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$*$-CONFORMAL CURVATURE OF CONTACT METRIC MANIFOLDS

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Abstract. We introduce the $*$-conformal curvature tensor and ${}^*\eta$-Einstien manifolds in contact manifolds. We investigate this tensor in the three main classes of contact manifolds: Sasakian manifolds, Kenmotsu manifolds, and cosymplectic manifolds. We prove that a manifold is η-Einstienian if and only if be ${}^*\eta$-Einstienian manifold.
Keywords: $*$-conformal curvature, ${}^*\eta$-Einstien manifolds, Sasakian manifolds, Kenmotsu manifolds, Cosymplectic manifolds.

1. Introduction

There are many similar concepts in complex geometry and contact geometry. Tachibana introduces $*$-Ricci tensor within the framework of an almost Hermitian manifold in their work [23]. Afterward, Hamada introduces the $*$-Ricci tensor for the real hypersurfaces embedded in a non-flat complex space form [16]. This notion on an almost contact metric manifold $(M, g, \eta, \xi, \varphi)$ is defined as

$$ {}^*Ric(X_1, X_2) = \frac{1}{2} trace\{\mathbf{X_3} \ \to K(X_1, \varphi X_2)\varphi \mathbf{X_3}\}, \tag{1.1} $$

for any vector field X_1, X_2. The $*$-Ricci operator ${}^*L$ is characterized by the relation $g({}^*LX_1, X_2) = {}^*Ric(X_1, X_2)$. With the help of the $*$-Ricci tensor, several authors have investigated $*$-Ricci soliton in contact geometry (see [14], [10], [25], [2]). In

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general, the equality ${}^*Ric(X_1, X_2) = {}^*Ric(X_2, X_1)$ does not always hold.

In a Riemannian manifold (M^{2n+1}, g), the conformal curvature tensor C is expressed as

$$\begin{aligned} C(X_1,X_2)X_3 = K(X_1,X_2)X_3 \quad &- \quad \frac{1}{2n-1}\Big(Ric(X_2,X_3)X_1 - Ric(X_1,X_3)X_2 \\ &+ \quad g(X_2,X_3)\,LX_1 - g(X_1,X_3)\,LX_2\Big) \\ &+ \quad \frac{r}{2n(2n-1)}\Big(g(X_2,X_3)X_1 - g(X_1,X_3)X_2\Big), \end{aligned} \tag{1.2}$$

where K represents the curvature tensor of (1,3) type, Ric indicates the Ricci tensor, r is the scalar curvature and L is the Ricci operator of (M, g).

The paper is organized as follows: In Section 2, we express some preliminary definitions, then we proceed to investigate $*$-conformal curvature tensor of the contact manifolds. We examine some features of $*$-conformal curvature tensor.

In Section 3, we considered the Sasakian structure. Then, having the $*$-Ricci, we determined the relationship between η-Einstien and ${}^*\eta$-Einstien manifold.

Theorem 1.1. *Let M^{2n+1} be a manifold with a Sasakian structure (g, η, ξ, φ). The manifold $(M^{2n+1}, g, \eta, \xi, \varphi)$ is an η-Einstien manifold if and only if it is a ${}^*\eta$-Einstien manifold.*

Then, we investigate the $*$-conformal curvature tensor of the Sasakian manifolds. In addition, we show that ξ-conformally flat and ξ-$*$conformally flat will not co-occur in Sasakian manifolds. By the condition ${}^*Ric(X_1, X_2)$ and ${}^*r$ for a $2n+1$-dimensional Sasakian manifold, we get the following $(0,2)$-tensor

$${}^*T(X_1,X_2) = -\frac{{}^*Ric(X_1,X_2)}{2n-1} + \frac{{}^*r\ g(X_1,X_2)}{4n(2n-1)}.$$

We conclude that if $n > 1$, then $*$-conformal curvature tensor ans and ${}^*D(X_1, X_2)X_3$ do not vanish simultaneously.

In Section 4, we find some conditions for a Kenmotsu 3-manifold to have vanishing $*$-conformal curvature tensor. We show that for a special case, the $*$-conformal tensor of this manifold becomes zero as in the following Theorem.

Theorem 1.2. *If a Kenmotsu 3-manifold is of quasi-constant curvature of the form*

$$\begin{aligned} K(X_1,X_2)X_3 = \alpha(X_1\wedge X_2)(X_3) \quad &- \quad \alpha\big[\eta(X_2)\eta(X_3)X_1 - \eta(X_1)\eta(X_3)X_2 \\ &+ \quad g(X_2,X_3)\eta(X_1)\xi - g(X_1,X_3)\eta(X_2)\xi\big], \end{aligned}$$

where $\alpha = \frac{r}{2} + 2$, then $*$-conformal curvature tensor vanishes.

But in general, we show that on Kenmotsu manifolds, the $*$-conformal tensor cannot vanish identically. Similarly, the equivalence of η-Einstien and ${}^*\eta$-Einstien is also established in Kenmotsu manifolds. The same result about $*$-conformal curvature tensor and ${}^*D(X_1, X_2)X_3$ on the Sasakian manifold is obtained for the Kenmotsu manifold.

In the last section, we prove the $*$-conformal curvature tensor is identically zero on the 3-dimensional cosymplectic manifolds. We confirm a conformally flat cosymplectic manifold is an ${}^*\eta$-Einstien manifold. We prove the following theorem:

Theorem 1.3. *Let $(M^{2n+1}, g, \eta, \xi, \varphi)$ be a $2n+1$-dimension cosymplectic manifold with $n \geqslant 1$. If M is a $*$-conformally flat manifold, then ${}^*D = 0$.*

2. Preliminaries

Definition 2.1. Consider a contact metric manifold $(M, g, \eta, \xi, \varphi)$ of dimension $2n+1$. The $*$-conformal curvature tensor for $(M, g, \eta, \xi, \varphi)$ is expressed as

$$\begin{aligned} {}^*C(X_1, X_2)X_3 = K(X_1, X_2)X_3 \quad &- \quad \frac{1}{2n-1}\Big({}^*Ric(X_2, X_3)X_1 - {}^*Ric(X_1, X_3)X_2 \\ &+ \quad g(X_2, X_3)\,{}^*LX_1 - g(X_1, X_3)\,{}^*LX_2\Big) \\ &+ \quad \frac{{}^*r}{2n(2n-1)}\Big(g(X_2, X_3)X_1 - g(X_1, X_3)X_2\Big), \end{aligned} \tag{2.1}$$

where ${}^*r$ represents the $*$-scalar curvature, which is the trace of the $*$-Ricci tensor.

Definition 2.2. A contact metric manifold is named ${}^*\eta$-Einstien if

$$ {}^*Ric(X_1, X_2) = c\, g(X_1, X_2) + d\, \eta(X_1)\eta(X_2), \qquad c, d \in C^\infty(M). \tag{2.2}$$

A differentiable manifold M^{2n+1} has an almost contact structure [2] if it admits a 1-form η, a characteristic vector field ξ, and a $(1,1)$-tensor field φ, which satisfy

$$\varphi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1, \tag{2.3}$$

where I indicates the identity endomorphism. Then, by (2.3), can see that

$$\varphi\xi = 0, \qquad \eta \circ \varphi = 0. \tag{2.4}$$

If an almost contact manifold M^{2n+1} admits a Riemannian metric g with the property:

$$g(\varphi X_1, \varphi X_2) = g(X_1, X_2) - \eta(X_1)\eta(X_2), \qquad \forall X_1, X_2 \in \chi(M), \tag{2.5}$$

then $(M^{2n+1}, g, \eta, \xi, \varphi)$ is called an almost contact metric manifold. The 2-form $\Phi(X_1, X_2) = g(X_1, \varphi X_2)$ is called the fundamental 2-form on the almost contact metric manifold $(M^{2n+1}, g, \eta, \xi, \varphi)$. An almost contact metric manifold is called normal if the (1,2)-type torsion tensor N_φ vanishes, where $N_\varphi = [\varphi, \varphi] + 2d\eta \otimes \xi$ is the Nijenhuis tensor of φ. A normal almost contact metric manifold is called a Sasakian manifold. A Sasakian manifold is also characterized by

$$\big(\nabla_{X_1}\varphi\big)X_2 = g(X_1, X_2)\xi - \eta(X_2)X_1, \qquad \forall X_1, X_2 \in \chi(M).$$

On a Sasakian manifold beside (2.3)-(2.5), we also have

$$\nabla_{X_1}\xi = -\varphi X_1, \qquad K(X_1, X_2)\xi = \eta(X_2)X_1 - \eta(X_1)X_2, \tag{2.6}$$

where K denotes the curvature tensor of (1,3) type. The importance and application of Sasakian structures are in holomorphic statistical structures and are also related to string theory (see [1]).

If the 1-form η is closed and $d\Phi = 2\eta \wedge \Phi$, then the almost contact metric manifold is called almost Kenmotsu manifold. A normal almost Kenmutsu manifold is a Kenmutsu manifold, which is equivalent to:

$$(\nabla_{X_1}\varphi)X_2 = g(\varphi X_1, X_2)\xi - \eta(X_2)\varphi X_1, \qquad \forall X_1, X_2 \in \chi(M).$$

It is known that every Kenmotsu manifold is locally a warped product $I \times_f N^{2n}$, where N^{2n} is a Kahler manifold, I is an open interval with coordinate t, and the warping function f defined by $f = ce^t$ for some positive constant c [19]. For a $(2n+1)$-dimensional Kenmotsu manifold, we have

$$\begin{aligned}
\nabla_{X_1}\xi &= X_1 - \eta(X_1)\xi, && (2.7)\\
K(X_1, X_2)\xi &= \eta(X_1)X_2 - \eta(X_2)X_1, && (2.8)\\
Ric(X_1, \xi) &= -2n\,\eta(X_1), && (2.9)\\
K(\xi, X_1)X_2 &= \eta(X_2)X_1 - g(X_1, X_2)\xi, && (2.10)\\
Ric(\phi X_1, \phi X_2) &= Ric(X_1, X_2) + 2n\,\eta(X_1)\eta(X_2). && (2.11)
\end{aligned}$$

An almost contact metric manifold is termed an almost cosymplectic manifold when both the 1-form η and 2-form Φ are closed. A normal almost cosymplectic manifold is called a cosymplectic manifold [3], [15]. Every cosymplectic manifold satisfies the following:

$$\nabla_{X_1}\xi = 0, \quad K(X_1, X_2)\xi = 0, \quad Ric(X_1, \xi) = 0. \tag{2.12}$$

The cosymplectic structure is a tool for time-dependent Hamiltonian mechanics. It has some applications in string theory, which shows the importance of cosymplectic manifolds.

Suppose that $(M^{2n+1}, g, \eta, \xi, \varphi)$ is an almost contact metric manifold and ${}^*C$ is its $*$-conformal curvature tensor, which is defined by (2.1). A direct computation shows some symmetries of ${}^*C$.

Proposition 2.1. *In a contact metric manifold, the $*$-conformal curvature tensor obeys the following:*

1. ${}^*C(X_1, X_2)X_3 = -{}^*C(X_2, X_1)X_3$,
2. ${}^*C(X_1, X_2)X_3 + {}^*C(X_2, X_3)X_1 + {}^*C(X_3, X_1)X_2$
$= -\frac{1}{2n-1}\{{}^*Ric(X_1, X_2)X_3 + {}^*Ric(X_2, X_3)X_1 + {}^*Ric(X_3, X_1)X_2 - {}^*Ric(X_1, X_3)X_2 - {}^*Ric(X_2, X_1)X_3 - {}^*Ric(X_3, X_2)X_1\}$.

Definition 2.3. A contact metric manifold is called ξ-conformally flat and ξ-$*$conformally flat, respectively, if $C(X_1, X_2)\xi = 0$ and ${}^*C(X_1, X_2)\xi = 0$, respectively.

3. $*$-conformal curvature tensor in Sasakian manifolds

In [14], Ghash and Patra obtained the $*$-Ricci tensor in a $(2n+1)$-dimensional Sasakian manifold as follows

$$ {}^*Ric(X_1, X_2) = Ric(X_1, X_2) - (2n-1)\, g(X_1, X_2) - \eta(X_1)\eta(X_2). \tag{3.1} $$

Equation (3.1) provides

$$ {}^*LX_1 = LX_1 - (2n-1)\, X_1 - \eta(X_1)\xi, \tag{3.2} $$

and

$$ {}^*r = r - 4n^2. \tag{3.3} $$

Theorem 3.1. *Let M^{2n+1} be a manifold with a Sasakian structure (g, η, ξ, φ). The manifold $(M^{2n+1}, g, \eta, \xi, \varphi)$ is an η-Einstien manifold if and only if it is a ${}^*\eta$-Einstien manifold.*

Proof. If $(M^{2n+1}, g, \eta, \xi, \varphi)$ is an η-Einstien manifold, then

$$ \exists c, d \in C^\infty(M), \quad Ric(X_1, X_2) = c\, g(X_1, X_2) + d\, \eta(X_1)\eta(X_2). \tag{3.4} $$

From (3.1) and (3.4), we have

$$ {}^*Ric(X_1, X_2) = \tilde{c}\, g(X_1, X_2) + \tilde{d}\, \eta(X_1)\eta(X_2), \tag{3.5} $$

where $\tilde{c} = c - (2n-1)$ and $\tilde{d} = d - 1$. Thus, $(M^{2n+1}, g, \eta, \xi, \varphi)$ is a ${}^*\eta$-Einstien manifold. In this case, there are smooth scalar functions $\tilde{c}$ and $\tilde{d}$

$$ {}^*Ric(X_1, X_2) = \tilde{c}\, g(X_1, X_2) + \tilde{d}\, \eta(X_1)\eta(X_2). \tag{3.6} $$

By (3.6) and (3.1), we conclude that M is a η-Einstien manifold. $\square$

A Sasakian manifold is said to be a $\phi-$recurrent manifold if there exists a nonzero $1-$form A such that

$$\phi^2((\nabla_{X_1}K)(X_2,X_3)X_4) = A(X_1)\,K(X_2,X_3)X_4, \tag{3.7}$$

for arbitrary vector fields X_1, X_2, X_3, and X_4 on the manifold M [11]. As a result, a $\phi-$recurrent Sasakian manifold is an Einstein manifold. Thus, by Theorem 3.1, it follows that every $\phi-$recurrent Sasakian manifold is a $^*\eta$-Einstein manifold.

In 1968, Yano and Sawaki [27] defined quasi-conformal curvature tensor as follows:

$$\begin{aligned} W(X_1,X_2)X_3 &= [-(n-2)\,d]\,C(X_1,X_2)X_3 \\ &+ [c+(n-2)\,d]\,\tilde{C}(X_1,X_2)X_3, \end{aligned} \tag{3.8}$$

where c and d are arbitrary constants, C is the conformal curvature tensor, and $\tilde{C}$ given by

$$\begin{aligned} \tilde{C}(X_1,X_2)X_3 &= K(X_1,X_2)X_3 \\ &- \frac{r}{n(n-1)}\big[g(X_2,X_3)X_1 - g(X_1,X_3)X_2\big], \end{aligned} \tag{3.9}$$

where K is the Riemannian curvature tensor.

A quasi-conformally flat Sasakian manifold or a quasi-conformally semi-symmetric Sasakian manifold is an η-Einstein manifold [9]. Using Theorem 3.1, we infer every quasi-conformally flat or quasi-conformally semi-symmetric Sasakian manifold is a $^*\eta$-Einstein manifold.

By using (3.1), (3.2) and (3.3), from (2.1), we get

$$\begin{aligned} {}^*C(X_1,X_2)X_3 = C(X_1,X_2)X_3 &+ \frac{2n-2}{2n-1}\Big(g(X_2,X_3)X_1 - g(X_1,X_3)X_2\Big) \\ &+ \frac{1}{2n-1}\Big(\eta(X_2)\eta(X_3)X_1 - \eta(X_1)\eta(X_3)X_2 \\ &+ g(X_2,X_3)\eta(X_1)\xi - g(X_1,X_3)\eta(X_2)\xi\Big). \end{aligned} \tag{3.10}$$

In Sasakian manifolds, Proposition 2.1 reduces to Proposition 3.1.

Proposition 3.1. *In a Sasakian manifold, the $*$-conformal curvature tensor obeys the following:*

$$^*C(X_1,X_2)X_3 + {}^*C(X_2,X_3)X_1 + {}^*C(X_3,X_1)X_2 = 0.$$

In a 3-dimensional manifold, C vanishes identically, and hence, we have:

$$\begin{aligned} {}^*C(X_1,X_2)X_3 &= \eta(X_2)\eta(X_3)X_1 - \eta(X_1)\eta(X_3)X_2 \\ &+ g(X_2,X_3)\eta(X_1)\xi - g(X_1,X_3)\eta(X_2)\xi, \end{aligned} \tag{3.11}$$

In this case, (3.11) infers ${}^*C$ does not vanish identically. Indeed, for any non-zero vector filed $\tilde{X}$ in the kernel of η, we have

$$ {}^*C(2\tilde{X}+\xi,\tilde{X}+\xi)\xi=\tilde{X}. $$

Suppose $(M^{2n+1},g,\eta,\xi,\varphi)$ is a Sasakian manifold. By putting $X_3=\xi$ in (3.10), we have

$$ {}^*C(X_1,X_2)\xi=C(X_1,X_2)\xi+K(X_1,X_2)\xi. \tag{3.12} $$

Based on (3.12) and $K(X_1,X_2)\xi\neq 0$, we infer the Sasakian manifold does not become ξ-conformally flat and ξ-$*$conformally flat simultaneously.

Every Sasakian manifold is K-contact, but in general, every K-contact manifold is not Sasakian. For 3-dimensional manifolds, these are equivalent. In [28], the authors prove that a K-contact manifold is ξ-conformally flat if and only if it is an η-Einstien Sasakian manifold. From Theorem 3.1, we can say that a K-contact manifold is ξ-conformally flat if and only if it is a ${}^*\eta$-Einstien Sasakian manifold.

In [8], the authors defined the $(0,2)$-tensor field T on M^{2n+1} as follows:

$$ T(X_1,X_2)=-\frac{Ric(X_1,X_2)}{2n-1}+\frac{r\,g(X_1,X_2)}{4n(2n-1)}. \tag{3.13} $$

The conformal curvature tensor is given by

$$ \begin{aligned} C(X_1,X_2)X_3=K(X_1,X_2)X_3 \quad &+\quad T(X_2,X_3)\cdot X_1-T(X_1,X_3)\cdot X_2\\ &+\quad g(X_2,X_3)\,\hat{T}(X_1)-g(X_1,X_3)\,\hat{T}(X_2), \end{aligned} \tag{3.14} $$

where $T(X_1,X_2)=g(\hat{T}(X_1),X_2)$. For $n>1$, If $C=0$, then

$$ \nabla_{X_1}T(X_2,X_3)-\nabla_{X_2}T(X_1,X_3)=0. \tag{3.15} $$

We put $D(X_1,X_2)X_3:=\nabla_{X_1}T(X_2,X_3)-\nabla_{X_2}T(X_1,X_3)$. Now, we define $(0,2)$-tensor field ${}^*T$ on a Sasakian manifold M^{2n+1} as follows:

$$ {}^*T(X_1,X_2)=-\frac{{}^*Ric(X_1,X_2)}{2n-1}+\frac{{}^*r\,g(X_1,X_2)}{4n(2n-1)}. \tag{3.16} $$

By (3.1) and (3.3), we can write (3.16) as follows

$$ {}^*T(X_1,X_2)=T(X_1,X_2)+\frac{n-1}{2n-1}\,g(X_1,X_2)+\frac{1}{2n-1}\,\eta(X_1)\eta(X_2). \tag{3.17} $$

Also, we define the conformal curvature tensor as follows:

$$ \begin{aligned} {}^*C(X_1,X_2)X_3=K(X_1,X_2)X_3 \quad &+\quad {}^*T(X_2,X_3)\cdot X_1-{}^*T(X_1,X_3)\cdot X_2\\ &+\quad g(X_2,X_3){}^*\hat{T}(X_1)-g(X_1,X_3){}^*\hat{T}(X_2), \end{aligned} \tag{3.18} $$

where ${}^*T(X_1,X_2)=g({}^*\hat{T}(X_1),X_2)$. So $(0,1)$-tensor field ${}^*\hat{T}$ is given by

$$ {}^*\hat{T}(X_1)=\hat{T}(X_1)+\frac{n-1}{2n-1}\,X_1+\frac{1}{2n-1}\,\eta(X_1)\xi. \tag{3.19} $$

By putting (3.17) and (3.19) in (3.18), we have

$$
\begin{aligned}
{}^*C(X_1,X_2)X_3 = C(X_1,X_2)X_3 \quad &+ \quad \frac{2(n-1)}{2n-1}\big[g(X_2,X_3)X_1 - g(X_1,X_3)X_2\big] \\
&+ \quad \frac{1}{2n-1}\big[g(X_2,X_3)\xi - \eta(X_3)X_2\big]\eta(X_1) \\
&- \quad \frac{1}{2n-1}\big[g(X_1,X_3)\xi - \eta(X_3)X_1\big]\eta(X_2). \qquad (3.20)
\end{aligned}
$$

We consider

$$
{}^*D(X_1,X_2)X_3 := \nabla_{X_1}{}^*T(X_2,X_3) - \nabla_{X_2}{}^*T(X_1,X_3). \qquad (3.21)
$$

A direct computation shows that

$$
\begin{aligned}
\nabla_{X_1}{}^*T(X_2,X_3) \quad &= \quad \nabla_{X_1}T(X_2,X_3) + \mu\ \nabla_{X_1}g(X_2,X_3) + \lambda\ \nabla_{X_1}(\eta(X_2)\eta(X_3)) \\
&= \quad \nabla_{X_1}T(X_2,X_3) + \mu\ \nabla_{X_1}g(X_2,X_3) \\
&+ \quad \lambda\ \big[\big(\nabla_{X_1}\eta(X_2)\big)\ \eta(X_3) + \eta(X_2)\ \big(\nabla_{X_1}\eta(X_3)\big)\big], \qquad (3.22)
\end{aligned}
$$

and

$$
\begin{aligned}
\nabla_{X_2}{}^*T(X_1,X_3) \quad &= \quad \nabla_{X_2}T(X_1,X_3) + \mu\ \nabla_{X_2}g(X_1,X_3) + \lambda\ \nabla_{X_2}(\eta(X_1)\eta(X_3)) \\
&= \quad \nabla_{X_2}T(X_1,X_3) + \mu\ \nabla_{X_2}g(X_1,X_3) \\
&+ \quad \lambda\ \big[\big(\nabla_{X_2}\eta(X_1)\big)\eta(X_3) + \eta(X_1)\ \big(\nabla_{X_2}\eta(X_3)\big)\big], \qquad (3.23)
\end{aligned}
$$

where $\mu = \frac{2n-2}{2n-1}$ and $\lambda = \frac{1}{2n-1}$. By putting (3.22) and (3.23) in (3.21), we have

$$
\begin{aligned}
{}^*D(X_1,X_2)X_3 = D(X_1,X_2)X_3 \quad &+ \quad \lambda\ \Big\{2g(X_1,\phi X_2)\eta(X_3) \\
&+ \quad (\nabla_{X_1}\eta)(X_3)\eta(X_2) - (\nabla_{X_2}\eta)(X_3)\eta(X_1)\Big\}. \qquad (3.24)
\end{aligned}
$$

If M^{2n+1} is a conformally flat Sasakian manifold with $n > 1$, then

$$
{}^*D(X_1,X_2)X_3 = \lambda\ \Big\{2g(X_1,\phi X_2)\eta(X_3) + (\nabla_{X_1}\eta)(X_3)\eta(X_2) - (\nabla_{X_2}\eta)(X_3)\eta(X_1)\Big\}. \qquad (3.25)
$$

From (3.24), it can be concluded that, if M^{2n+1} is a Sasakian manifold of dimension greater than 3, then $D(X_1,X_2)X_3 = 0$ and ${}^*D(X_1,X_2)X_3 = 0$ do not hold simultaneously, because otherwise, we have $d\eta = 0$, which is a contradiction with the Sasakian structure.

Example 3.1. We consider the Sasakian manifold $(\mathbb{R}^3, g, \eta, \xi, \varphi)$, where the 1-form η, vector field ξ, Riemannian metric g, and $(1,1)$-tensor field φ respectively as follows

$$
\eta = \frac{1}{2}(dz - ydx), \qquad \xi = 2\frac{\partial}{\partial z}, \qquad g = \eta\otimes\eta + \frac{1}{4}\Big((dx)^2 + (dy)^2\Big),
$$

and $\varphi == dx \otimes \frac{\partial}{\partial y} - dy \otimes \frac{\partial}{\partial x} + ydz \otimes \frac{\partial}{\partial y}$. Also, the vector fields are given by

$$X_1 = 2\frac{\partial}{\partial y}, \qquad X_2 = 2(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}), \qquad X_3 = \xi.$$

So, we have

$$\varphi X_1 = X_2, \qquad \varphi X_2 = -X_1, \qquad \varphi\xi = 0.$$

We know that, $\mathbb{R}^3$ is a conformally flat manifold, then $C = 0$. By (3.10) and $C(X_1, X_2)X_3 = 0$, we have ${}^*C(X_1, X_2)X_3 = -y\,X_1$. Therefore, for this 3-dimensional Sasakian manifold, the tensor ${}^*C$ will not be zero. On the other hand, we know that since $C(X_1, X_2)X_3 = 0$, then $D(X_1, X_2)X_3 = 0$. Therefore, having (3.25), we calculate the tensor ${}^*D$ as follows:

$${}^*D(X_1, X_2)X_3 = -2.$$

4. $*$-conformal curvature tensor in Kenmotsu manifolds

In [25], the author proves that in a Kenmotsu 3-manifold the $*$-Ricci tensor is given by

$$\begin{aligned}
{}^*Ric(X_1, X_2) &= (\frac{r}{2} + 2)g(\varphi X_1, \varphi X_2), && (4.1)\\
{}^*r &= r + 4, && (4.2)\\
{}^*LX_1 &= (\frac{r}{2} + 2)\big[X_1 - \eta(X_1)\xi\big]. && (4.3)
\end{aligned}$$

By substituting (4.1), (4.2), and (4.3) into (2.1) yields

$$\begin{aligned}
{}^*C(X_1, X_2)X_3 = K(X_1, X_2)X_3 &- (\frac{r}{2} + 2)\big[g(X_2, X_3)X_1 - g(X_1, X_3)X_2\big]\\
&+ (\frac{r}{2} + 2)\big[\eta(X_2)\eta(X_3)X_1 - \eta(X_1)\eta(X_3)X_2\\
&+ g(X_2, X_3)\eta(X_1)\xi - g(X_1, X_3)\eta(X_2)\xi\big]. \qquad (4.4)
\end{aligned}$$

Definition 4.1. [18] If the curvature tensor K of an almost contact metric manifold obeys the subsequent condition, then is called quasi-constant curvature:

$$\begin{aligned}
K(X_1, X_2)X_3 = \alpha(X_1 \wedge X_2)(X_3) &+ \beta\big[\eta(X_2)\eta(X_3)X_1 - \eta(X_1)\eta(X_3)X_2\\
&+ g(X_2, X_3)\eta(X_1)\xi - g(X_1, X_3)\eta(X_2)\xi\big], \qquad (4.5)
\end{aligned}$$

where $(X_1 \wedge X_2)(X_3) := g(X_2, X_3)X_1 - g(X_1, X_3)X_2$, α and β are smooth functions.

By some calculation, one concludes that the following holds.

Theorem 4.1. *If a Kenmotsu 3-manifold is of quasi-constant curvature of the form*

$$\begin{aligned}
K(X_1, X_2)X_3 = \alpha(X_1 \wedge X_2)(X_3) &- \alpha\big[\eta(X_2)\eta(X_3)X_1 - \eta(X_1)\eta(X_3)X_2\\
&+ g(X_2, X_3)\eta(X_1)\xi - g(X_1, X_3)\eta(X_2)\xi\big], \qquad (4.6)
\end{aligned}$$

where $\alpha = \frac{r}{2} + 2$, then $*$-conformal curvature tensor vanishes.

Suppose $(M^{2n+1}, g, \eta, \xi, \varphi)$ is a Kenmostu manifold. By [21], we have

$$\begin{aligned}
&(4.7) \quad & {}^*Ric(X_1, X_2) &= Ric(X_1, X_2) + (2n-1)g(X_1, X_2) + \eta(X_1)\eta(X_2), \\
&(4.8) \quad & {}^*r &= r + 4n^2, \\
&(4.9) \quad & {}^*LX_1 &= LX_1 + (2n-1)X_1 + \eta(X_1)\xi.
\end{aligned}$$

By putting $X_2 = \xi$ in (4.7) and from (2.9), we have

$$ {}^*Ric(X_1, \xi) = 0, \tag{4.10}$$

from (2.11) and (4.7), we have

$$ {}^*Ric(\phi X_1, \phi X_2) = {}^*Ric(X_1, X_2). \tag{4.11}$$

Theorem 4.2. *Suppose M^{2n+1} is a manifold and (g, η, ξ, φ) is a Kenmotsu structure on M. The M is an η-Einstien manifold if and only if it is a ${}^*\eta$-Einstien manifold.*

Proof. In [5], the contact metric structure is said to be η-Einstein if

$$L = c\, I + d\, \eta \otimes \xi, \qquad c, d \in C^\infty(M). \tag{4.12}$$

Let $(M^{2n+1}, g, \eta, \xi, \varphi)$ be a η-Einstein Kenmotsu manifold. By (4.9) and (4.12), we have

$$ {}^*L = \tilde{c}\, I + \tilde{d}\, \eta \otimes \xi, \tag{4.13}$$

where $\tilde{c} = c + (2n-1)$ and $\tilde{d} = c + 1$.

Suppose $(M^{2n+1}, g, \eta, \xi, \varphi)$ is a ${}^*\eta$-Einstein Kenmotsu manifold, then there are smooth functions $\tilde{c}$, and $\tilde{d}$ such that

$$ {}^*Ric(X_1, X_2) = \tilde{c}\, g(X_1, X_2) + \tilde{d}\, \eta(X_1)\eta(X_2). \tag{4.14}$$

By (4.14) and (4.7), we have

$$ Ric(X_1, X_2) = c\, g(X_1, X_2) + d\, \eta(X_1)\eta(X_2), \tag{4.15}$$

where $c = \tilde{c} - (2n-1)$ and $d = \tilde{d} - 1$. $\square$

By substituting (4.7), (4.8), and (4.9) into (2.1) yields

$$\begin{aligned}
{}^*C(X_1, X_2)X_3 = C(X_1, X_2)X_3 \quad &- \quad \frac{2n-2}{2n-1}\big[g(X_2, X_3)X_1 - g(X_1, X_3)X_2\big] \\
&- \quad \frac{1}{2n-1}\big[g(X_2, X_3)\eta(X_1)\xi - g(X_1, X_3)\eta(X_2)\xi \\
&+ \quad \eta(X_2)\eta(X_3)X_1 - \eta(X_1)\eta(X_3)X_2\big].
\end{aligned} \tag{4.16}$$

By putting $X_3 = \xi$ in (4.16), we obtain

$$ {}^*C(X_1, X_2)\xi = C(X_1, X_2)\xi + K(X_1, X_2)\xi. \tag{4.17}$$

From (4.17), we conclude that if $C(X_1, X_2)\xi = 0$ then ${}^*C(X_1, X_2)\xi \neq 0$. In other words, the Kenmotsu manifold cannot be ξ-conformally flat and ξ-$*$conformally flat simultaneously.

In the Kenmotsu manifold, (2) results in ${}^*Ric(X_1, X_2) = {}^*Ric(X_2, X_1)$. By Proposition 2.1 and ${}^*Ric(X_1, X_2) = {}^*Ric(X_2, X_1)$, the $*$-conformal curvature tensor satisfies in Bianchi type identity, which leads to the next proposition.

Proposition 4.1. *In a Kenmotsu manifold, the $*$-conformal curvature tensor obeys the relation:*

$$ {}^*C(X_1, X_2)X_3 + {}^*C(X_2, X_3)X_1 + {}^*C(X_3, X_1)X_2 = 0. $$

Let us define

$$ C(X_1, X_2, X_3, X_4) := g\big(C(X_1, X_2)X_3, X_4\big), \qquad \forall X_1, X_2, X_3, X_4 \in \chi(M). $$

By substituting (4.7) into (2.1), we have

$$ \begin{aligned} {}^*C(X_1, X_2, X_3, X_4) &= C(X_1, X_2, X_3, X_4) \\ &- \frac{2n-2}{2n-1}\big[g(X_2, X_3)g(X_1, X_4) - g(X_1, X_3)g(X_2, X_4)\big] \\ &- \frac{1}{2n-1}\big[g(X_2, X_3)\eta(X_1)\eta(X_4) - g(X_1, X_3)\eta(X_2)\eta(X_4) \\ &+ g(X_1, X_4)\eta(X_2)\eta(X_3) - g(X_2, X_4)\eta(X_1)\eta(X_3)\big]. \end{aligned} \tag{4.18} $$

Proposition 4.2. *For a Kenmotsu manifold, the $*$-conformal tensor cannot vanish identically.*

Proof. One can see that

$$ C(X_1, X_2, X_3, X_4) = -C(X_1, X_2, X_4, X_3). \tag{4.19} $$

Suppose that ${}^*C$ vanishes identically. Therefore, by (4.18) and (4.19), we have

$$ \begin{aligned} 2(2n-2) \Big[& g(X_2, X_3)g(X_1, X_4) - g(X_1, X_3)g(X_2, X_4)\Big] \\ &+ 2\big[g(X_2, X_3)\eta(X_1)\eta(X_4) - g(X_1, X_3)\eta(X_2)\eta(X_4) \\ &+ g(X_1, X_4)\eta(X_2)\eta(X_3) - g(X_2, X_4)\eta(X_1)\eta(X_3)\big] = 0. \end{aligned} \tag{4.20} $$

Putting $X_3 = X_1 = \xi$ into (4.20) implies that

$$ (2n-1)\Big(g(X_2, X_4) - \eta(X_2)\eta(X_4)\Big) = 0. \tag{4.21} $$

Since $2n-1$ is an odd number, we have

$$ g(X_2, X_4) - \eta(X_2)\eta(X_4) = 0, \qquad \forall X_2, X_4 \in \chi(M), \tag{4.22} $$

which is impossible. $\square$

Using Propositions 4.1 and 4.2, one concludes that a Kenmotsu 3-manifold cannot be of quasi-constant curvature of the form (4.6).

Now, we consider $(0,2)$-tensor field ${}^*T$ on Kenmotsu manifold M^{2n+1} as follows:

$$ {}^*T(X_1,X_2) = -\frac{{}^*Ric(X_1,X_2)}{2n-1} + \frac{{}^*r\ g(X_1,X_2)}{4n(2n-1)}. \tag{4.23}$$

By (4.8) and (4.7), we can write (4.23) as follows:

$$ {}^*T(X_1,X_2) = T(X_1,X_2) + \frac{(1-n)}{(2n-1)}\ g(X_1,X_2) + \frac{-1}{2n-1}\ \eta(X_1)\eta(X_2). \tag{4.24}$$

Also, we define the conformal curvature tensor as follows:

$$\begin{aligned} {}^*C(X_1,X_2)X_3 = K(X_1,X_2)X_3 \quad &+ \quad {}^*T(X_2,X_3)\cdot X_1 - {}^*T(X_1,X_3)\cdot X_2 \\ &+ \quad g(X_2,X_3){}^*\hat{T}(X_1) - g(X_1,X_3){}^*\hat{T}(X_2), \end{aligned} \tag{4.25}$$

where ${}^*T(X_1,X_2) = g({}^*\hat{T}(X_1),X_2)$. So ${}^*\hat{T}$ is given by

$$ {}^*\hat{T}(X_1) = \hat{T}(X_1) + \frac{(1-n)}{(2n-1)}\ X_1 + \frac{-1}{2n-1}\ \eta(X_1)\xi. \tag{4.26}$$

By putting (4.24) and (4.26) in (4.25), we have

$$\begin{aligned} {}^*C(X_1,X_2)X_3 = C(X_1,X_2)X_3 \quad &+ \quad \frac{2(1-n)}{(2n-1)}\big[g(X_2,X_3)X_1 - g(X_1,X_3)X_2\big] \\ &+ \quad (\frac{-1}{2n-1})\big[g(X_2,X_3)\xi - \eta(X_3)X_2\big]\eta(X_1) \\ &- \quad (\frac{-1}{2n-1})\big[g(X_1,X_3)\xi - \eta(X_3)X_1\big]\eta(X_2). \end{aligned} \tag{4.27}$$

We consider

$$ {}^*D(X_1,X_2)X_3 := \nabla_{X_1}{}^*T(X_2,X_3) - \nabla_{X_2}{}^*T(X_1,X_3). \tag{4.28}$$

Now, we consider can we conclude ${}^*D(X_1,X_2)X_3 = 0$ if ${}^*C(X_1,X_2)X_3 = 0$. So

$$\begin{aligned} \nabla_{X_1}{}^*T(X_2,X_3) \quad &= \quad \nabla_{X_1}T(X_2,X_3) + \mu\ \nabla_{X_1}g(X_2,X_3) + \lambda\ \nabla_{X_1}(\eta(X_2)\eta(X_3)) \\ &= \quad \nabla_{X_1}T(X_2,X_3) + \mu\ \nabla_{X_1}g(X_2,X_3) \\ &+ \quad \lambda\ \big[\big(\nabla_{X_1}\eta(X_2)\big)\ \eta(X_3) + \eta(X_2)\ \big(\nabla_{X_1}\eta(X_3)\big)\big], \end{aligned} \tag{4.29}$$

and

$$\begin{aligned} \nabla_{X_2}{}^*T(X_1,X_3) \quad &= \quad \nabla_{X_2}T(X_1,X_3) + \mu\ \nabla_{X_2}g(X_1,X_3) + \lambda\ \nabla_{X_2}(\eta(X_1)\eta(X_3)) \\ &= \quad \nabla_{X_2}T(X_1,X_3) + \mu\ \nabla_{X_2}g(X_1,X_3) \\ &+ \quad \lambda\ \big[\big(\nabla_{X_2}\eta(X_1)\big)\eta(X_3) + \eta(X_1)\ \big(\nabla_{X_2}\eta(X_3)\big)\big], \end{aligned} \tag{4.30}$$

where $\mu = \frac{2(1-n)}{(2n-1)}$ and $\lambda = \frac{-1}{2n-1}$. By putting (4.29) and (4.30) in (4.28), we have

$$\begin{aligned} {}^*D(X_1,X_2)X_3 &= D(X_1,X_2)X_3 \\ &+ \lambda\Big\{(\nabla_{X_1}\eta)(X_3)\eta(X_2) - (\nabla_{X_2}\eta)(X_3)\eta(X_1)\Big\}. \end{aligned} \tag{4.31}$$

Theorem 4.3. *Let M be a $2n+1$-dimension manifold with $n > 1$ and (g, η, ξ, φ) is a Kenmotsu structure on M. Then $D(X_1, X_2)X_3 = 0$ and ${}^*D(X_1, X_2)X_3 = 0$ do not hold at the same time.*

Proof. From (4.31), it is easily proved. □

Example 4.1. We consider the Kenmotsu manifold $(\mathbb{R}^3 - (0,0,0), g, \eta, \xi, \varphi)$, where the 1-form η, vector field ξ, Riemannian metric g, and $(1,1)$-tensor field φ respectively as follows

$$\eta = -\frac{1}{z}dz, \qquad \xi = -z\frac{\partial}{\partial z}, \qquad g = (dx)^2 + (dy)^2 + (dz)^2,$$

and $\varphi = dx \otimes \frac{\partial}{\partial y} - dy \otimes \frac{\partial}{\partial x}$. Also, the vector fields are given by

$$X_1 = z\frac{\partial}{\partial x}, \qquad X_2 = z\frac{\partial}{\partial y}, \qquad X_3 = \xi.$$

So, we have

$$\varphi X_1 = -X_2, \qquad \varphi X_2 = X_1, \qquad \varphi\xi = 0.$$

By conformally flat manifold $\mathbb{R}^3$, we have $C = 0$. By (4.16) and $C = 0$, then ${}^*C(X_1, X_2)X_3 = 0$. We know that since $C(X_1, X_2)X_3 = 0$, then $D(X_1, X_2)X_3 = 0$. Therefore, having (4.31), ${}^*D(X_1, X_2)X_3 = 0$.

5. ∗-conformal curvature of the cosymplectic manifolds

Let (g, η, ξ, φ) be a cosymplectic structure on M^{2n+1}. In [17], it is proved that for a cosymplectic manifold

$${}^*Ric(X_1, X_2) = Ric(X_1, X_2), \tag{5.1}$$

and

$${}^*r = r. \tag{5.2}$$

Theorem 5.1. *Suppose $(M^{2n+1}, g, \eta, \xi, \varphi)$ is a cosymplectic manifold. Then M is an η-Einstien manifold if and only if it is a ${}^*\eta$-Einstien manifold.*

Proof. It is easy to conclude from (5.1) that for the cosymplectic manifold, the η-Einstien manifold and ${}^*\eta$-Einstien manifold are equivalent. □

Substituting (5.1) and (5.2) into (2.1) yields

$${}^*C(X_1, X_2)X_3 = C(X_1, X_2)X_3. \tag{5.3}$$

Proposition 5.1. *In a cosymplectic manifold, the $*$-conformal curvature tensor obeys the relation:*

$$^*C(X_1, X_2)X_3 + {}^*C(X_2, X_3)X_1 + {}^*C(X_3, X_1)X_2 = 0.$$

The following results are obtained from (5.3).

Corollary 5.1. *Let $(M^{2n+1}, g, \eta, \xi, \varphi)$ be a cosymplectic manifold. Then M is a conformally flat if and only if it is a $*$-conformally flat.*

Corollary 5.2. *Let $(M^{2n+1}, g, \eta, \xi, \varphi)$ be a cosymplectic manifold. Then M is a ξ-conformally flat if and only if it is a ξ-$*$conformally flat.*

The conformal curvature tensor is zero in dimension 3. Thus we have:

Proposition 5.2. *For a 3-dimensional cosymplectic manifold, $^*C$ is identically zero.*

We consider $(0, 2)$-tensor field $^*T$ on cosymplectic manifold M^{2n+1} as follows:

$$^*T(X_1, X_2) = -\frac{^*Ric(X_1, X_2)}{2n-1} + \frac{^*r\ g(X_1, X_2)}{4n(2n-1)}. \tag{5.4}$$

By (5.1) and (5.2), we can

$$^*T(X_1, X_2) = T(X_1, X_2). \tag{5.5}$$

Also, define the conformal curvature tensor as follows:

$$\begin{aligned} {}^*C(X_1, X_2)X_3 = K(X_1, X_2)X_3 \quad &+ \quad {}^*T(X_2, X_3) \cdot X_1 - {}^*T(X_1, X_3) \cdot X_2 \\ &+ \quad g(X_2, X_3){}^*\hat{T}(X_1) - g(X_1, X_3){}^*\hat{T}(X_2), \end{aligned} \tag{5.6}$$

where $^*T(X_1, X_2) = g(^*\hat{T}(X_1), X_2)$. So $(0, 1)$-tensor field $^*\hat{T}$ is given by

$$^*\hat{T}(X_1) = \hat{T}(X_1). \tag{5.7}$$

By putting (5.5) and (5.7) in (5.6), we have

$$^*C(X_1, X_2)X_3 = C(X_1, X_2)X_3. \tag{5.8}$$

We consider

$$^*D(X_1, X_2)X_3 := \nabla_{X_1}{}^*T(X_2, X_3) - \nabla_{X_2}{}^*T(X_1, X_3). \tag{5.9}$$

On the other hand, we have

$$\nabla_{X_1}{}^*T(X_2, X_3) = \nabla_{X_1}T(X_2, X_3), \tag{5.10}$$

and

$$\nabla_{X_2}{}^{*}T(X_1, X_3) = \nabla_{X_2}T(X_1, X_3). \tag{5.11}$$

By putting (5.10) and (5.11) in (5.9), we have

$$^{*}D(X_1, X_2)X_3 = D(X_1, X_2)X_3. \tag{5.12}$$

We know that if $C = 0$ for a $2n+1$-dimension cosymplectic manifold with $n \geqslant 1$, then $D = 0$. Now, if we assume $^{*}C = 0$, then according to (5.12), the following theorem is obtained.

Theorem 5.2. *Let $(M^{2n+1}, g, \eta, \xi, \varphi)$ be a $2n+1$-dimension cosymplectic manifold with $n \geqslant 1$. If M is a $*$-conformally flat manifold, then $^{*}D = 0$.*

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