoted as TM. Then we have the following:*

$$\begin{array}{ll} 1) & {}^{H}X(g(u, \varphi u)) = 0, \\ 2) & {}^{V}X(g(u, \varphi u)) = 2g(X, \varphi u), \\ 3) & {}^{H}X(g(Y, \varphi u)) = g(\nabla_X Y, \varphi u), \\ 4) & {}^{V}X(g(Y, \varphi u)) = g(X, \varphi Y), \\ 5) & {}^{H}(\varphi u)(g(Y, \varphi u)) = g(\nabla_{\varphi u} Y, \varphi u), \\ 6) & {}^{V}(\varphi u)(g(Y, \varphi u)) = g(Y, u) \end{array}$$

for any vector fields X, Y on M.

Proof. From (2.1) we obtain

$$\begin{aligned}
1)\ {}^{H}X(g(u,\varphi u)) &= X^i\frac{\partial}{\partial x^i}(g(u,\varphi u)) - u^s\Gamma^i_{sk}X^k\frac{\partial}{\partial u^i}(g(u,\varphi u))\\
&= X^i\frac{\partial}{\partial x^i}(g_{lj}u^l\varphi^j_t u^t) - u^s\Gamma^i_{sk}X^k\frac{\partial}{\partial u^i}(g_{lj}u^l\varphi^j_t u^t)\\
&= Xg(u,\varphi u) - g_{ij}u^s\Gamma^i_{sk}X^k\varphi^j_t u^t - g_{lj}u^l\varphi^j_i u^s\Gamma^i_{sk}X^k\\
&= Xg(u,\varphi u) - g(\nabla_X u,\varphi u) - g(u,\varphi\nabla_X u)\\
&= 0.\\
2)\ {}^{V}X(g(u,\varphi u)) &= X^i\frac{\partial}{\partial u^i}(g(u,\varphi u))\\
&= X^i\frac{\partial}{\partial u^i}(g_{lj}u^l\varphi^j_t u^t)\\
&= X^i\big(g_{ij}\varphi^j_t u^t + g_{lj}u^l\varphi^j_i\big)\\
&= g(X,\varphi u) + g(u,\varphi X)\\
&= 2g(X,\varphi u).
\end{aligned}$$

$$\begin{aligned}
3)\ {}^{H}X(g(Y,\varphi u)) &= X^i\frac{\partial}{\partial x^i}\big(g_{lj}Y^l\varphi^j_t u^t\big) - u^s\Gamma^i_{sk}X^k\frac{\partial}{\partial u^i}\big(g_{lj}Y^l\varphi^j_t u^t\big)\\
&= X(g(Y,\varphi u)) - u^s\Gamma^i_{sk}X^k g_{lj}Y^l\varphi^j_t\delta^t_i\\
&= g\big(\nabla_X Y,\varphi u\big) + g\big(Y,\nabla_X(\varphi u)\big) - g_{lj}Y^l\varphi^j_i u^s\Gamma^i_{sk}X^k\\
&= g\big(\nabla_X Y,\varphi u\big) + g\big(Y,\nabla_X(\varphi u)\big) - g\big(Y,\varphi(\nabla_X u)\big)\\
&= g\big(\nabla_X Y,\varphi u\big).
\end{aligned}$$

The remaining formulas are derived through analogous calculations. □

Lemma 3.3. *Consider an anti-paraKähler manifold (M,φ,g) and its tangent bundle $(TM,\tilde{g})$ equipped with the Berger-type Cheeger-Gromoll metric. In this context, the following relationships hold:*

$$\begin{aligned}
1)\ {}^{H}X\big(\tilde{g}({}^{H}Y,{}^{H}Z)\big) &= X(g(Y,Z)),\\
2)\ {}^{V}X\big(\tilde{g}({}^{H}Y,{}^{H}Z)\big) &= 0,\\
3)\ {}^{H}X\big(\tilde{g}({}^{V}Y,{}^{V}Z)\big) &= \tilde{g}({}^{V}(\nabla_X Y),{}^{V}Z) + \tilde{g}({}^{V}Y,{}^{V}(\nabla_X Z)),\\
4)\ {}^{V}X\big(\tilde{g}({}^{V}Y,{}^{V}Z)\big) &= \frac{\delta^2}{\alpha}\big(g(X,\varphi Y)g(Z,\varphi u) + g(Y,\varphi u)g(X,\varphi Z)\big)\\
&\quad -\frac{2\delta^2}{\alpha}g(X,u)\tilde{g}({}^{V}Y,{}^{V}Z)
\end{aligned}$$

for all vector fields X, Y and Z on M.

Proof. The results can be directly deduced from Definition 3.1, Lemma 3.1, and Lemma 3.2. □

Let us derive the Levi-Civita connection $\widetilde{\nabla}$ for the tangent bundle TM endowed with the Berger-type Cheeger-Gromoll metric $\tilde{g}$. This connection is defined by the Koszul formula, which expresses the metric compatibility of the connection:

$$(3.1)\qquad \begin{aligned} 2\tilde{g}(\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y},\widetilde{Z}) &= \widetilde{X}\big(\tilde{g}(\widetilde{Y},\widetilde{Z})\big)+\widetilde{Y}\big(\tilde{g}(\widetilde{Z},\widetilde{X})\big)\\ &\quad -\widetilde{Z}\big(\tilde{g}(\widetilde{X},\widetilde{Y})\big)+\tilde{g}(\widetilde{Z},[\widetilde{X},\widetilde{Y}])\\ &\quad +\tilde{g}(\widetilde{Y},[\widetilde{Z},\widetilde{X}])-\tilde{g}(\widetilde{X},[\widetilde{Y},\widetilde{Z}]) \end{aligned}$$

for all vector fields $\widetilde{X},\widetilde{Y}$ and $\widetilde{Z}$ on TM.

Theorem 3.1. *Consider an anti-paraKähler manifold (M,φ,g) along with its tangent bundle (TM,g) equipped with the Berger-type Cheeger-Gromoll metric. In this context, we can establish the following relationships:*

$$\begin{aligned} 1.\ \widetilde{\nabla}_{^HX}{}^HY &= {}^H(\nabla_XY)-\frac{1}{2}{}^V(R(X,Y)u),\\ 2.\ \widetilde{\nabla}_{^HX}{}^VY &= \frac{1}{2\alpha}{}^H\big(R(u,Y)X\big)+{}^V(\nabla_XY),\\ 3.\ \widetilde{\nabla}_{^VX}{}^HY &= \frac{1}{2\alpha}{}^H\big(R(u,X)Y\big),\\ 4.\ \widetilde{\nabla}_{^VX}{}^VY &= -\frac{\delta^2}{\alpha}(g(X,u)^VY+g(Y,u)^VX)+\delta^2\tilde{g}(X^V,Y^V)u^V\\ &+\frac{\delta^2}{\alpha}(g(X,\varphi Y)-\delta^2g(u,\varphi u)\tilde{g}(X^V,Y^V))^V(\varphi u) \end{aligned}$$

for all vector fields X,Y on M, where ∇ denotes the Levi-Civita connection and R represents its Riemannian curvature tensor of (M,φ,g).

Proof. In the proof, we will make use of (2.2), Koszul formula (3.1) and Lemma 3.3. 1. By performing direct calculations, we obtain:

$$\begin{aligned} 2\tilde{g}(\widetilde{\nabla}_{^HX}{}^HY,{}^HZ) &= {}^HX\big(\tilde{g}({}^HY,{}^HZ)\big)+{}^HY\big(\tilde{g}({}^HZ,{}^HX)\big)\\ &\quad -{}^HZ\big(\tilde{g}({}^HX,{}^HY)\big)+\tilde{g}({}^HZ,[{}^HX,{}^HY])\\ &\quad +\tilde{g}({}^HY,[{}^HZ,{}^HX])-\tilde{g}({}^HX,[{}^HY,{}^HZ])\\ &= Xg(Y,Z)+Yg(Z,X)-Zg(X,Y)+g(Z,[X,Y])\\ &\quad +g(Y,[Z,X])-g(X,[Y,Z])\\ &= 2g(\nabla_XY,Z)\\ &= 2\tilde{g}({}^H(\nabla_XY),{}^HZ) \end{aligned}$$

and

$$\begin{aligned} 2\tilde{g}(\widetilde{\nabla}_{^HX}{}^HY,{}^VZ) &= {}^HX\big(\tilde{g}({}^HY,{}^VZ)\big)+{}^HY\big(\tilde{g}({}^VZ,{}^HX)\big)-{}^VZ\big(\tilde{g}({}^HX,{}^HY)\big)\\ &\quad +\tilde{g}({}^VZ,[{}^HX,{}^HY])+\tilde{g}({}^HY,[{}^VZ,{}^HX])-\tilde{g}({}^HX,[{}^HY,{}^VZ])\\ &= \tilde{g}({}^VZ,[{}^HX,{}^HY])\\ &= -\tilde{g}({}^V(R(X,Y)u),{}^VZ), \end{aligned}$$

from which we find

$$\widetilde{\nabla}_{^HX}{}^HY = {}^H(\nabla_X Y) - \frac{1}{2}{}^V(R(X,Y)u).$$

2. Similar calculations as those provided above lead to:

$$\begin{aligned}
2\tilde{g}(\widetilde{\nabla}_{^HX}{}^VY, {}^HZ) &= {}^HX\big(\tilde{g}({}^VY, {}^HZ)\big) + {}^VY\big(\tilde{g}({}^HZ, {}^HX)\big) \\
&\quad -{}^HZ\big(\tilde{g}({}^HX, {}^VY)\big) + \tilde{g}({}^HZ, [{}^HX, {}^VY]) \\
&\quad +\tilde{g}({}^VY, [{}^HZ, {}^HX]) - \tilde{g}({}^HX, [{}^VY, {}^HZ]) \\
&= \tilde{g}({}^VY, [{}^HZ, {}^HX]) \\
&= -\tilde{g}({}^VY, {}^V(R(Z,X)u)) \\
&= -\frac{1}{\alpha}(g(Y, R(Z,X)u) + \delta^2 g(Y, \varphi u) g(R(Z,X)u, \varphi u)).
\end{aligned}$$

As the Riemannian curvature tensor field is pure with respect to φ, we can conclude that:

$$g(R(Z,X)u, \varphi u) = g(R(\varphi Z, X)u, u) = 0$$

and

$$-g(Y, R(Z,X)u) = g(R(u,Y)X, Z) = \tilde{g}({}^H(R(u,Y)X), {}^HZ),$$

then

$$2\tilde{g}(\widetilde{\nabla}_{^HX}{}^VY, {}^HZ) = \frac{1}{\alpha}\tilde{g}({}^H(R(u,Y)X), {}^HZ).$$

Also, it follows that

$$\begin{aligned}
2\tilde{g}(\widetilde{\nabla}_{^HX}{}^VY, {}^VZ) &= {}^HX\big(\tilde{g}({}^VY, {}^VZ)\big) + {}^VY\big(\tilde{g}({}^VZ, {}^HX)\big) \\
&\quad -{}^VZ\big(\tilde{g}({}^HX, {}^VY)\big) + \tilde{g}({}^VZ, [{}^HX, {}^VY]) \\
&\quad +\tilde{g}({}^VY, [{}^VZ, {}^HX]) - \tilde{g}({}^HX, [{}^VY, {}^VZ]) \\
&= {}^HX\tilde{g}({}^VY, {}^VZ) + \tilde{g}({}^VZ, [{}^HX, {}^VY]) + \tilde{g}({}^VY, [{}^VZ, {}^HX]) \\
&= \tilde{g}({}^V(\nabla_X Y), {}^VZ) + \tilde{g}({}^VY, {}^V(\nabla_X Z)) \\
&\quad +\tilde{g}({}^VZ, {}^V(\nabla_X Y)) - \tilde{g}({}^VY, {}^V(\nabla_X Z)) \\
&= 2\tilde{g}({}^V(\nabla_X Y), {}^VZ).
\end{aligned}$$

So, we see that

$$\widetilde{\nabla}_{^HX}{}^VY = \frac{1}{2\alpha}{}^H\big(R(u,Y)X\big) + {}^V(\nabla_X Y).$$

The remaining formulas can be derived through similar calculations. □

In the following, we will provide some lemmas without proofs since their proofs can be easily obtained through standard calculations. Additionally, these lemmas are required for proving certain results in the future.

Lemma 3.4. *In the context where we have an anti-paraKähler manifold* (M, φ, g) *and its tangent bundle* $(TM, \tilde{g})$ *endowed with the Berger-type Cheeger-Gromoll metric, the following results hold:*

$$
\begin{aligned}
1.\ \widetilde{\nabla}_{{}^HX}{}^Vu &= \widetilde{\nabla}_{{}^Vu}{}^HX = 0,\\
2.\ \widetilde{\nabla}_{{}^VX}{}^Vu &= \frac{1}{\alpha}{}^VX + \frac{\delta^4}{\alpha}g(u,\varphi u)g(X,\varphi u){}^Vu + \frac{\delta^2}{\alpha}g(X,\varphi u){}^V(\varphi u)\\
&\quad -\frac{\delta^4}{\alpha}g(u,\varphi u)\tilde{g}({}^VX,{}^Vu){}^V(\varphi u),\\
3.\ \widetilde{\nabla}_{{}^Vu}{}^VX &= \frac{1-\alpha}{\alpha}{}^VX + \frac{\delta^4}{\alpha}g(u,\varphi u)g(X,\varphi u){}^Vu + \frac{\delta^2}{\alpha}g(X,\varphi u){}^V(\varphi u)\\
&\quad -\frac{\delta^4}{\alpha}g(u,\varphi u)\tilde{g}({}^VX,{}^Vu){}^V(\varphi u),\\
4.\ \widetilde{\nabla}_{{}^Vu}{}^Vu &= \frac{1}{\alpha}(1+\delta^4 g(u,\varphi u)^2){}^Vu + \frac{\delta^2}{\alpha}g(u,\varphi u)(1-\delta^2\tilde{g}({}^Vu,{}^Vu)){}^V(\varphi u)
\end{aligned}
$$

for any vector field X *on* M*.*

Lemma 3.5. *In the context where we have an anti-paraKähler manifold* (M, φ, g) *and its tangent bundle* $(TM, \tilde{g})$ *endowed with the Berger-type Cheeger-Gromoll metric, we have the following results:*

$$
\begin{aligned}
1.\ \widetilde{\nabla}_{{}^HX}{}^V(\varphi u) &= \widetilde{\nabla}_{{}^V(\varphi u)}{}^HX = 0,\\
2.\ \widetilde{\nabla}_{{}^VX}{}^V(\varphi u) &= {}^V(\varphi X) - \frac{\delta^2}{\alpha}g(u,\varphi u){}^VX + \delta^2 g(X,\varphi u){}^Vu\\
&\quad -\frac{\delta^4}{\alpha}g(u,\varphi u)g(X,\varphi u){}^V(\varphi u),\\
3.\ \widetilde{\nabla}_{{}^V(\varphi u)}{}^VX &= -\frac{\delta^2}{\alpha}g(u,\varphi u){}^VX + \delta^2 g(X,\varphi u){}^Vu - \frac{\delta^4}{\alpha}g(u,\varphi u)g(X,\varphi u){}^V(\varphi u),\\
4.\ \widetilde{\nabla}_{{}^V(\varphi u)}{}^V(\varphi u) &= \alpha{}^Vu - \delta^2 g(u,\varphi u){}^V(\varphi u)
\end{aligned}
$$

for any vector field X *on* M*.*

Furthermore, in order to calculate the curvature tensors of the tangent bundle TM with the Berger-type Cheeger-Gromoll metric $\tilde{g}$, we will provide the following definitions and propositions. We would like to note that the proof of the proposition can be readily obtained through standard calculations.

Definition 3.2. Consider a Riemannian manifold (M, g) and a smooth bundle endomorphism $F : TM \to TM$. We can respectively define the vertical and horizontal vector fields ${}^VF$ and ${}^HF$ on TM as follows:

$$
\begin{array}{rccc}
{}^VF: & TM & \to & TTM\\
 & (x,u) & \mapsto & {}^V(Fu)_x,
\end{array}
\quad ,\quad
\begin{array}{rccc}
{}^HF: & TM & \to & TTM\\
 & (x,u) & \mapsto & {}^H(Fu)_x.
\end{array}
$$

Locally, we have

$$\begin{aligned} {}^{V}(Fu) &= u^{j}\,{}^{V}(F(\frac{\partial}{\partial x^{j}})), \\ {}^{H}(Fu) &= u^{j}\,{}^{H}(F(\frac{\partial}{\partial x^{j}})). \end{aligned}$$

Proposition 3.1. *In the context of an anti-paraKähler manifold (M, φ, g) and its tangent bundle $(TM, \tilde{g})$ equipped with the Berger-type Cheeger-Gromoll metric, the following formulas hold:*

$$\begin{aligned} 1.\ \widetilde{\nabla}_{{}^{H}X}{}^{H}(Fu) &= {}^{H}((\nabla_X F)u) - \frac{1}{2}{}^{V}(R(X, Fu)u), \\ 2.\ \widetilde{\nabla}_{{}^{H}X}{}^{V}(Fu) &= \frac{1}{2\alpha}{}^{H}(R(u, Fu)X) + {}^{V}((\nabla_X F)u), \\ 3.\ \widetilde{\nabla}_{{}^{V}X}{}^{H}(Fu) &= {}^{H}(FX) + \frac{1}{2\alpha}{}^{H}(R(u, X)Fu), \\ 4.\ \widetilde{\nabla}_{{}^{V}X}{}^{V}(Fu) &= {}^{V}(FX) - \frac{\delta^2}{\alpha}(g(X, u){}^{V}(Fu) + g(Fu, u){}^{V}X) + \delta^2\tilde{g}({}^{V}X, {}^{V}(Fu)){}^{V}u \\ &\quad + \frac{\delta^2}{\alpha}(g(\varphi X, Fu) - \delta^2 g(u, \varphi u)\tilde{g}({}^{V}X, {}^{V}(Fu))){}^{V}(\varphi u) \end{aligned}$$

for any vector field X on M.

Proof. The results can be directly derived from Theorem 3.1. □

4. The Riemannian curvatures of the Berger-type Cheeger-Gromoll metric

We will perform the computation of the Riemannian curvature tensor $\widetilde{R}$ for TM using the Berger-type Cheeger-Gromoll metric $\tilde{g}$. The Riemannian curvature tensor is characterized by the formula:

$$\widetilde{R}(\widetilde{X}, \widetilde{Y})\widetilde{Z} = \widetilde{\nabla}_{\widetilde{X}}\widetilde{\nabla}_{\widetilde{Y}}\widetilde{Z} - \widetilde{\nabla}_{\widetilde{Y}}\widetilde{\nabla}_{\widetilde{X}}\widetilde{Z} - \widetilde{\nabla}_{[\widetilde{X}, \widetilde{Y}]}\widetilde{Z},$$

where $\widetilde{X}, \widetilde{Y}$ and $\widetilde{Z}$ are vector fields defined on TM.

Theorem 4.1. *In the context of an anti-paraKähler manifold (M, φ, g) and its tangent bundle $(TM, \tilde{g})$ equipped with the Berger-type Cheeger-Gromoll metric, the following formulas hold:*

(4.1)

$$\begin{aligned} \widetilde{R}({}^{H}X, {}^{H}Y){}^{H}Z &= {}^{H}(R(X, Y)Z) + \frac{1}{2\alpha}{}^{H}(R(u, R(X, Y)u)Z) \\ &\quad + \frac{1}{4\alpha}{}^{H}(R(u, R(X, Z)u)Y) - \frac{1}{4\alpha}{}^{H}(R(u, R(Y, Z)u)X) \\ &\quad + \frac{1}{2}{}^{V}((\nabla_Z R)(X, Y)u), \end{aligned}$$

(4.2)
$$\begin{aligned}\widetilde{R}({}^{H}X,{}^{V}Y){}^{H}Z &= \frac{1}{2\alpha}{}^{H}((\nabla_X R)(u,Y)Z) - \frac{1}{4\alpha}{}^{V}(R(X,R(u,Y)Z)u)\\ &+\frac{1}{2}{}^{V}(R(X,Z)Y) - \frac{\delta^2}{2\alpha}g(Y,u){}^{V}(R(X,Z)u)\\ &+\frac{\delta^2}{2\alpha}g(R(X,Z)u,Y){}^{V}u + \frac{\delta^2}{2\alpha}g(R(X,Z)u,\varphi Y){}^{V}(\varphi u)\\ &-\frac{\delta^4}{2\alpha^2}g(u,\varphi u)g(R(X,Z)u,Y){}^{V}(\varphi u),\end{aligned}$$

(4.3)
$$\begin{aligned}\widetilde{R}({}^{H}X,{}^{H}Y){}^{V}Z &= \frac{1}{2\alpha}{}^{H}((\nabla_X R)(u,Z)Y) - \frac{1}{2\alpha}{}^{H}((\nabla_Y R)(u,Z)X)\\ &+{}^{V}(R(X,Y)Z) - \frac{1}{4\alpha}{}^{V}(R(X,R(u,Z)Y)u)\\ &+\frac{1}{4\alpha}{}^{V}(R(Y,R(u,Z)X)u) - \frac{\delta^2}{\alpha}g(Z,u){}^{V}(R(X,Y)u)u\\ &+\frac{\delta^2}{\alpha}\ {}^{V}g(R(X,Y)u,Z) + (\frac{\delta^2}{\alpha}g(R(X,Y)u,\varphi Z)\\ &-\frac{\delta^4}{\alpha^2}g(u,\varphi u)g(R(X,Y)u,Z)){}^{V}(\varphi u),\end{aligned}$$

(4.4)
$$\begin{aligned}\widetilde{R}({}^{H}X,{}^{V}Y){}^{V}Z &= -\frac{1}{2\alpha}{}^{H}(R(Y,Z)X) - \frac{1}{4\alpha^2}{}^{H}(R(u,Y)(R(u,Z)X)\\ &+\frac{\delta^2}{2\alpha^2}g(Y,u){}^{H}(R(u,Z)X) - \frac{\delta^2}{2\alpha^2}g(Z,u){}^{H}(R(u,Y)X),\end{aligned}$$

(4.5)
$$\begin{aligned}\widetilde{R}({}^{V}X,{}^{V}Y){}^{H}Z &= \frac{1}{4\alpha^2}{}^{H}(R(u,X)R(u,Y)Z)\\ &-\frac{1}{4\alpha^2}{}^{H}(R(u,Y)R(u,X)Z) + \frac{\delta^2}{\alpha^2}g(Y,u){}^{H}(R(u,X)Z)\\ &-\frac{\delta^2}{\alpha^2}g(X,u){}^{H}(R(u,Y)Z) + \frac{1}{\alpha}{}^{H}(R(X,Y)Z),\end{aligned}$$

(4.6)
$$\begin{aligned}\widetilde{R}({}^{V}X,{}^{V}Y){}^{V}Z &= A(Y,Z){}^{V}X - A(X,Z){}^{V}Y + B(Y,Z){}^{V}(\varphi X)\\ &-B(X,Z){}^{V}(\varphi Y) + C(X,Y,Z){}^{V}u + D(X,Y,Z){}^{V}(\varphi u),\end{aligned}$$

for all vector fields X, Y *and* Z *on* M*, where*

$$A(*,Z) = \left(\frac{\delta^2}{\alpha}+\frac{\delta^6}{\alpha^2}g(u,\varphi u)^2\right)\tilde{g}({}^V*,{}^VZ)+\frac{\delta^2}{\alpha}g(*,Z) - \frac{\delta^4}{\alpha^2}g(*,u)g(Z,u)-\frac{\delta^4}{\alpha^2}g(u,\varphi u)g(*,\varphi Z), \tag{4.7}$$

$$B(*,Z)=\frac{\delta^2}{\alpha}g(*,\varphi Z)-\frac{\delta^4}{\alpha}g(u,\varphi u)\tilde{g}({}^V*,{}^VZ), \tag{4.8}$$

$$C(X,Y,Z)=\frac{\delta^4}{\alpha}\big(g(Y,u)\tilde{g}({}^VX,{}^VZ)-g(X,u)\tilde{g}({}^VY,{}^VZ)\big), \tag{4.9}$$

$$\begin{aligned} D(X,Y,Z) &= \frac{\delta^4}{\alpha^2}\big(g(Y,u)g(X,\varphi Z)-g(X,u)g(Y,\varphi Z)\big) \\ &+\frac{\delta^4}{\alpha^2}\big(g(Y,\varphi u)g(X,Z)-g(X,\varphi u)g(Y,Z)\big) \\ &-\frac{2\delta^2}{\alpha}g(u,\varphi u)\,C(X,Y,Z). \end{aligned} \tag{4.10}$$

Proof. In the proof, we will utilize the following mathematical results: Theorem 3.1, Lemma 3.4, Lemma 3.5 and Theorem 3.1.

1) Consider the bundle endomorphism $F : TM \to TM$ defined as follows: for any vector u in TM, we have $Fu = R(Y,Z)u$. Let us perform explicit calculations:

$$\begin{aligned} \widetilde{\nabla}_{{}^HX}\widetilde{\nabla}_{{}^HY}{}^HZ &= \widetilde{\nabla}_{{}^HX}\big({}^H(\nabla_Y Z)-\frac{1}{2}{}^V(Fu)\big) \\ &= {}^H(\nabla_X\nabla_Y Z)-\frac{1}{2}{}^V(R(X,\nabla_Y Z)u)-\frac{1}{2}{}^V(\nabla_X(R(Y,Z)u)) \\ &+\frac{1}{2}{}^V(R(Y,Z)(\nabla_X u))-\frac{1}{4\alpha}{}^H(R(u,R(Y,Z)u)X). \end{aligned} \tag{4.11}$$

By permuting the symbols X and Y in the formula (4.11), we obtain the following expression:

$$\begin{aligned} \widetilde{\nabla}_{{}^HY}\widetilde{\nabla}_{{}^HX}{}^HZ &= {}^H(\nabla_Y\nabla_X Z)-\frac{1}{2}{}^V(R(Y,\nabla_X Z)u) \\ &-\frac{1}{2}{}^V(\nabla_Y(R(X,Z)u))+\frac{1}{2}{}^V(R(X,Z)(\nabla_Y u)) \\ &-\frac{1}{4\alpha}{}^H(R(u,R(X,Z)u)Y). \end{aligned} \tag{4.12}$$

Also, we find

$$\begin{aligned} \widetilde{\nabla}_{[{}^HX,{}^HY]}{}^HZ &= \widetilde{\nabla}_{{}^H[X,Y]}{}^HZ-\widetilde{\nabla}_{{}^V(R(X,Y)u)}{}^HZ \\ &= {}^H(\nabla_{[X,Y]}Z)-\frac{1}{2}{}^V(R([X,Y],Z)u) \\ &-\frac{1}{2\alpha}{}^H(R(u,R(X,Y)u)Z). \end{aligned} \tag{4.13}$$

From (4.11), (4.12) and (4.13) we find

$$\begin{aligned}\widetilde{R}({}^HX,{}^HY){}^HZ &= {}^H(R(X,Y)Z)+\frac{1}{2\alpha}{}^H(R(u,R(X,Y)u)Z)\\ &+\frac{1}{4\alpha}{}^H(R(u,R(X,Z)u)Y)-\frac{1}{4\alpha}{}^H(R(u,R(Y,Z)u)X)\\ &-\frac{1}{2}{}^V((\nabla_XR)(Y,Z)u)+\frac{1}{2}{}^V((\nabla_YR)(X,Z)u).\end{aligned}$$

Utilizing the second Bianchi identity:

$$(\nabla_XR)(Y,Z)+(\nabla_YR)(Z,X)+(\nabla_ZR)(X,Y)=0,$$

we can derive the formula (4.1).
2) Given the bundle endomorphism $F:TM\to TM$ defined as $Fu=R(u,Y)Z$, we can deduce the following result:

$$\begin{aligned}\widetilde{\nabla}_{{}^HX}\widetilde{\nabla}_{{}^VY}{}^HZ &= \widetilde{\nabla}_{{}^HX}\big(\frac{1}{2\alpha}{}^H(Fu)\big)\\ &= \frac{1}{2\alpha}{}^H(\nabla_X(R(u,Y)Z))-\frac{1}{2\alpha}{}^H(R(\nabla_Xu,Y)Z)\\ &-\frac{1}{4\alpha}{}^V(R(X,R(u,Y)Z)u).\end{aligned}$$

Consider the bundle endomorphism $F:TM\to TM$ given by $Fu=R(X,Z)u$. We obtain

$$\begin{aligned}\widetilde{\nabla}_{{}^VY}\widetilde{\nabla}_{{}^HX}{}^HZ &= \widetilde{\nabla}_{{}^VY}\big({}^H(\nabla_XZ)-\frac{1}{2}{}^V(Fu)\big)\\ &= \frac{1}{2\alpha}{}^H(R(u,Y)\nabla_XZ)-\frac{1}{2}{}^V(R(X,Z)Y)+\frac{\delta^2}{2\alpha}g(Y,u)^V(R(X,Z)u)\\ &-\frac{\delta^2}{2\alpha}g(Y,R(X,Z)u)^Vu-\frac{\delta^2}{2\alpha}g(\varphi Y,R(X,Z)u)^V(\varphi u)\\ &+\frac{\delta^4}{2\alpha^2}g(u,\varphi u)g(Y,R(X,Z)u)^V(\varphi u).\end{aligned}$$

Also,

$$\widetilde{\nabla}_{[{}^HX,{}^VY]}{}^HZ=\widetilde{\nabla}_{{}^V(\nabla_XY)}{}^HZ=\frac{1}{2\alpha}{}^H(R(u,\nabla_XY)Z),$$

which gives the formula (4.2).
3) By applying formula (4.2) and the first Bianchi identity, we can express the result as follows:

$$\begin{aligned}\widetilde{R}({}^HX,{}^VZ){}^HY &= \frac{1}{2\alpha}{}^H((\nabla_XR)(u,Z)Y)-\frac{1}{4\alpha}{}^V(R(X,R(u,Z)Y)u)\\ &+\frac{1}{2}{}^V(R(X,Y)Z)-\frac{\delta^2}{2\alpha}g(Z,u)^V(R(X,Y)u)\\ &+\frac{\delta^2}{2\alpha}g(R(X,Y)u,Z)^Vu+\frac{\delta^2}{2\alpha}g(R(X,Y)u,\varphi Z)^V(\varphi u)\\ &-\frac{\delta^4}{2\alpha^2}g(u,\varphi u)g(R(X,Y)u,Z)^V(\varphi u)\end{aligned}$$

and

$$\begin{aligned}\widetilde{R}({}^{H}Y,{}^{V}Z){}^{H}X &= \frac{1}{2\alpha}{}^{H}((\nabla_Y R)(u,Z)X)-\frac{1}{4\alpha}{}^{V}(R(Y,R(u,Z)X)u)\\ &+\frac{1}{2}{}^{V}(R(Y,X)Z)-\frac{\delta^2}{2\alpha}g(Z,u){}^{V}(R(Y,X)u)\\ &+\frac{\delta^2}{2\alpha}g(R(Y,X)u,Z){}^{V}u+\frac{\delta^2}{2\alpha}g(R(Y,X)u,\varphi Z){}^{V}(\varphi u)\\ &-\frac{\delta^4}{2\alpha^2}g(u,\varphi u)g(R(Y,X)u,Z){}^{V}(\varphi u),\end{aligned}$$

which gives the formula (4.3).

Similar calculations yield the other formulas, but to avoid redundancy, we will omit them in this presentation. □

Theorem 4.2. *Consider an anti-paraKähler manifold (M,φ,g) and its tangent bundle $(TM,\tilde{g})$ equipped with the Berger-type Cheeger-Gromoll metric. If the tangent bundle $(TM,\tilde{g})$ is flat, then it follows that (M,φ,g) must also be flat.*

Proof. We can readily observe from equation (4.1) that if we assume $\widetilde{R}=0$ and compute the Riemann curvature tensor for three horizontal vector fields at the point $(x,0)$, we obtain:

$$\widetilde{R}_{(x,0)}({}^{H}X,{}^{H}Y){}^{H}Z={}^{H}(R_x(X,Y)Z)=0.$$

□

Now, let $(x,u)\in TM$ with $u\neq 0$ and $\{E_i\}_{i=\overline{1,2m}}$ be a local orthonormal frame on (M,φ,g) at x, such that $E_1=\dfrac{\varphi u}{|\varphi u|}=\dfrac{\varphi u}{|u|}$. Then

$$\big\{F_i={}^{H}E_i,F_{2m+1}={}^{V}E_1,F_{2m+j}=\sqrt{\alpha}{}^{V}E_j\big\}_{i=\overline{1,2m},j=\overline{2,2m}} \tag{4.14}$$

denotes a local orthonormal frame on TM at (x,u), where $\alpha=1+\delta^2 g(u,u)$.

Theorem 4.3. *Consider an anti-paraKähler manifold (M,φ,g) and its tangent bundle $(TM,\tilde{g})$ equipped with the Berger-type Cheeger-Gromoll metric. If we denote the Ricci curvature of (M,φ,g) as Ric and the Ricci curvature of $(TM,\tilde{g})$ as $\widetilde{Ric}$, then we can state the relationship as follows:*

$$\widetilde{Ric}({}^{H}X,{}^{H}Y)=Ric(X,Y)-\frac{1}{2\alpha}\sum_{a=1}^{2m}g(R(E_a,X)u,R(E_a,Y)u), \tag{4.15}$$

$$\widetilde{Ric}({}^{H}X,{}^{V}Y)=\frac{1}{2\alpha}\sum_{a=1}^{2m}g((\nabla_{E_a}R)(u,Y)X,E_a), \tag{4.16}$$

$$\widetilde{Ric}(^VX, ^VY) = \frac{1}{4\alpha^2}\sum_{a=1}^{2m} g(R(u,X)E_a, R(u,Y)E_a) + (2m-1)A(X,Y) + \lambda\, B(X,Y) - B(\varphi X, Y) + C(u,X,Y) + D(\varphi u, X, Y), \tag{4.17}$$

where A, B, C, D are defined in Theorem 4.1 and $\lambda = \sum_{a=1}^{2m} g(E_a, \varphi E_a)$.

Proof. Utilizing the local orthonormal frame (4.14) on TM, we can express the following:

$$\begin{aligned}\widetilde{Ric}(^HX, ^HY) &= \sum_{a=1}^{2m} \tilde{g}(\widetilde{R}(^HE_a, {}^HX)^HY, {}^HE_a) + \tilde{g}(\widetilde{R}(^VE_1, {}^HX)^HY, {}^VE_1) \\ &\quad + \alpha\sum_{a=2}^{2m} \tilde{g}(\widetilde{R}(^VE_a, {}^HX)^HY, {}^VE_a).\end{aligned}$$

By considering equations (4.1) and (4.2), we can deduce the following result:

$$\begin{aligned}\widetilde{Ric}(^HX, ^HY) &= Ric(X,Y) - \frac{3}{4\alpha}\sum_{a=1}^{2m} g(R(E_a,X)u, R(E_a,Y)u) \\ &\quad + \frac{1}{4\alpha}\sum_{a=1}^{2m} g(R(u,E_a)X, R(u,E_a)Y).\end{aligned}$$

Using

$$\sum_{a=1}^{2m} g(R(E_a,X)u, R(E_a,Y)u) = \sum_{a=1}^{2m} g(R(u,E_a)X, R(u,E_a)Y),$$

we get

$$\begin{aligned}\widetilde{Ric}(^HX, ^HY) &= Ric(X,Y) - \frac{3}{4\alpha}\sum_{a=1}^{2m} g(R(E_a,X)u, R(E_a,Y)u) \\ &\quad + \frac{1}{4\alpha}\sum_{a=1}^{2m} g(R(u,E_a)X, R(u,E_a)Y).\end{aligned}$$

Similar calculations lead to the derivation of the other formulas. □

In the following, we denote $\widetilde{Q}(V,W)$ as the square of the area of the parallelogram with sides V and W, given by:

$$\widetilde{Q}(V,W) = \tilde{g}(V,V)\tilde{g}(W,W) - \tilde{g}(V,W)^2.$$

If we have a point $p \in TM$ and vectors V_p and W_p that are linearly independent at point p, then the sectional curvature $\widetilde{K}(V_p, W_p)$ of the plane spanned by V_p and W_p can be calculated as:

$$\widetilde{K}(V_p, W_p) = \frac{\tilde{g}(\widetilde{R}_p(V,W)W, V_p)}{\widetilde{Q}(V_p, W_p)},$$

Here, $\widetilde{K}(V_p, W_p)$ represents the sectional curvature for the plane defined by the vectors V_p and W_p.

Let $\widetilde{K}({}^HX, {}^HY)$, $\widetilde{K}({}^HX, {}^VY)$ and $\widetilde{K}({}^VX, {}^VY)$ denote the sectional curvature of the plane spanned by the sets: $\{{}^HX, {}^HY\}$, $\{{}^HX, {}^VY\}$ and $\{{}^VX, {}^VY\}$ on the tangent bundle $(TM, \tilde{g})$, where X and Y are vector fields on M.

Theorem 4.4. *In the context of an anti-paraKähler manifold (M, φ, g) and its tangent bundle $(TM, \tilde{g})$ equipped with the Berger-type Cheeger-Gromoll metric, the sectional curvature $\widetilde{K}$ satisfies the following equations:*

$$\begin{aligned}
1)\ \widetilde{K}({}^HX, {}^HY) &= K(X,Y) - \frac{3}{4\alpha(|X|^2|Y|^2 - g(X,Y)^2)}|R(X,Y)u|^2, \\
2)\ \widetilde{K}({}^HX, {}^VY) &= \frac{1}{4\alpha|X|^2\big(|Y|^2 + \delta^2 g(Y,\varphi u)^2\big)}|R(u,Y)X|^2, \\
3)\ \widetilde{K}({}^VX, {}^VY) &= \frac{1}{\widetilde{Q}({}^VX, {}^VY)}\big(\tilde{g}({}^VX, {}^VX)A(Y,Y) - \tilde{g}({}^VX, {}^VY)A(X,Y) \\
&\quad + \tilde{g}({}^VX, {}^V\varphi X)B(Y,Y) - \tilde{g}({}^VX, {}^V\varphi Y)B(X,Y) \\
&\quad + \tilde{g}({}^VX, {}^Vu)C(X,Y,Y) + g(X,\varphi u)D(X,Y,Y)\big),
\end{aligned}$$

where K denotes the sectional curvature of (M, φ, g).

Proof. $i)$ From the formula (4.1), we can deduce the following:

$$\begin{aligned}
\tilde{g}(\widetilde{R}({}^HX, {}^HY){}^HY, {}^HX) &= g(R(X,Y)Y,X) + \frac{1}{2\alpha}g(R(u,R(X,Y)u)Y,X) \\
&\quad + \frac{1}{4\alpha}g(R(u,R(X,Y)u)Y,X) \\
&= g(R(X,Y)Y,X) - \frac{3}{4\alpha}g(R(X,Y)u, R(X,Y)u) \\
&= g(R(X,Y)Y,X) - \frac{3}{4\alpha}|R(X,Y)u|^2.
\end{aligned}$$

$ii)$ From the formula (4.4), we have

$$\begin{aligned}
\tilde{g}(\widetilde{R}({}^HX, {}^VY){}^VY, {}^HX) &= -\frac{1}{4\alpha^2}g(R(u,Y)R(u,Y)X,X) \\
&= \frac{1}{4\alpha^2}g(R(u,Y)X, R(u,Y)X) \\
&= \frac{1}{4\alpha^2}|R(u,Y)X|^2.
\end{aligned}$$

iii) It follows immediately from the formula (4.6) that

$$\begin{aligned}\tilde{g}(\widetilde{R}({}^VX,{}^VY){}^VY,{}^VX) &= \tilde{g}({}^VX,{}^VX)A(Y,Y)-\tilde{g}({}^VX,{}^VY)A(X,Y)\\&\quad-\tilde{g}({}^VX,{}^V\varphi X)B(Y,Y)-\tilde{g}({}^VX,{}^V\varphi Y)B(X,Y)\\&\quad+\tilde{g}({}^VX,{}^Vu)C(X,Y,Y)+g(X,\varphi u)D(X,Y,Y).\end{aligned}$$

On the other hand, we have the followings:

$$\begin{aligned}i)\ \widetilde{Q}({}^HX,{}^HY) &= \tilde{g}({}^HX,{}^HX)\tilde{g}({}^HY,{}^HY)-\tilde{g}({}^HX,{}^HY)^2\\&= |X|^2|Y|^2-g(X,Y)^2,\end{aligned}$$

$$\begin{aligned}ii)\ \widetilde{Q}({}^HX,{}^VY) &= \tilde{g}({}^HX,{}^HX)\tilde{g}({}^VY,{}^VY)-\tilde{g}({}^HX,{}^VY)^2\\&= \frac{1}{\alpha}|X|^2\big(|Y|^2+\delta^2g(Y,\varphi u)^2\big),\end{aligned}$$

$$\begin{aligned}iii)\ \widetilde{Q}({}^VX,{}^VY) &= \tilde{g}({}^VX,{}^VX)\tilde{g}({}^VY,{}^VY)-\tilde{g}({}^VX,{}^VY)^2\\&= \frac{1}{\alpha^2}\big(|X|^2+\delta^2g(X,\varphi u)^2\big)+\big(|Y|^2+\delta^2g(Y,\varphi u)^2\big)\\&\quad-\frac{1}{\alpha^2}\big(g(X,Y)+\delta^2g(X,\varphi u)g(Y,\varphi u)\big)^2\\&= \frac{1}{\alpha^2}\big(|X|^2|Y|^2-g(X,Y)^2+\delta^2|X|^2g(Y,\varphi u)^2+\delta^2|Y|^2g(X,\varphi u)^2\\&\quad-2\delta^2g(X,Y)g(X,\varphi u)g(Y,\varphi u)\big).\end{aligned}$$

The division of $\tilde{g}(\widetilde{R}(X^i,Y^j)Y^j,X^i)$ by $\widetilde{Q}(X^i,Y^j)$ for $i,j\in\{H,V\}$ gives the result. □

Now, let us consider the scalar curvature $\widetilde{\sigma}$ of $(TM,\tilde{g})$. With standard calculations, we obtain the following result:

Theorem 4.5. *Consider an anti-paraKähler manifold (M,φ,g) and its tangent bundle $(TM,\tilde{g})$ equipped with the Berger-type Cheeger-Gromoll metric. If we denote the scalar curvature of (M,φ,g) as σ and the scalar curvature of $(TM,\tilde{g})$ as $\widetilde{\sigma}$, then we can state the relationship as follows:*

$$\begin{aligned}\widetilde{\sigma} &= \sigma-\frac{1}{4\alpha}\sum_{a,b=1}^{2m}|R(E_a,E_b)u|^2\\&\quad+\delta^2\lambda^2+\frac{\alpha-4m+1}{\alpha}\delta^4g(u,\varphi u)\lambda\\&\quad-\frac{\alpha^2-(4m^2-2m+3)\alpha+4m^2-2m}{\alpha^2(\alpha-1)}\delta^2g(u,\varphi u)^2\\&\quad-\frac{(2m-1)\alpha^2-(4m^2-4m)\alpha-4m^2+1}{\alpha}\delta^2,\end{aligned}$$

where $\lambda = \sum_{a=1}^{2m} g(E_a, \varphi E_a)$ *and* $\{E_a\}_{a=\overline{1,2m}}$ *is a local orthonormal frame on* M.

Proof. Let $(F_k)_{k=\overline{1,4m}}$ be a local orthonormal frame on $(TM, \tilde{g})$ defined by equation (4.14). By applying Theorem 4.3 and utilizing the definition of scalar curvature, we can derive the following:

$$\widetilde{\sigma} = \sum_{b=1}^{2m} \widetilde{Ric}(F_b, F_b) + \widetilde{Ric}(F_{2m+1}, F_{2m+1}) + \sum_{b=2}^{2m} \widetilde{Ric}(F_{2m+b}, F_{2m+b}).$$

Using (4.15) and (4.16), we get

$$\sum_{b=1}^{2m} \widetilde{Ric}(F_b, F_b) = \sigma - \frac{1}{2\alpha} \sum_{a,b=1}^{2m} |R(E_a, E_b)u|^2,$$

$$\begin{aligned} \widetilde{Ric}(F_{2m+1}, F_{2m+1}) &= (2m-1)A(E_1, E_1) + W.B(E_1, E_1) \\ &\quad -B(E_1, \varphi E_1) + C(u, E_1, E_1) \end{aligned}$$

and

$$\begin{aligned} \sum_{b=2}^{2m} \widetilde{Ric}(F_{m+b}, F_{m+b}) &= \frac{1}{4\alpha} \sum_{a,b=1}^{2m} |R(u, E_b)E_a|^2 + (2m-1)\alpha \sum_{b=2}^{2m} A(E_b, E_b) \\ &\quad +\alpha\lambda \sum_{b=2}^{2m} B(E_b, E_b) - \alpha \sum_{b=2}^{2m} B(E_b, \varphi E_b) \\ &\quad +\alpha \sum_{b=2}^{2m} C(u, E_b, E_b) - \alpha \sum_{b=2}^{2m} D(\varphi u, E_b, E_b). \end{aligned}$$

So, the expression for $\widetilde{\sigma}$ takes the following form

$$\begin{aligned} \widetilde{\sigma} &= \sigma - \frac{1}{4\alpha} \sum_{a,b=1}^{2m} |R(E_a, E_b)\tilde{p}|^2 + (2m-1)\alpha \sum_{b=2}^{2m} A(E_b, E_b)) \\ &\quad +\alpha\lambda \sum_{b=2}^{2m} B(E_b, E_b) - \alpha \sum_{b=2}^{2m} B(E_b, \varphi E_b) + \alpha \sum_{b=2}^{2m} C(u, E_b, E_b) \\ &\quad -\alpha \sum_{b=2}^{2m} D(\varphi u, E_b, E_b) + (2m-1)A(E_1, E_1) + \lambda B(E_1, E_1) \\ &\quad -B(E_1, \varphi E_1) + C(u, E_1, E_1). \end{aligned}$$

To simplify this last expression, we can make use of equations (4.7), (4.8), (4.9) and

(4.10). This yields the following result:

$$\begin{aligned}
\widetilde{\sigma} &= \sigma - \frac{1}{4\alpha}\sum_{a,b=1}^{2m}|R(E_a,E_b)u|^2 + \delta^2\lambda^2 + \frac{\alpha-4m+1}{\alpha}\delta^4 g(u,\varphi u)\lambda \\
&\quad - \frac{\alpha^2-(4m^2-2m+3)\alpha+4m^2-2m}{\alpha^2(\alpha-1)}\delta^2 g(u,\varphi u)^2 \\
&\quad - \frac{(2m-1)\alpha^2-(4m^2-4m)\alpha-4m^2+1}{\alpha}\delta^2.
\end{aligned}$$

□

5. The Berger-type Cheeger-Gromoll metric on the φ-unit tangent bundle $T_1^{\varphi}M$

The hypersurface that corresponds to the φ-unit tangent (sphere) bundle over an anti-paraKähler manifold (M,φ,g) can be expressed as follows:

$$T_1^{\varphi}M = \big\{(x,u)\in TM,\ g(u,\varphi u)=1\big\}.$$

We have the function f defined as:

$$\begin{aligned}
f: TM &\to \mathbb{R} \\
(x,u) &\mapsto f(x,u) = g(u,\varphi u)-1,
\end{aligned}$$

where x is a point in the base manifold M, and u is a tangent vector in the tangent space at x. This function is used to define the hypersurface $T_1^{\varphi}M$ as:

$$T_1^{\varphi}M = \{(x,u)\in TM,\ f(x,u)=0\}.$$

Let $\widetilde{grad}f$ (the gradient of f with respect to the metric $\tilde{g}$) be a normal vector field to $T_1^{\varphi}M$. From the Lemma 3.2, for any vector field X on M, we have the following relationships:

$$\begin{aligned}
\tilde{g}({}^HX,\widetilde{grad}f) &= {}^HX(f) = {}^HX(g(u,\varphi u)-1) = 0, \\
\tilde{g}({}^VX,\widetilde{grad}f) &= {}^VX(f) = {}^VX(g(u,\varphi u)-1) = 2g(X,\varphi u) = 2\tilde{g}({}^VX,{}^V(\varphi u)).
\end{aligned}$$

From these equations, we conclude that:

$$\widetilde{grad}f = 2\,{}^V(\varphi u).$$

The unit normal vector field to $T_1^{\varphi}M$ is given by

$$\mathcal{N} = \frac{\widetilde{grad}f}{|\widetilde{grad}f|_{\tilde{g}}} = \frac{{}^V(\varphi u)}{|{}^V(\varphi u)|_{\tilde{g}}} = \frac{{}^V(\varphi u)}{|u|} = \sqrt{\frac{\delta^2}{\alpha-1}}\,{}^V(\varphi u).$$

where $\alpha = 1 + \delta^2 g(u,u)$.

The tangential lift ${}^TX$ of a vector $X \in T_xM$ with respect to $\tilde{g}$ at the point $(x,u) \in T_1^{\varphi}M$ is defined as the tangential projection of the vertical lift of X to (x,u) with respect to $\mathcal{N}$, which can be expressed as:

$$ {}^TX = {}^VX - \tilde{g}_{(x,u)}({}^VX, \mathcal{N}_{(x,u)})\mathcal{N}_{(x,u)} = {}^VX - \frac{\delta^2}{\alpha - 1} g_x(X, \varphi u)^V(\varphi u)_{(x,u)}. $$

To simplify the notation for clarity, let us use $\bar{X} = X - \dfrac{\delta^2}{\alpha - 1} g(X, \varphi u)\varphi u$, so we have ${}^TX = {}^V\bar{X}$. From the above equation, we can derive a direct sum decomposition as follows:

$$ \begin{aligned} T_{(x,u)}TM &= T_{(x,u)}T_1^{\varphi}M \oplus span\{\mathcal{N}_{(x,u)}\} \\ &= T_{(x,u)}T_1^{\varphi}M \oplus span\{{}^V(\varphi u)_{(x,u)}\}, \end{aligned} \tag{5.1} $$

where $(x,u) \in T_1^{\varphi}M$.

Indeed, if $V \in T_{(x,u)}TM$, then they exist $X, Y \in T_xM$, such that

$$ \begin{aligned} V &= {}^HX + {}^VY \\ &= {}^HX + {}^TY + \tilde{g}_{(x,u)}({}^VY, \mathcal{N}_{(x,u)})\mathcal{N}_{(x,u)} \\ &= {}^HX + {}^TY + \frac{\delta^2}{\alpha - 1} g_x(Y, \varphi u)^V(\varphi u)_{(x,u)}. \end{aligned} \tag{5.2} $$

From (5.2), we can conclude that the tangent space $T_{(x,u)}T_1^{\varphi}M$ of $T_1^{\varphi}M$ at (x,u) is given by

$$ T_{(x,u)}T_1^{\varphi}M = \{{}^HX + {}^TY \,/\, X \in T_xM, Y \in \{\varphi u\}^{\perp} \subset T_xM\}, $$

where $\{\varphi u\}^{\perp} = \big\{Y \in T_xM,\ g(Y, \varphi u) = 0\big\}$. Hence $T_{(x,u)}T_1^{\varphi}M$ is spanned by vectors of the form ${}^HX$ and ${}^TY$.

Given a vector field X on M, the tangential lift ${}^TX$ of X is given by

$$ {}^TX_{(x,u)} = \big({}^VX - \tilde{g}({}^VX, \mathcal{N})\mathcal{N}\big)_{(x,u)} = \big({}^VX - \frac{\delta^2}{\alpha - 1} g(X, \varphi u)^V(\varphi u)\big)_{(x,u)}. $$

Lemma 5.1. *Consider an anti-paraKähler manifold (M, φ, g) and its tangent bundle $(TM, \tilde{g})$ equipped with the Berger-type Cheeger-Gromoll metric. In this context, we can derive the following results:*

$$ \begin{array}{ll} 1) & \tilde{g}({}^HX, \mathcal{N}) = 0, \\ 2) & \tilde{g}({}^TX, \mathcal{N}) = 0, \\ 3) & {}^TX = {}^VX \Leftrightarrow g(X, \varphi u) = 0, \\ 4) & {}^T(\varphi u) = 0, \\ 5) & g(\bar{X}, \varphi u) = 0 \end{array} $$

for any vector field X on M.

Definition 5.1. In the context of an anti-paraKähler manifold (M, φ, g) and its tangent bundle $(TM, \tilde{g})$ equipped with the Berger-type Cheeger-Gromoll metric, the Riemannian metric $\widehat{g}$ on $T_1^{\varphi}M$, induced by $\tilde{g}$, is completely determined by the following identities:

$$\begin{aligned} \widehat{g}({}^{H}X, {}^{H}Y) &= g(X,Y), \\ \widehat{g}({}^{T}X, {}^{H}Y) &= \widehat{g}({}^{H}X, {}^{T}Y) = 0, \\ \widehat{g}({}^{T}X, {}^{T}Y) &= \frac{1}{\alpha}\big(g(X,Y) - \frac{\delta^2}{\alpha-1} g(X,\varphi u) g(Y,\varphi u)\big). \end{aligned}$$

We will compute the Levi-Civita connection $\widehat{\nabla}$ of $T_1^{\varphi}M$ equipped with the Berger-type Cheeger-Gromoll metric $\widehat{g}$. This connection is defined by the formula:

$$\widehat{\nabla}_U V = \widetilde{\nabla}_U V - \tilde{g}(\widetilde{\nabla}_U V, \mathcal{N})\mathcal{N}, \tag{5.3}$$

where U and V are vector fields on $T_1^{\varphi}M$.

Theorem 5.1. *Consider an anti-paraKähler manifold (M, φ, g) and its φ-unit tangent bundle equipped with the Berger-type Cheeger-Gromoll metric. In this context, we can express the following formulas:*

$$\begin{aligned} \widehat{\nabla}_{{}^{H}X}{}^{H}Y &= {}^{H}(\nabla_X Y) - \frac{1}{2}{}^{T}(R(X,Y)u), \\ \widehat{\nabla}_{{}^{H}X}{}^{T}Y &= \frac{1}{2\alpha}{}^{H}\big(R(u,Y)X\big) + {}^{T}(\nabla_X Y), \\ \widehat{\nabla}_{{}^{T}X}{}^{H}Y &= \frac{1}{2\alpha}{}^{H}\big(R(u,X)Y\big), \\ \widehat{\nabla}_{{}^{T}X}{}^{T}Y &= \frac{-\delta^2}{\alpha}\big(g(Y,u) - \frac{\delta^2}{\alpha-1} g(Y,\varphi u)\big){}^{T}X \\ &\quad -\frac{\delta^2}{\alpha}\big(g(X,u) - \frac{\delta^2}{\alpha-1} g(X,\varphi u)\big){}^{T}Y \\ &\quad -\frac{\delta^2}{\alpha-1} g(Y,\varphi u){}^{T}(\varphi X) + \frac{\delta^2}{\alpha}\big(g(X,Y) \\ &\quad +\frac{\delta^2}{(\alpha-1)^2} g(X,\varphi u) g(Y,\varphi u)\big){}^{T}u \end{aligned}$$

for all vector fields X, Y on M, where ∇ represents the Levi-Civita connection and R denotes its Riemannian curvature tensor of (M, φ, g).

Proof. 1. By direct calculations, we have

$$\begin{aligned} \widehat{\nabla}_{{}^{H}X}{}^{H}Y &= \widetilde{\nabla}_{{}^{H}X}{}^{H}Y - \tilde{g}(\widetilde{\nabla}_{{}^{H}X}{}^{H}Y, \mathcal{N})\mathcal{N} \\ &= {}^{H}(\nabla_X Y) - \frac{1}{2}{}^{V}(R(X,Y)u) - \tilde{g}(-\frac{1}{2}{}^{V}(R(X,Y)u), \mathcal{N})\mathcal{N} \\ &= {}^{H}(\nabla_X Y) - \frac{1}{2}{}^{T}(R(X,Y)u). \end{aligned}$$

2. We have $\widehat{\nabla}_{^H X}{}^T Y = \widetilde{\nabla}_{^H X}{}^T Y - \tilde{g}(\widetilde{\nabla}_{^H X}{}^T Y, \mathcal{N})\mathcal{N}$, direct calculations give

$$\widetilde{\nabla}_{^H X}{}^T Y = \frac{1}{2\alpha}{}^H\big(R(u,Y)X\big) + {}^T(\nabla_X Y) \text{ and } \tilde{g}(\widetilde{\nabla}_{^H X}{}^T Y, \mathcal{N}) = 0.$$

Hence

$$\widehat{\nabla}_{^H X}{}^T Y = \frac{1}{2\alpha}{}^H\big(R(u,Y)X\big) + {}^T(\nabla_X Y).$$

3. Also, we have $\widehat{\nabla}_{^T X}{}^H Y = \widetilde{\nabla}_{^T X}{}^H Y - \tilde{g}(\widetilde{\nabla}_{^T X}{}^H Y, \mathcal{N})\mathcal{N}$. It follows that

$$\widetilde{\nabla}_{^T X}{}^H Y = \frac{1}{2\alpha}{}^H\big(R(u,Y)X\big) \text{ and } \tilde{g}(\widetilde{\nabla}_{^H X}{}^T Y, \mathcal{N}) = 0.$$

Hence

$$\widehat{\nabla}_{^H X}{}^T Y = \frac{1}{2\alpha}{}^H\big(R(u,Y)X\big).$$

4. Similarly, in the context mentioned above, we can present the following formulas: $\widehat{\nabla}_{^T X}{}^T Y = \widetilde{\nabla}_{^T X}{}^T Y - \tilde{g}(\widetilde{\nabla}_{^T X}{}^T Y, \mathcal{N})\mathcal{N}$, which give

$$\begin{aligned}
\widetilde{\nabla}_{^T X}{}^T Y &= \frac{-\delta^2}{\alpha}\big(g(Y,u) - \frac{\delta^2}{\alpha-1}g(Y,\varphi u)\big)^V X \\
&\quad -\frac{\delta^2}{\alpha}\big(g(X,u) - \frac{\delta^2}{\alpha-1}g(X,\varphi u)\big)^V Y \\
&\quad -\frac{\delta^2}{\alpha-1}g(Y,\varphi u)^V(\varphi X) + \frac{\delta^2}{\alpha}\big(g(X,Y) \\
&\quad +\frac{\delta^2}{(\alpha-1)^2}g(X,\varphi u)g(Y,\varphi u)\big)^V u \\
&\quad +\Big(\frac{-\delta^2}{\alpha(\alpha-1)}g(X,\varphi Y) - \frac{\delta^4}{\alpha^2}g(X,Y) \\
&\quad +\frac{2\delta^4}{(\alpha-1)^2}g(X,u)g(Y,\varphi u) + \frac{\delta^4}{(\alpha-1)^2}g(X,\varphi u)g(Y,u) \\
&\quad -\frac{(2\alpha^2+\alpha-1)\delta^6}{\alpha^2(\alpha-1)^3}g(X,\varphi u)g(Y,\varphi u)\Big)^V(\varphi u)
\end{aligned}$$

and

$$\begin{aligned}
\tilde{g}(\widetilde{\nabla}_{^TX}{}^TY,\mathcal{N})\mathcal{N} &= -\frac{\delta^2}{\alpha}\big(g(Y,u)-\frac{\delta^2}{\alpha-1}g(Y,\varphi u)\big)\tilde{g}(^VX,\mathcal{N})\mathcal{N}\\
&\quad -\frac{\delta^2}{\alpha}\big(g(X,u)-\frac{\delta^2}{\alpha-1}g(X,\varphi u)\big)\tilde{g}(^VY,\mathcal{N})\mathcal{N}\\
&\quad +\frac{\delta^2}{\alpha}\big(g(X,Y)+\frac{\delta^2}{(\alpha-1)^2}g(X,\varphi u)g(Y,\varphi u)\big)\tilde{g}(^Vu,\mathcal{N})\mathcal{N}\\
&\quad -\frac{\delta^2}{\alpha-1}g(Y,\varphi u)\tilde{g}(^V(\varphi X),\mathcal{N})\mathcal{N}+\Big(\frac{-\delta^2}{\alpha(\alpha-1)}g(X,\varphi Y)\\
&\quad +\frac{2\delta^4}{(\alpha-1)^2}g(X,u)g(Y,\varphi u)+\frac{\delta^4}{(\alpha-1)^2}g(X,\varphi u)g(Y,u)\\
&\quad -\frac{\delta^4}{\alpha^2}g(X,Y)-\frac{(2\alpha^2+\alpha-1)\delta^6}{\alpha^2(\alpha-1)^3}g(X,\varphi u)g(Y,\varphi u)\Big)^V(\varphi u).
\end{aligned}$$

Hence

$$\begin{aligned}
\widehat{\nabla}_{^TX}{}^TY &= \frac{-\delta^2}{\alpha}\big(g(Y,u)-\frac{\delta^2}{\alpha-1}g(Y,\varphi u)\big)^TX\\
&\quad -\frac{\delta^2}{\alpha}\big(g(X,u)-\frac{\delta^2}{\alpha-1}g(X,\varphi u)\big)^TY\\
&\quad -\frac{\delta^2}{\alpha-1}g(Y,\varphi u)^T(\varphi X)+\frac{\delta^2}{\alpha}\big(g(X,Y)\\
&\quad +\frac{\delta^2}{(\alpha-1)^2}g(X,\varphi u)g(Y,\varphi u)\big)^Tu.
\end{aligned}$$

When providing proofs, we use the Lemma 3.1, Theorem 3.1, Lemma 3.5 and the formula (5.3). □

Next, we will compute the Riemannian curvature tensor of $T_1^{\varphi}M$ with the Berger-type Cheeger-Gromoll metric $\widehat{g}$. Denoting $\widehat{R}$ as the Riemannian curvature tensor of $(T_1^{\varphi}M,\widehat{g})$, we can derive the following expression from the Gauss equation for hypersurfaces:

$$\widehat{R}(U,V)W={}^t(\widetilde{R}(U,V)W)-B(U,W).A_{\mathcal{N}}V+B(V,W).A_{\mathcal{N}}U, \tag{5.4}$$

where U,V and W are vector fields on $T_1^{\varphi}M$ and ${}^t(\widetilde{R}(U,V)W)$ represents the tangential component of (5.1). $A_{\mathcal{N}}$ is the shape operator of $T_1^{\varphi}M$ in $(TM,\tilde{g})$ derived from $\mathcal{N}$, and B is the second fundamental form of $T_1^{\varphi}M$ as a hypersurface immersed in TM, associated to $\mathcal{N}$ on $T_1^{\varphi}M$. The tangential component of $-\widetilde{\nabla}_U\mathcal{N}$, is given by:

$$A_{\mathcal{N}}U=-{}^t(\widetilde{\nabla}_U\mathcal{N}).$$

Furthermore, $B(U,V)$ can be expressed using Gauss's formula: $\widetilde{\nabla}_UV=\widehat{\nabla}_UV+B(U,V).\mathcal{N}$, which allows us to calculate $B(U,V)$ as

$$B(U,V)=\tilde{g}(\widetilde{\nabla}_UV,\mathcal{N}).$$

Theorem 5.2. *Consider an anti-paraKähler manifold* (M,φ,g) *and its* φ*-unit tangent bundle equipped with the Berger-type Cheeger-Gromoll metric. In this context, we can express the following formulas:*

$$\begin{aligned}\widehat{R}({}^HX,{}^HY){}^HZ &= {}^H(R(X,Y)Z)+\frac{1}{2\alpha}{}^H(R(u,R(X,Y)u)Z)\\ &\quad+\frac{1}{4\alpha}{}^H(R(u,R(X,Z)u)Y)-\frac{1}{4\alpha}{}^H(R(u,R(Y,Z)u)X)\\ &\quad+\frac{1}{2}{}^T((\nabla_Z R)(X,Y)u),\end{aligned}$$

$$\begin{aligned}\widehat{R}({}^HX,{}^TY){}^HZ &= \frac{1}{2\alpha}{}^H((\nabla_X R)(u,Y)Z)-\frac{1}{4\alpha}{}^T(R(X,R(u,Y)Z)u)\\ &\quad-\frac{\delta^2}{2\alpha}g(\bar{Y},u){}^T(R(X,Z)u)+\frac{1}{2}{}^T(R(X,Z)\bar{Y})\\ &\quad+\frac{\delta^2}{2\alpha}g(R(X,Z)u,Y){}^Tu,\end{aligned}$$

$$\begin{aligned}\widehat{R}({}^HX,{}^HY){}^TZ &= \frac{1}{2\alpha}{}^H((\nabla_X R)(u,Z)Y)-\frac{1}{2\alpha}{}^H((\nabla_Y R)(u,Z)X)+{}^T(R(X,Y)\bar{Z})\\ &\quad-\frac{1}{4\alpha}{}^T(R(X,R(u,Z)Y)u)+\frac{1}{4\alpha}{}^T(R(Y,R(u,Z)X)u)\\ &\quad-\frac{\delta^2}{\alpha}g(\bar{Z},u){}^T(R(X,Y)u)+\frac{\delta^2}{\alpha}g(R(X,Y)u,Z){}^Tu,\end{aligned}$$

$$\begin{aligned}\widehat{R}({}^HX,{}^TY){}^TZ &= -\frac{1}{2\alpha}{}^H(R(\bar{Y},\bar{Z})X)-\frac{1}{4\alpha^2}{}^H(R(u,Y)R(u,Z)X)\\ &\quad+\frac{\delta^2}{2\alpha^2}g(\bar{Y},u){}^H(R(u,Z)X)-\frac{\delta^2}{2\alpha^2}g(\bar{Z},u){}^H(R(u,Y)X),\end{aligned}$$

$$\begin{aligned}\widehat{R}({}^TX,{}^TY){}^HZ &= \frac{1}{4\alpha^2}{}^H(R(u,X)R(u,Y)Z)-\frac{1}{4\alpha^2}{}^H(R(u,Y)R(u,X)Z)\\ &\quad+\frac{\delta^2}{\alpha^2}g(\bar{Y},u){}^H(R(u,X)Z)-\frac{\delta^2}{\alpha^2}g(\bar{X},u){}^H(R(u,Y)Z)\\ &\quad+\frac{1}{\alpha}{}^H(R(\bar{X},\bar{Y})Z),\end{aligned}$$

$$\begin{aligned}\widehat{R}({}^TX,{}^TY){}^TZ &= (A(\bar{Y},\bar{Z})+\frac{\delta^2}{\alpha}L(Y,Z)){}^TX-(A(\bar{X},\bar{Z})+\frac{\delta^2}{\alpha}L(X,Z)){}^TY\\ &\quad+(B(\bar{Y},\bar{Z})-L(Y,Z)){}^T(\varphi X)-(B(\bar{X},\bar{Z})-L(X,Z)){}^T(\varphi Y)\\ &\quad+\big(C(\bar{X},\bar{Y},\bar{Z})+\frac{\delta^2}{\alpha-1}g(Y,\varphi u)(B(\bar{X},\bar{Z})-L(X,Z))\\ &\quad-\frac{\delta^2}{\alpha-1}g(X,\varphi u)(B(\bar{Y},\bar{Z})-L(Y,Z))\big){}^Tu\end{aligned}$$

for all vector fields X, Y *and* Z *on* M*, where* $\bar{X} = X - \frac{\delta^2}{\alpha - 1} g(X, \varphi u)\varphi u$ *and*

$$L(*, Z) = \frac{\delta^2}{\alpha - 1}\Big(\frac{1}{\alpha} g(\bar{*}, \varphi \bar{Z}) - \frac{\delta^2}{\alpha^2} g(*, Z) + \frac{(2\alpha^4 - 3\alpha^3 - 3\alpha^2 + 3\alpha - 1)\delta^4}{\alpha^2(\alpha - 1)^2} g(*, \varphi u) g(Y, \varphi u)\Big).$$

Proof. By utilizing Theorem 3.1 and Lemma 3.5, we can derive the following results:

(5.5)
$$A_{\mathcal{N}}{}^{H}X = 0,$$

(5.6)
$$A_{\mathcal{N}}{}^{T}X = \sqrt{\frac{\delta^2}{\alpha - 1}}\Big(\frac{\delta^2}{\alpha}\, {}^{T}X - {}^{T}(\varphi X) + \frac{\delta^2}{\alpha - 1} g(X, \varphi u)^{T}u\Big),$$

(5.7)
$$B({}^{H}X, {}^{H}Y) = B({}^{H}X, {}^{T}Y) = B({}^{T}X, {}^{H}Y) = 0$$

and

(5.8)
$$B({}^{T}X, {}^{T}Y) = -\sqrt{\frac{\delta^2}{\alpha - 1}}\Big(\frac{1}{\alpha} g(\bar{X}, \varphi \bar{Y}) - \frac{\delta^2}{\alpha^2} g(X, Y) + \frac{(2\alpha^4 - 3\alpha^3 - 3\alpha^2 + 3\alpha - 1)\delta^4}{\alpha^2(\alpha - 1)^2} g(X, \varphi u) g(Y, \varphi u)\Big).$$

We can obtain the necessary formulas for the curvature tensor by simply applying Theorem 4.1 along with equations (5.4) through (5.8). □

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