

SOME RESULTS RELATED TO SOFT TOPOLOGICAL SPACES

E. Peyghan, B. Samadi and A. Tayebi

Abstract. The notion of soft sets is introduced as a general mathematical tool for dealing with uncertainty. In this paper, we consider the concepts of soft compactness, countably soft compactness and obtain some results. We study some soft separation axioms that have been studied by Min and Shabir-Naz. By constructing a special soft topological space, we show that some classical results in general topology are not true about soft topological spaces, for instance, every compact Housdorff space need not be normal.

Keywords: Soft closed, Soft compact space, Soft open, Soft topological spaces.

1. Introduction

During recent years General Topology has been developed by many mathematicians. The theory of generalized topological spaces, which was founded by Á. Császár is one of these developments [6]. Recently, in [16] Shabir-Naz introduced and studied the concepts of soft topological spaces and some related concepts. The generalized topology is different from topology by its axioms (A collection of subsets of X is a generalized topology on X if and only if it contains an empty set and an arbitrary union of its elements). Soft topology, however, is based on soft sets theory and not sets.

Some notions in Mathematics can be considered as mathematical tools for dealing with uncertainties, namely, theory of fuzzy sets, theory of intuitionistic fuzzy sets, theory of vague sets, theory of rough sets, etc. But all of these theories have their own difficulties. In [11], Molodtsov introduced the concept of a soft set in order to solve complicated problems in economics, engineering and environmental areas because no mathematical tool can successfully deal with the various kinds of uncertainties in these problems. He successfully applied the soft theory in several directions, such as game theory, probability, Perron integration, Riemann integration and theory of measurement [11, 12].

In [9], Maji-Biswas-Roy defined and studied operations of soft sets. Then Pei-Miao [14] and Chen [5] improved the work of Maji-Biswas-Roy [8, 9]. The

properties and applications of soft set theory were studied increasingly in [1]. In [3], Çağman-Enginoglu redefined the operations of soft sets and constructed a uni-int decision-making method by using these new operations, and developed soft set theory. Then to make compaction with the operations of soft sets easy, they presented soft matrix theory and set up the soft max ϵ min decision-making method [4]. These decision-making methods can be successfully applied to many problems that contain uncertainties. In [16], the authors studied some concepts related to soft spaces such as soft interior, soft subspace and soft separation axioms. Recently, Aygunoglu-Aygun introduced soft product topology and defined a version of compactness in soft spaces named soft compactness [2].

In this paper, we consider the concepts of soft compactness and countably soft compact and get some results. Then, we study some soft separation axioms that have been studied by Min and Shabir-Naz. By constructing some examples we show that some classical results in General Topology are not true about soft topological spaces, for instance, every compact Housdorff space need not be normal.

2. Preliminaries

In this section, we recall some definitions and concepts discussed in [7, 10, 16, 17]. Let U be an initial universe and E be a set of parameters. Let $\mathbb{P}(U)$ denote the power set of U and A be a nonempty subset of E . A pair (F, A) is called a *soft set* over U , where F is a mapping given by $F : A \rightarrow \mathbb{P}(U)$. For two soft sets (F, A) and (G, B) over common universe U , we say that (F, A) is a *soft subset* (G, B) if $A \subseteq B$ and $F(e) \subseteq G(e)$, for all $e \in A$. In this case, we write $(F, A) \widetilde{\subseteq} (G, B)$ and (G, B) is said to be a *soft super set* of (F, A) . Two soft sets (F, A) and (G, B) over a common universe U are said to be *soft equal* if $(F, A) \widetilde{\subseteq} (G, B)$ and $(G, B) \widetilde{\subseteq} (F, A)$. A soft set (F, A) over U is called a *null soft set*, denoted by Φ_A , if for each $e \in A$, $F(e) = \emptyset$. Similarly, it is called *absolute soft set*, denoted by \widetilde{U} , if for each $e \in A$, $F(e) = U$.

The *union* of two soft sets (F, A) and (G, B) over the common universe U is the soft set (H, C) , where $C = A \cup B$ and for each $e \in C$,

$$H(e) = \begin{cases} F(e) & e \in A - B \\ G(e) & e \in B - A \\ F(e) \cup G(e) & e \in A \cap B \end{cases}$$

We write $(F, A) \cup (G, B) = (H, C)$. Moreover, the *intersection* (H, C) of two soft sets (F, A) and (G, B) over a common universe U , denoted by $(F, A) \cap (G, B)$, is defined as $C = A \cap B$ and $H(e) = F(e) \cap G(e)$ for each $e \in C$. The *difference* (H, E) of two soft sets (F, E) and (G, E) over X , denoted by $(F, E) - (G, E)$, is defined as $H(e) = F(e) - G(e)$, for each $e \in E$. Let Y be a nonempty subset of X . Then \widetilde{Y} denotes the soft set (Y, E) over X where $Y(e) = Y$, for each $e \in E$. In particular, (X, E) will be denoted by \widetilde{X} . Let (F, E) be a soft set over X and $x \in X$. We say that $x \in (F, E)$, whenever $x \in F(e)$, for each $e \in E$ [15].

The relative complement of a soft set (F, A) is denoted by $(F, A)'$ and is defined by $(F, A)' = (F', A)$ where $F' : A \rightarrow \mathbb{P}(U)$ is defined by following

$$F'(e) = U - F(e), \quad \forall e \in A.$$

In this paper, for convenience, let $SS(X)_E$ be the family of soft sets over X with set of parameters E . We will apply two next propositions so much in the proofs.

Proposition 2.1. *Let (F, E) , (G, E) , (H, E) and (I, E) be soft sets in $SS(X)_E$. Then the following holds.*

- (i) $(F, E) \widetilde{\subseteq} (G, E)$ if and only if $(F, E) \cap (G, E) = (F, E)$;
- (ii) $(F, E) \widetilde{\subseteq} (G, E)$, (H, E) if and only if $(F, E) \widetilde{\subseteq} (G, E) \cap (H, E)$;
- (iii) If $(F, E) \widetilde{\subseteq} (H, E)$ and $(G, E) \widetilde{\subseteq} (I, E)$, then $(F, E) \cup (G, E) \widetilde{\subseteq} (H, E) \cup (I, E)$;
- (iv) $(F, E) \cap (F, E)' = \Phi_E$;
- (v) $(F, E) \cap (G, E) = \Phi_E$ if and only if $(F, E) \widetilde{\subseteq} (G, E)'$;
- (vi) $(F, E) \widetilde{\subseteq} (G, E)$ if and only if $(G, E)' \widetilde{\subseteq} (F, E)'$.

Proof. Here, we only prove the (iii). Let $(F, E) \cup (G, E) = (J, E)$ and $(H, E) \cup (I, E) = (K, E)$. Since $(F, E) \widetilde{\subseteq} (H, E)$ and $(G, E) \widetilde{\subseteq} (I, E)$; then

$$F(e) \subseteq H(e) \text{ and } G(e) \subseteq I(e), \quad \forall e \in E.$$

Therefore

$$J(e) = F(e) \cup G(e) \subseteq H(e) \cup I(e) = K(e).$$

Hence $(J, E) \widetilde{\subseteq} (K, E)$. \square

Also we can obtain the following easily.

Proposition 2.2. *Let (F, E) , (G, E) and (H, E) be soft sets and $\{(F_\alpha, E)\}_{\alpha \in J}$ be a family of soft sets in $SS(X)_E$. Then the following holds.*

- (i) $(F, E) \cap (F, E)' = \Phi_E$;
- (ii) $(F, E) \cup \Phi_E = (F, E)$;
- (iii) $(F, E) \cap (\cup_{\alpha \in J} (F_\alpha, E)) = \cup_{\alpha \in J} ((F, E) \cap (F_\alpha, E))$;
- (iv) If $(F, E) \widetilde{\subseteq} (G, E)$ and $(G, E) \cap (H, E) = \Phi_E$, then $(F, E) \cap (H, E) = \Phi_E$;
- (v) $\Phi'_E = \widetilde{X}$;
- (vi) $\widetilde{X}' = \Phi_E$.

Let τ be the collection of soft sets over X . Then τ is called a soft topology on X if τ satisfies the following axioms:

- (i) Φ_E and \widetilde{X} belong to τ .
- (ii) The union of any number of soft sets in τ belongs to τ .
- (iii) The intersection of any two soft sets in τ belongs to τ .

The triple (X, τ, E) is called a soft topological space over X . The members of τ are said to be soft open in X , and the soft set (F, E) is called soft closed in X if its relative component $(F, E)'$ belongs to τ .

The proof of the following proposition is an easy application of De Morgan's laws with the definition of a soft topology on X (see Proposition 3.3 of [17]).

Proposition 2.3. Let (X, τ, E) be a soft space over X . Then

- 1) Φ_E and \widetilde{X} are closed soft sets over X ;
- 2) The intersection of any number of soft closed sets is a soft closed set over X ;
- 3) The union of any two soft closed sets is a soft closed set over X .

3. Soft Compactness

In this section, we are going to introduce the concept of soft compactness concerning soft topological spaces and study some properties related to these spaces (also, see [17]).

A family $\mathcal{A} = \{(F_\alpha, E)\}_{\alpha \in J}$ of soft sets is a cover of a soft set (F, E) if

$$(F, E) \widetilde{\subseteq} \bigcup_{\alpha \in J} (F_\alpha, E).$$

It is a soft open cover if each member of \mathcal{A} is a soft open set. A subcover of \mathcal{A} is a subfamily of \mathcal{A} which is also a cover. A soft topological space (X, τ, E) is said to be soft compact if each soft open cover of (X, E) has a finite subcover.

Let (X, τ_1, E) and (X, τ_2, E) be soft topological spaces. If $\tau_1 \subseteq \tau_2$, then τ_2 is soft finer than τ_1 . If $\tau_1 \subseteq \tau_2$ or $\tau_2 \subseteq \tau_1$, then τ_1 is soft comparable with τ_2 . Then, we have the following.

Proposition 3.1. Let (X, τ_2, E) be a soft compact space and $\tau_1 \subseteq \tau_2$. Then (X, τ_1, E) is soft compact.

Proof. Let $\{(F_\alpha, E)\}_{\alpha \in J}$ be a soft open cover of \widetilde{X} by soft open sets of (X, τ_1, E) . Since $\tau_1 \subseteq \tau_2$, then $\{(F_\alpha, E)\}_{\alpha \in J}$ is a soft open cover of \widetilde{X} by soft open sets of (X, τ_2, E) . But (X, τ_2, E) is soft compact. Therefore

$$(X, E) \widetilde{\subseteq} (F_{\alpha_1}, E) \cup \dots \cup (F_{\alpha_n}, E),$$

for some $\alpha_1, \dots, \alpha_n \in J$. Hence (X, τ_1, E) is soft compact. \square

Let (F, E) be a soft set over X and Y be a nonempty subset of X . Then the sub-soft set of (F, E) over Y denoted by $({}^Y F, E)$ is defined as follows

$${}^Y F(e) = Y \cap F(e),$$

for each $e \in E$. In other words $({}^Y F, E) = \widetilde{Y} \cap (F, E)$. Now, suppose that (X, τ, E) is a soft topological space over X and Y is a nonempty subset of X . Then

$$\tau_Y = \{({}^Y F, E) \mid (F, E) \in \tau\},$$

is said to be soft relative topology on Y and (Y, τ_Y, E) is called a soft subspace of (X, τ, E) . Here, we exhibit a criterion that implies \widetilde{Y} is soft compact by soft open covers of \widetilde{Y} , that all of members are soft open sets in X .

Theorem 3.1. *Let (Y, τ_Y, E) be a soft subspace of a soft space (X, τ, E) . Then (Y, τ_Y, E) is soft compact if and only if every cover of \widetilde{Y} by soft open sets in X contains a finite subcover.*

Proof. Let (Y, τ_Y, E) be soft compact and $\{(F_{\alpha}, E)\}_{\alpha \in J}$ be a cover of \widetilde{Y} by soft open sets in X . By Propositions 2.1 and 2.2, we can see that $\{{}^Y F_{\alpha}, E\}_{\alpha \in J}$ is a soft open cover of \widetilde{Y} . Therefore

$$(Y, E) \widetilde{\subseteq} ({}^Y F_{\alpha_1}, E) \cup \dots \cup ({}^Y F_{\alpha_n}, E),$$

for some $\alpha_1, \dots, \alpha_n \in J$. This implies that $\{(F_{\alpha_i}, E)\}_{i=1}^n$ is a subcover of \widetilde{Y} by soft open sets in X . Conversely, let $\{{}^Y F_{\alpha}, E\}_{\alpha \in J}$ be a soft open cover of \widetilde{Y} . It is easy to see that $\{(F_{\alpha}, E)\}_{\alpha \in J}$ is a cover of \widetilde{Y} by soft open sets in X . Then we can write

$$\widetilde{Y} \widetilde{\subseteq} (F_{\alpha_1}, E) \cup \dots \cup (F_{\alpha_n}, E),$$

for some $\alpha_1, \dots, \alpha_n \in J$. Therefore $\{{}^Y F_{\alpha_i}, E\}_{i=1}^n$ is a subcover of \widetilde{Y} . Hence (Y, τ_Y, E) is soft compact. \square

Definition 3.1. A soft space (X, τ, E) is said to be soft Hausdorff if for each pair x, y of distinct points of X , there exist disjoint soft open sets containing x and y , respectively.

Theorem 3.2. *Every soft compact subspace of a soft Hausdorff space is soft closed.*

Proof. Let (Y, τ_Y, E) be a soft compact subspace of soft Hausdorff space (X, τ, E) . Let $x \in (X, E) - (Y, E)$. Then for all $y \in (Y, E)$, $x \neq y$. Therefore, there exist soft open sets (U_y, E) and (U_{xy}, E) containing x and y , respectively such that $(U_y, E) \cap (U_{xy}, E) = \Phi_E$. Obviously, $\{(U_{xy}, E)\}_{y \in Y}$ is a cover of \widetilde{Y} by soft open sets in X . By Theorem 3.1, we have $(Y, E) \widetilde{\subseteq} (U_{xy_1}, E) \cup \dots \cup (U_{xy_n}, E)$ for some $y_1, \dots, y_n \in Y$. Now, $x \in (U_{y_1}, E) \cap \dots \cap (U_{y_n}, E) = (U_x, E)$ and Proposition 2.2 implies that $(U_x, E) \cap (Y, E) = \Phi_E$. Hence $x \in (U_x, E) \subseteq (X, E) - (Y, E)$. Then $(X, E) - (Y, E) = \bigcup_{x \in X - Y} (U_x, E)$. Therefore $(X, E) - (Y, E)$ is soft open. Hence (Y, E) is soft closed. \square

Using Propositions 2.1 and 2.2, we are going to prove that every soft closed subspace of a soft compact space is soft compact.

Theorem 3.3. *Every soft closed subset of a soft compact space is soft compact.*

Proof. Let (Y, τ_Y, E) be a soft subspace of a soft compact space (X, τ, E) such that (Y, E) is a soft closed in X . Let $\{(F_{\alpha}, E)\}_{\alpha \in J}$ be a cover of \widetilde{Y} by soft open sets in X . Since $(Y, E)'$ is a soft open set in X then Propositions 2.1 and Proposition 2.2 show that $\{(F_{\alpha}, E)\}_{\alpha \in J} \cup \{(Y', E)\}$ form a soft open cover of \widetilde{X} . Therefore

$$(X, E) \widetilde{\subseteq} (F_{\alpha_1}, E) \cup \dots \cup (F_{\alpha_n}, E) \cup (Y', E),$$

for some $\alpha_1, \dots, \alpha_n \in J$. Applying the propositions 2.1 and 2.2, we can see that $\{(F_{\alpha_i}, E)\}_{i=1}^n$ is a subcover of \widetilde{Y} . This completes the proof. \square

Let (X, τ, E) be a soft topological space and $\mathcal{B} \subseteq \tau$. If every element of τ can be written as a union of elements of \mathcal{B} , then \mathcal{B} is called a soft basis for the soft topology τ . Each element of \mathcal{B} is called a soft basis element.

We can characterize soft compact spaces in terms of basis elements as follows:

Theorem 3.4. *A soft topological space (X, τ, E) is soft compact if and only if there is a soft basis \mathcal{B} for τ such that every cover of \widetilde{X} by elements of \mathcal{B} has a finite subcover.*

Proof. Let (X, τ, E) be soft compact. Obviously, τ is a soft basis for τ . Therefore, every cover of \widetilde{X} by elements of τ has finite subcover. Conversely, let $\{(U_{\alpha}, E)\}_{\alpha \in J}$ be a soft open cover of \widetilde{X} . We can write (U_{α}, E) as a union of basis elements, for each $\alpha \in J$. These elements form a soft open cover of \widetilde{X} such as $\{(F_{\mathcal{B}}, E)\}_{\mathcal{B} \in I}$. Therefore $\widetilde{X} = (F_{\mathcal{B}_1}, E) \cup \dots \cup (F_{\mathcal{B}_n}, E)$, for some $\mathcal{B}_1, \dots, \mathcal{B}_n \in I$. Let $(F_{\mathcal{B}_i}, E) \widetilde{\subseteq} (U_{\alpha_i}, E)$, for each $1 \leq i \leq n$. This implies that $\{(U_{\alpha_i}, E)\}_{i=1}^n$ is a finite subcover of \widetilde{X} . Hence, (X, τ, E) is soft compact. \square

Remark 3.1. *Clearly, a soft set is not a set. Indeed, the differences between soft topological spaces and topological spaces arise from this fact. In a sense, when $|E| = 1$, a soft set (F, E) behaves similar to a set. In fact, in this case the soft set (F, E) is the same as the set $F(e)$, where $E = \{e\}$. Therefore when $|E| = 1$, soft topological spaces are the same as topological spaces. Nevertheless, in this paper we will see some differences between these two concepts when $|E| \geq 2$.*

Now, we consider the countably soft compact spaces in soft topology. A soft topological space (X, τ, E) is said to be *countably soft compact* if every countable soft open cover of \widetilde{X} contains a finite subcover of \widetilde{X} . Obviously, every soft compact space is countably soft compact but the following example shows that the converse is not true in general.

Example 3.1. We consider the (topological) space S_Ω , the minimal uncountable well-ordered set with order topology (see [13]). Let $X = S_\Omega$, $E = \{e\}$ and $\tau = \{(F, E) | F(e) \text{ is open in } S_\Omega\}$. Considering Remark 3.1, the soft topological space (X, τ, E) is countably soft compact but not soft compact.

There is a criterion for a soft space to be countable soft compact in terms of soft closed sets rather than soft open sets. First we have a definition.

A collection \mathcal{A} of soft set is said to have the finite intersection property if for every finite sub-collection $\{(A_1, E), \dots, (A_n, E)\}$ of \mathcal{A} , the intersection $(A_1, E) \cap \dots \cap (A_n, E)$ is non-null.

Theorem 3.5. *A soft topological space is countably soft compact if and only if every countable family of soft closed sets with the finite intersection property has a non-null intersection.*

Proof. Let the soft space (X, τ, E) be countably soft compact. Let the family $\{(F_n, E)\}_{n=1}^\infty$ of soft closed sets have the finite intersection property. If $\bigcap_{n=1}^\infty (F_n, E) = \Phi_E$ by Proposition 2.2, $\{(F_n, E)'\}_{n=1}^\infty$ is a countable soft open cover of \widetilde{X} . Therefore $\widetilde{X} = (F_{n_1}, E)' \cup \dots \cup (F_{n_k}, E)'$, for some $n_1, \dots, n_k \in N$. Now, De Morgan's laws and Proposition 2.2 imply that $(F_{n_1}, E) \cap \dots \cap (F_{n_k}, E) = \Phi_E$. This is a contradiction. Conversely, Let $\{(F_n, E)\}_{n=1}^\infty$ be a countable soft open cover of \widetilde{X} without any subcover. Then $\{(F_n, E)'\}_{n=1}^\infty$ is a family of soft closed sets over X such that $\bigcap_{n=1}^\infty (F_n, E)' = \Phi_E$. Let n_1, \dots, n_k be arbitrary positive integers. If $(F_{n_1}, E)' \cap \dots \cap (F_{n_k}, E)' = \Phi_E$ then $\widetilde{X} = (F_{n_1}, E) \cup \dots \cup (F_{n_k}, E)$, that is impossible. Therefore $(F_{n_1}, E)' \cap \dots \cap (F_{n_k}, E)' \neq \Phi_E$, for each $n_1, \dots, n_k \in N$. This shows that $\{(F_n, E)'\}_{n=1}^\infty$ have the finite intersection property. Therefore $\bigcap_{n=1}^\infty (F_n, E)' \neq \Phi_E$. This is a contradiction. \square

An immediate result of previous theorem is the following.

Corollary 3.1. *A soft space (X, τ, E) is countably soft compact if and only if every nested sequence $(F_1, E) \supseteq (F_2, E) \supseteq \dots$ of nonnull soft closed sets over X has a non-null intersection.*

Proof. Let (X, τ, E) is countably soft compact. The collection $\{(F_n, E)\}_{n=1}^\infty$ have the finite intersection property. Therefore $\bigcap_{n=1}^\infty (F_n, E) \neq \Phi_E$. Conversely, let $\{(C_n, E)\}_{n=1}^\infty$ be a collection of soft closed sets with the finite intersection property. By Proposition 2.3, we construct nested sequence $(F_1, E) \supseteq (F_2, E) \supseteq \dots$ of non-null soft closed sets by setting $(F_n, E) = (C_1, E) \cap \dots \cap (C_n, E)$, for each positive integer n . By the hypothesis $\bigcap_{n=1}^\infty (F_n, E) = \bigcap_{n=1}^\infty (C_n, E) \neq \Phi_E$. Now, Theorem 3.8 implies that (X, τ, E) is countably soft compact. \square

4. Soft Separation Axioms

In this section, we study some soft separation axioms that were studied in [10, 16]. First, we recall the definitions.

A soft topological space (X, τ, E) over X is called a soft T_0 -space if for each pair of distinct points, at least one has a neighborhood not containing the other, and a soft T_1 -space if for each pair of distinct points, each one has a neighborhood not containing the other. Also, a soft space (X, τ, E) is said to be a soft T_2 -space (or soft Hausdorff) if for each pair x, y of distinct points of X , there exist disjoint soft open sets containing x and y , respectively.

Obviously, every soft T_i -space ($i = 1, 2$) is a soft T_{i-1} -space. But by Remark 3.6 and general topology the converse is not true. In [16], the authors have shown that if (x, E) is a soft closed set in soft set (X, τ, E) , for all $x \in X$, then (X, τ, E) is soft T_1 , but the converse does not hold in general.

A soft space (X, τ, E) over X is called soft regular if for each soft closed set (G, E) and $x \in X$ such that $x \notin (G, E)$ there exist soft open sets (F_1, E) and (F_2, E) such that $x \in (F_1, E)$, $(G, E) \widetilde{\subseteq} (F_2, E)$ and $(F_1, E) \cap (F_2, E) = \Phi_E$. The soft space (X, τ, E) is said to be soft T_3 -space if it is soft regular and soft T_1 -space.

Before proceeding, we introduce the concept of soft closure of a soft set (see [7]). Let (X, τ, E) be a soft topological space and (F, E) be a soft set over X . Then the soft closure of (F, E) , denoted by $(\overline{F, E})$, is the intersection of all soft closed super sets of (F, E) . First, we prove the following.

Lemma 4.1. *Let (X, τ, E) be a soft topological space and (F, E) be a soft set over X . If $x \in (\overline{F, E})$, then every soft open set (G, E) containing x intersects (F, E) .*

Proof. Let $x \in (\overline{F, E})$. Let there be a soft open set (G, E) containing x such that $(F, E) \cap (G, E) = \Phi_E$. By Proposition 2.1, we have $(F, E) \widetilde{\subseteq} (G, E)'$. Therefore $(\overline{F, E}) \widetilde{\subseteq} (G, E)'$. Hence $x \in (G, E) \cap (G, E)'$. This is a contradiction. Therefore $(F, E) \cap (G, E) \neq \Phi_E$, for each soft open set (G, E) containing x . \square

The following example shows that the converse of Lemma 4.1 is not true.

Example 4.1. Suppose that the following sets are given: $X = \{h_1, h_2, h_3\}$, $E = \{e_1, e_2\}$ and $\tau = \{\Phi_E, \overline{X}, (F_1, E), (F_2, E), \dots, (F_{30}, E)\}$ where F_1, F_2, \dots, F_{30} are given in Example 9 of [16]. Then (X, τ) is a soft topological space over X . We consider the soft set (F_{25}, E) , where

$$F_{25}(e_1) = \{h_2\}, \quad F_{25}(e_2) = X.$$

It is easy to see that

$$(\overline{F_{25}, E}) = (F_{25}, E), \quad h_1 \notin (\overline{F_{25}, E}).$$

But for every soft open set (F, E) containing h_1 , we have $(F, E) \cap (F_{25}, E) \neq \Phi_E$.

Proposition 4.1. *Let (X, τ, E) be a soft regular space. Then, for each point x of X and a soft open set (F, E) containing x , there exists a soft open set (G, E) containing x such that $(\overline{G, E}) \widetilde{\subseteq} (F, E)$.*

Proof. Since $(F, E)'$ is a soft closed set not containing x , then there exist soft open sets (G, E) and (H, E) such that $x \in (G, E)$, $(F, E)' \widetilde{\subseteq} (H, E)$ and $(G, E) \cap (H, E) = \Phi_E$. Proposition 2.1 implies that $(G, E) \widetilde{\subseteq} (H, E)'$. Therefore $(\overline{G, E}) \widetilde{\subseteq} (H, E)' \widetilde{\subseteq} ((F, E)')' = (F, E)$. \square

The following example shows that the converse of Proposition 4.1 does not hold in general.

Example 4.2. Let $X = \{h\}$, $E = \{e_1, e_2\}$ and $\tau = \{\Phi_E, \widetilde{X}, (F_1, E), (F_2, E)\}$, where

$$F_1(e_1) = \{h\}, F_1(e_2) = \emptyset \text{ \& } F_2(e_1) = \emptyset, F_2(e_2) = \{h\}$$

It is easy to see that (X, τ, E) is not soft regular. Nevertheless, for $h \in X$ and soft open set \widetilde{X} containing h , \widetilde{X} itself is a soft open set containing h such that $h \in \overline{\widetilde{X}} \widetilde{\subseteq} \widetilde{X}$.

Now, we exhibit a necessary and sufficient condition for a soft space to be a soft regular space.

Theorem 4.1. *A soft space (X, τ, E) is soft regular if and only if for each $x \in X$ and soft closed set (F, E) not containing x , there is a soft open set (G, E) containing x such that $(\overline{G, E}) \cap (F, E) = \Phi_E$.*

Proof. Let (X, τ, E) be soft regular. There exist soft open sets (G, E) and (H, E) such that $x \in (G, E)$, $(F, E)' \widetilde{\subseteq} (H, E)$ and $(G, E) \cap (H, E) = \Phi_E$. Then $(G, E) \widetilde{\subseteq} (H, E)' \widetilde{\subseteq} (F, E)'$. This implies that $(\overline{G, E}) \widetilde{\subseteq} (H, E)' \widetilde{\subseteq} (F, E)'$. Therefore $(\overline{G, E}) \cap (F, E) = \Phi_E$.

Conversely, Proposition 2.1 implies that $(F, E) \widetilde{\subseteq} (\overline{G, E})'$. Therefore there is a soft open set $(\overline{G, E})'$ containing (F, E) such that $(G, E) \cap (\overline{G, E})' = \Phi_E$. This completes the proof. \square

A soft space topological space (X, τ, E) is said to be soft normal if for each soft closed sets (F, E) and (G, E) over X with null intersection there exist soft open sets (F_1, E) and (F_2, E) containing (F, E) and (G, E) respectively, such that $(F_1, E) \cap (F_2, E) = \Phi_E$. Also, a soft topological space (X, τ, E) is said to be a soft T_4 -space if it is soft normal and soft T_1 -space.

Theorem 4.2. *Let (X, τ, E) be a soft space. Let for each soft closed set (F, E) and soft open set (G, E) containing (F, E) there is a soft open set (H, E) containing (F, E) such that $(\overline{H, E}) \widetilde{\subseteq} (G, E)$. Then (X, τ, E) is soft normal.*

Proof. For each soft closed sets (F, E) and (I, E) with null intersection $(I, E)'$ is a soft open set containing (F, E) . Therefore there exists a soft open set (H, E) containing (F, E) such that $(\overline{H, E}) \widetilde{\subseteq} (I, E)'$. By Proposition 2.1, $(I, E) \widetilde{\subseteq} (\overline{H, E})'$. Since $(H, E) \widetilde{\subseteq} ((\overline{H, E})')'$, we have $(H, E) \cap (\overline{H, E})' = \Phi_E$. Hence (X, τ, E) is soft normal. \square

There is an obvious question to ask at this point. Is a soft T_4 -space a soft T_3 -space? The soft space (X, τ, E) in Example 4.2, shows that the answer is "NO". In fact, it is easy to see that (X, τ, E) is a soft T_4 -space and not a soft T_3 -space.

Remark 4.1. In Theorem 3.17 of [10], the following is proved:

Theorem. ([10]) Let (X, τ, E) be a soft topological space over X and $x \in X$. Then the following are equivalent:

- (1) (X, τ, E) is a soft regular space;
- (2) For each soft closed set (G, E) such that $(x, E) \cap (G, E) = \Phi_E$, there exist two soft open sets (F_1, E) and (F_2, E) such that $(x, E) \subseteq (F_1, E)$, $(G, E) \subseteq (F_2, E)$ and $(F_1, E) \cap (F_2, E) = \Phi_E$.

By Example 4.2, we can see that this theorem is incorrect. In fact, the soft space (X, τ, E) in this example satisfies (2), but it is not soft regular. We note that $(x, E) \cap (G, E) = \Phi_E$ is not equivalent to $x \notin (G, E)$ but $(x, E) \not\subseteq (G, E)$ is. Therefore, we must replace the condition $(x, E) \not\subseteq (G, E)$ with $(x, E) \cap (G, E) = \Phi_E$ in Theorem 3.17 of [10].

Remark 4.2. In Theorem 3.25 of [10], the following is proved:

Theorem. ([10]) Let (X, τ, E) be a soft topological space over X . If (X, τ, E) is a soft normal space and (x, E) is a soft closed set for each $x \in X$, then (X, τ, E) is a soft T_3 -space.

This theorem is incorrect. The soft space (X, τ, E) in Example 4.2 satisfies the conditions of the theorem, but it is not a soft T_3 -space.

There are some familiar results on the applications of compactness in separation axioms in General Topology such as: *Every compact Hausdorff space is normal*. But it is not true about soft topological spaces. Consider the following example.

Example 4.3. Let $X = \{h\}$, $E = \{e_i\}_{i=1}^5$ and $\tau = \{\Phi_E, \widetilde{X}, (F_1, E), (F_2, E), (F_3, E)\}$, where

$$F_1(e_1) = \emptyset, F_1(e_2) = X, F_1(e_3) = \emptyset, F_1(e_4) = X, F_1(e_5) = \emptyset;$$

$$F_2(e_1) = X, F_2(e_2) = X, F_2(e_3) = X, F_2(e_4) = \emptyset, F_2(e_5) = X;$$

$$F_3(e_1) = \emptyset, F_3(e_2) = X, F_3(e_3) = \emptyset, F_3(e_4) = \emptyset, F_3(e_5) = \emptyset.$$

It is easy to see that (X, τ, E) is not soft normal. Nevertheless, it is soft compact.

It is remarkable that a soft compact Hausdorff space need not be soft normal, even if we consider (X, τ, E) as a soft regular space. Indeed, the example 4.2 is a counterexample.

5. Conclusion

Here we investigated soft topological spaces with an emphasis on soft compactness. A soft space $(X, \mathcal{C}\tau, E\mathcal{C})$ can be considered as a topological space when $|E| = 1$. Therefore we can consider the results of soft spaces for topological spaces. But there are some classical results in general topology that they fail if we use soft topology. For example, a soft $T\mathcal{C}_4\mathcal{C}$ -space need not be a soft $T\mathcal{C}_3\mathcal{C}$ -space and a soft compact Hausdorff space need not be a soft $T\mathcal{C}_4\mathcal{C}$ -space.

REFERENCES

1. M.I. ALI, F. FENG, X. LIU, W.K. MIN AND M. SHABIR: *On some new operations in soft set theory*. Comput. Math. Appl. **57** (2009), 1547-1553.
2. A. AYGUNOGLU AND H. AYGUN: *Some notes on soft topological spaces*. NEURAL. COMPUT. APPL. (2011), 1-7.
3. N. ÇAĞMAN AND S. ENGINOGLU: *Soft set theory and uniçint decision making*. EUROP. J. OPERAT. RES. **207** (2010), 848-855.
4. N. ÇAĞMAN AND S. ENGINOGLU: *Soft matrix theory and its decision making*. Comput. Math. Appl., **59** (2010), 3308-3314.
5. D. CHEN: *The parametrization reduction of soft sets and its applications*. Computers and Mathematics with Applications. **49** (2005), 757-763.
6. Á. CSÁSZÁR: *Generalized topology, generalized countiniuty*. Acta. Math. Hungar. **96** (2002), 351-357.
7. S. HUSSAIN AND B. AHMAD: *Some properties of soft topological spaces*. Comput. Math. Appl. **62** (2011), 4058-4067.
8. P. K. MAJI, R. BISWAS AND A. R. ROY: *An application of soft sets in a decision making problem*. Computers and Mathematics with Applications. **44** (2002), 1077-1083
9. P. K. MAJI, R. BISWAS AND A. R. ROY: *Soft set theory*. Comput. Math. Appl. **45** (2003), 555-562.
10. W. K. MIN: *A note on soft topological spaces*. Comput. Math. Appl. **62** (2011), 3524-3528.
11. D. A. MOLODTSOV: *Soft set theory-first results*. Comput. Math. Appl. **37** (1999), 19-31.
12. D. A. MOLODTSOV, V. Y. LEONOV AND D. V. KOVKOV: *Soft sets technique and its application*. Nechetkie Sistemy i Myagkie Vychisleniya. **1** (2006), 8-39.
13. J. MUNKRES: *Topology*. Prentice-Hall, Englewood Cliffs, NJ, 2000.
14. D. PIE AND D. MIAO: *From soft sets to information systems*. in: Granular computing, IEEE Inter. Conf. 2005, pp. 617-621.
15. E. PEYGHAN, B. SAMADI AND A. TAYEBI: *About soft topological spaces*. Journal of New Results in Science. **2** (2013), 60-75.
16. M. SHABIR AND M. NAZ: *On soft topological spaces*. Comput. Math. Appl. **61** (2011), 1786-1799.
17. I. ZORLUTUNA, M. AKDAG, W. K. MIN AND S. ATMACA: *Remarks on soft topological spaces*. Annal. Fuzzy. Math. Inform. **3** (2012), 171- 185.

Esmail Peyghan and Babak Samadi
Department of Mathematics, Faculty of Science
Arak University
Arak 38156-8-8349, Iran
e-peyghan@araku.ac.ir, b-samadi@araku.ac.ir

Akbar Tayebi
Department of Mathematics, Faculty of Science
University of Qom
Qom, Iran
akbar.tayebi@gmail.com