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A CLASS OF β-KENMOTSU MANIFOLD ADMITTING GENERALIZED TANAKA-WEBSTER CONNECTION

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Abstract. The objective of this paper is to investigate a class of β-Kenmotsu manifold admitting generalized Tanaka-Webster connection. We use the connection $\widetilde{\widetilde{\nabla}}$ to investigate some curvature properties in the manifold. Here we study the projective and ζ-projectively flat curvature tensors admitting the connection $\widetilde{\widetilde{\nabla}}$ in the manifold. Further, we discuss recurrent condition, conharmonic curvature tensor and Weyl conformal curvature tensor in the manifold admitting the connection $\widetilde{\widetilde{\nabla}}$. Likewise, we demonstrate Ricci pseudo-symmetric, quasi-concircularly flat and ζ-quasi-concircularly flat β-Kenmotsu manifold admitting the connection $\widetilde{\widetilde{\nabla}}$. Finally, we give an example of a β-Kenmotsu manifold admitting the connection $\widetilde{\widetilde{\nabla}}$ which support our results.
Keywords: β-Kenmotsu manifold, generalized Tanaka-Webster connection, projective curvature tensor, conharmonic curvature tensor, Weyl conformal curvature tensor, quasi-concircularly flat, recurrent.

1. Introduction

Tanno [19] introduced the Tanaka-Webster connection which is a generalization of the well-known connection defined by Tanaka [18] and Webster [21]. This connection is canonical affine connection defined on a non-degenerate pseudo-Hermitian

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CR-manifold [18,21]. Also, this connection coincides with the Tanaka-Webster connection if the associated CR-structure is integrable. Using the generalized Tanaka-Webster connection, few geometers have studied some characterizations of real hypersurfaces in complex space forms [17]. Recently many authors [8, 10, 12, 13] have studied generalized Tanaka-Webster connection in Kenmotsu manifold. A trans-Sasakian manifold of type $(0,0),(\alpha,0)$ and $(0,\beta)$ are called the cosympletic, α-Sasakian and β-Kenmotsu manifold respectively, where α and β are the scalar functions [3]. In particular if $\alpha=0$, $\beta=1$; $\alpha=0$, β is constant and $\alpha=1$, $\beta=0$ then the trans-Sasakian manifold are said to be a Kenmotsu manifold; a class of β-Kenmotsu manifold and Sasakian manifold respectively [9]. β-Kenmotsu manifold have been studied by several authors like Shaikh and Hui [15,16], De [4] and many others.

Motivated by above studies, the present work has been classified as follows: After introduction, we recall basic formulas and results of β-Kenmotsu manifold in section 2. In section 3. we study some curvature tensors and its properties with respect to the connection $\widetilde{\nabla}$ in the manifold. Section 4. deals with the study of recurrent condition, conharmonic curvature tensor and Weyl conformal curvature tensor in the manifold admitting the connection $\widetilde{\nabla}$. In section 5. we discuss Ricci pseudo-symmetric, quasi-concircularly flat and ξ-quasi-concircularly flat β-Kenmotsu manifold admitting the connection $\widetilde{\nabla}$. Finally, in section 6. we give an example of a 3-dimensional β-Kenmotsu manifold admitting the connection $\widetilde{\nabla}$ which verify our results.

2. Preliminaries

A $(2n+1)$-dimensional differentiable manifold $\mathfrak{M}^{2n+1}$ is said to be an almost contact metric manifold [2] if it admits a $(1,1)$-tensor field φ, a vector field ζ, a 1-form η and a Riemannian metric g which satisfies

$$\varphi^2(\mathcal{E}_1)=-\mathcal{E}_1+\eta(\mathcal{E}_1)\zeta, \quad \eta(\zeta)=1, \tag{2.1}$$

$$\varphi\zeta=0, \quad \eta(\varphi\mathcal{E}_1)=0, \quad g(\mathcal{E}_1,\zeta)=\eta(\mathcal{E}_1), \tag{2.2}$$

$$g(\varphi\mathcal{E}_1,\varphi\mathcal{E}_2)=g(\mathcal{E}_1,\mathcal{E}_2)-\eta(\mathcal{E}_1)\eta(\mathcal{E}_2), \quad g(\varphi\mathcal{E}_1,\mathcal{E}_2)=-g(\mathcal{E}_1,\varphi\mathcal{E}_2) \tag{2.3}$$

$\forall\ \mathcal{E}_1,\mathcal{E}_2\in\mathfrak{X}(\mathfrak{M})$; where $\mathfrak{X}(\mathfrak{M})$ is a set of all smooth vector fields on $\mathfrak{M}$.

An almost contact metric manifold $\mathfrak{M}^{2n+1}(\varphi,\zeta,\eta,g)$ is said to be β-Kenmotsu manifold if the following conditions hold:

$$\nabla_{\mathcal{E}_1}\zeta=\beta[\mathcal{E}_1-\eta(\mathcal{E}_1)\zeta], \tag{2.4}$$

and

$$(\nabla_{\mathcal{E}_1}\varphi)\mathcal{E}_2=\beta[g(\varphi\mathcal{E}_1,\mathcal{E}_2)\zeta-\eta(\mathcal{E}_2)\varphi\mathcal{E}_1], \tag{2.5}$$

where ∇ denotes the Riemannian connection of g. If $\beta = 1$ then β-Kenmotsu manifold becomes Kenmotsu manifold and if β is constant then it becomes a class of β-Kenmotsu manifold.

In a class of β-Kenmotsu manifold the following relations hold [9]

$$(\nabla_{\mathcal{E}_1}\eta)\mathcal{E}_2 = \beta[g(\mathcal{E}_1,\mathcal{E}_2) - \eta(\mathcal{E}_1)\eta(\mathcal{E}_2)], \tag{2.6}$$

$$\eta(\mathfrak{R}(\mathcal{E}_1,\mathcal{E}_2)\mathcal{E}_3) = \beta^2[g(\mathcal{E}_1,\mathcal{E}_3)\eta(\mathcal{E}_2) - g(\mathcal{E}_2,\mathcal{E}_3)\eta(\mathcal{E}_1)], \tag{2.7}$$

$$\mathfrak{R}(\mathcal{E}_1,\mathcal{E}_2)\zeta = \beta^2[\eta(\mathcal{E}_1)\mathcal{E}_2 - \eta(\mathcal{E}_2)\mathcal{E}_1], \tag{2.8}$$

$$\mathfrak{R}(\zeta,\mathcal{E}_1)\mathcal{E}_2 = \beta^2[\eta(\mathcal{E}_2)\mathcal{E}_1 - g(\mathcal{E}_1,\mathcal{E}_2)\zeta], \tag{2.9}$$

$$\mathfrak{R}(\zeta,\mathcal{E}_1)\zeta = \beta^2[\mathcal{E}_1 - \eta(\mathcal{E}_1)\zeta], \tag{2.10}$$

$$\mathcal{S}(\mathcal{E}_1,\mathcal{E}_2) = g(\mathcal{Q}\mathcal{E}_1,\mathcal{E}_2), \tag{2.11}$$

$$\mathcal{S}(\mathcal{E}_1,\zeta) = -2n\beta^2\eta(\mathcal{E}_1), \tag{2.12}$$

$$\mathcal{Q}\mathcal{E}_1 = -2n\beta^2\mathcal{E}_1, \tag{2.13}$$

$$\mathcal{Q}\zeta = -2n\beta^2\zeta, \tag{2.14}$$

$$\mathcal{S}(\varphi\mathcal{E}_1,\varphi\mathcal{E}_2) = g(\mathcal{Q}\varphi\mathcal{E}_1,\varphi\mathcal{E}_2), \tag{2.15}$$

Using (2.3), (2.11), (2.13) and $\mathcal{Q}\varphi = \varphi\mathcal{Q}$ in (2.15), we have

$$\mathcal{S}(\varphi\mathcal{E}_1,\varphi\mathcal{E}_2) = \mathcal{S}(\mathcal{E}_1,\mathcal{E}_2) - 2n\beta^2\eta(\mathcal{E}_1)\eta(\mathcal{E}_2), \tag{2.16}$$

$$\mathcal{S}(\zeta,\zeta) = -2n\beta^2 \tag{2.17}$$

$\forall\ \mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3 \in \mathfrak{X}(\mathfrak{M})$; $\mathfrak{R}$, $\mathcal{S}$ and $\mathcal{Q}$ denote the curvature tensor of type $(1,3)$, Ricci tensor of type $(0,2)$ and Ricci operator of the Levi-Civita connection ∇, respectively.

Definition 2.1. A β-Kenmotsu manifold $\mathfrak{M}^{2n+1}$ is said to be an η-Einstein manifold if its Ricci tensor $\mathcal{S}$ of type $(0,2)$ satisfies

$$\mathcal{S}(\mathcal{E}_1,\mathcal{E}_2) = \Theta_1 g(\mathcal{E}_1,\mathcal{E}_2) + \Theta_2\eta(\mathcal{E}_1)\eta(\mathcal{E}_2), \tag{2.18}$$

where Θ_1 and Θ_2 are smooth functions on $\mathfrak{M}^{2n+1}$. In particular, if $\Theta_2 = 0$, then the manifold $\mathfrak{M}^{2n+1}$ is an Einstein manifold.

Definition 2.2. The quasi-concircular curvature tensor $\mathcal{C}$ on a $(2n+1)$-dimensional β-Kenmotsu manifold $\mathfrak{M}$ with respect to the connection ∇ is given by [11, 14]

$$\mathcal{C}(\mathcal{E}_1,\mathcal{E}_2)\mathcal{E}_3 = a\mathfrak{R}(\mathcal{E}_1,\mathcal{E}_2)\mathcal{E}_3 + \frac{\mathfrak{r}}{2n+1}\left(\frac{a}{2n}+2b\right)[g(\mathcal{E}_2,\mathcal{E}_3)\mathcal{E}_1 - g(\mathcal{E}_1,\mathcal{E}_3)\mathcal{E}_2], \tag{2.19}$$

where a and b are constants such that $a, b \neq 0$ and $\mathfrak{R}$ is the curvature tensor, $\mathfrak{r}$ is the scalar curvature with respect to the connection ∇ on $\mathfrak{M}$. If $a = 1$ and $b = -\frac{1}{2n}$, then (2.19) takes the form

$$\begin{aligned} \mathcal{C}(\mathcal{E}_1,\mathcal{E}_2)\mathcal{E}_3 &= \mathfrak{R}(\mathcal{E}_1,\mathcal{E}_2)\mathcal{E}_3 - \frac{\mathfrak{r}}{2n(2n+1)}[g(\mathcal{E}_2,\mathcal{E}_3)\mathcal{E}_1 - g(\mathcal{E}_1,\mathcal{E}_3)\mathcal{E}_2] \\ &= \widetilde{\mathcal{C}}(\mathcal{E}_1,\mathcal{E}_2)\mathcal{E}_3, \end{aligned} \tag{2.20}$$

where $\widetilde{\mathcal{C}}$ is the concircular curvature tensor.

3. The generalized Tanaka-Webster connection (GTWC) $\widetilde{\nabla}$

The generalized Tanaka-Webster connection (GTWC) $\widetilde{\nabla}$ defined by Tanno for contact metric manifold is given by [19]

$$\widetilde{\nabla}_{\mathcal{E}_1}\mathcal{E}_2 = \nabla_{\mathcal{E}_1}\mathcal{E}_2 + (\nabla_{\mathcal{E}_1}\eta)(\mathcal{E}_2)\zeta - \eta(\mathcal{E}_2)\nabla_{\mathcal{E}_1}\zeta - \eta(\mathcal{E}_1)\varphi\mathcal{E}_2 \tag{3.1}$$

$\forall\ \mathcal{E}_1,\mathcal{E}_2 \in \mathfrak{X}(\mathfrak{M})$. By virtue of (2.4) and (2.6), (3.1) takes the form

$$\widetilde{\nabla}_{\mathcal{E}_1}\mathcal{E}_2 = \nabla_{\mathcal{E}_1}\mathcal{E}_2 + \beta g(\mathcal{E}_1,\mathcal{E}_2)\zeta - \beta\eta(\mathcal{E}_2)\mathcal{E}_1 - \eta(\mathcal{E}_1)\varphi\mathcal{E}_2. \tag{3.2}$$

Replacing $\mathcal{E}_2$ by ζ in (3.2) and using (2.1), (2.2) and (2.4), we have

$$\widetilde{\nabla}_{\mathcal{E}_1}\zeta = 0. \tag{3.3}$$

Now

$$(\widetilde{\nabla}_{\mathcal{E}_1}\eta)(\mathcal{E}_2) = \widetilde{\nabla}_{\mathcal{E}_1}(\eta\mathcal{E}_2) - \eta(\widetilde{\nabla}_{\mathcal{E}_1}\mathcal{E}_2). \tag{3.4}$$

Using (3.2) in (3.4), we have

$$(\widetilde{\nabla}_{\mathcal{E}_1}\eta)(\mathcal{E}_2) = (\nabla_{\mathcal{E}_1}\eta)(\mathcal{E}_2) - \beta g(\mathcal{E}_1,\mathcal{E}_2) + \beta\eta(\mathcal{E}_1)\eta(\mathcal{E}_2), \tag{3.5}$$

Using (2.6) in (3.5), we have

$$(\widetilde{\nabla}_{\mathcal{E}_1}\eta)(\mathcal{E}_2) = 0. \tag{3.6}$$

Now

$$(\widetilde{\nabla}_{\mathcal{E}_1}\varphi)(\mathcal{E}_2) = \widetilde{\nabla}_{\mathcal{E}_1}(\varphi\mathcal{E}_2) - \varphi(\widetilde{\nabla}_{\mathcal{E}_1}\mathcal{E}_2). \tag{3.7}$$

Using (3.2) in (3.7), we have

$$(\widetilde{\nabla}_{\mathcal{E}_1}\varphi)(\mathcal{E}_2) = (\nabla_{\mathcal{E}_1}\varphi)(\mathcal{E}_2) + \beta\eta(\mathcal{E}_2)\varphi\mathcal{E}_1 - \eta(\mathcal{E}_1)\mathcal{E}_2 + \eta(\mathcal{E}_1)\eta(\mathcal{E}_2)\zeta, \tag{3.8}$$

Using (2.5) in (3.8), we have

$$(\widetilde{\nabla}_{\mathcal{E}_1}\varphi)(\mathcal{E}_2) = \beta g(\varphi\mathcal{E}_1, \mathcal{E}_2)\zeta - \eta(\mathcal{E}_1)\mathcal{E}_2 + \eta(\mathcal{E}_1)\eta(\mathcal{E}_2)\zeta. \tag{3.9}$$

Now

$$(\widetilde{\nabla}_{\mathcal{E}_1}g)(\mathcal{E}_2, \mathcal{E}_3) = \widetilde{\nabla}_{\mathcal{E}_1}g(\mathcal{E}_2, \mathcal{E}_3) - g(\widetilde{\nabla}_{\mathcal{E}_1}\mathcal{E}_2, \mathcal{E}_3) - g(\mathcal{E}_2, \widetilde{\nabla}_{\mathcal{E}_1}\mathcal{E}_3). \tag{3.10}$$

Using (3.2) in (3.10), we have

$$(\widetilde{\nabla}_{\mathcal{E}_1}g)(\mathcal{E}_2, \mathcal{E}_3) = 0. \tag{3.11}$$

Hence, we have the following:

Theorem 3.1. *In a β-Kenmotsu manifold the GTWC $\widetilde{\nabla}$ is a metric connection.*

Theorem 3.2. *In a β-Kenmotsu manifold ζ, η and g are parallel with respect to the GTWC $\widetilde{\nabla}$.*

Proposition 3.1. *In a β-Kenmotsu manifold, the integral curves of the vector field ζ are geodesic with respect to the GTWC $\widetilde{\nabla}$.*

Now, the torsion tensor $\widetilde{\mathcal{T}}$ with respect to the GTWC $\widetilde{\nabla}$ is given by

$$\widetilde{\mathcal{T}}(\mathcal{E}_1, \mathcal{E}_2) = \widetilde{\nabla}_{\mathcal{E}_1}\mathcal{E}_2 - \widetilde{\nabla}_{\mathcal{E}_2}\mathcal{E}_1 - [\mathcal{E}_1, \mathcal{E}_2]. \tag{3.12}$$

Using (3.2) in (3.12), we have

$$\widetilde{\mathcal{T}}(\mathcal{E}_1, \mathcal{E}_2) = \beta\eta(\mathcal{E}_1)\mathcal{E}_2 - \beta\eta(\mathcal{E}_2)\mathcal{E}_1 - \eta(\mathcal{E}_1)\varphi\mathcal{E}_2 + \eta(\mathcal{E}_2)\varphi\mathcal{E}_1. \tag{3.13}$$

Hence, we have the following:

Theorem 3.3. *In a β-Kenmotsu manifold the GTWC $\widetilde{\nabla}$ associated with Levi-Civita connection ∇ is just the only one affine connection which is metric and its torsion tensor is of the form (3.13).*

Any metric connection can be expressed with the help of its torsion tensor $\widetilde{\mathcal{T}}$ in the following way:

$$\begin{aligned} g(\widetilde{\nabla}_{\mathcal{E}_1}\mathcal{E}_2, \mathcal{E}_3) &= g(\nabla_{\mathcal{E}_1}\mathcal{E}_2, \mathcal{E}_3) + \frac{1}{2}[g(\widetilde{\mathcal{T}}(\mathcal{E}_1, \mathcal{E}_2), \mathcal{E}_3) \\ &\quad - g(\widetilde{\mathcal{T}}(\mathcal{E}_1, \mathcal{E}_3), \mathcal{E}_2) - g(\widetilde{\mathcal{T}}(\mathcal{E}_2, \mathcal{E}_3), \mathcal{E}_1)]. \end{aligned} \tag{3.14}$$

Using (3.13) in (3.14), we have

$$g(\widetilde{\nabla}_{\mathcal{E}_1}\mathcal{E}_2,\mathcal{E}_3) = g(\nabla_{\mathcal{E}_1}\mathcal{E}_2,\mathcal{E}_3)+\beta g(\mathcal{E}_1,\mathcal{E}_2)g(\zeta,\mathcal{E}_3) -\beta g(\mathcal{E}_1,\mathcal{E}_3)\eta(\mathcal{E}_2)-g(\varphi\mathcal{E}_2,\mathcal{E}_3)\eta(\mathcal{E}_1). \tag{3.15}$$

Contracting $\mathcal{E}_3$ in above equation, we have

$$\widetilde{\nabla}_{\mathcal{E}_1}\mathcal{E}_2=\nabla_{\mathcal{E}_1}\mathcal{E}_2+\beta g(\mathcal{E}_1,\mathcal{E}_2)\zeta-\beta\eta(\mathcal{E}_2)\mathcal{E}_1-\eta(\mathcal{E}_1)\varphi\mathcal{E}_2. \tag{3.16}$$

Let $\mathfrak{R}$ and $\widetilde{\mathfrak{R}}$ denote the curvature tensors of ∇ and $\widetilde{\nabla}$ respectively, then we have

$$\widetilde{\mathfrak{R}}(\mathcal{E}_1,\mathcal{E}_2)\mathcal{E}_3=\widetilde{\nabla}_{\mathcal{E}_1}\widetilde{\nabla}_{\mathcal{E}_2}\mathcal{E}_3-\widetilde{\nabla}_{\mathcal{E}_2}\widetilde{\nabla}_{\mathcal{E}_1}\mathcal{E}_3-\widetilde{\nabla}_{[\mathcal{E}_1,\mathcal{E}_2]}\mathcal{E}_3. \tag{3.17}$$

Using (2.1), (2.2), (2.3), (2.4), (2.5), (2.6) and (3.2) in (3.17), we have

$$\widetilde{\mathfrak{R}}(\mathcal{E}_1,\mathcal{E}_2)\mathcal{E}_3=\mathfrak{R}(\mathcal{E}_1,\mathcal{E}_2)\mathcal{E}_3+\beta^2[g(\mathcal{E}_2,\mathcal{E}_3)\mathcal{E}_1-g(\mathcal{E}_1,\mathcal{E}_3)\mathcal{E}_2], \tag{3.18}$$

where

$$\mathfrak{R}(\mathcal{E}_1,\mathcal{E}_2)\mathcal{E}_3=\nabla_{\mathcal{E}_1}\nabla_{\mathcal{E}_2}\mathcal{E}_3-\nabla_{\mathcal{E}_2}\nabla_{\mathcal{E}_1}\mathcal{E}_3-\nabla_{[\mathcal{E}_1,\mathcal{E}_2]}\mathcal{E}_3 \tag{3.19}$$

is the curvature tensor with respect to the Levi-Civita connection ∇.
Contracting $\mathcal{E}_1$ in (3.18), we have

$$\widetilde{\mathcal{S}}(\mathcal{E}_2,\mathcal{E}_3)=\mathcal{S}(\mathcal{E}_2,\mathcal{E}_3)+2n\beta^2 g(\mathcal{E}_2,\mathcal{E}_3). \tag{3.20}$$

Using (2.11) in (3.20), we have

$$\widetilde{\mathcal{Q}}\mathcal{E}_2=\mathcal{Q}\mathcal{E}_2+2n\beta^2\mathcal{E}_2. \tag{3.21}$$

Contracting $\mathcal{E}_2$ and $\mathcal{E}_3$ in (3.20), we have

$$\widetilde{\mathfrak{r}}=\mathfrak{r}+2n(2n+1)\beta^2. \tag{3.22}$$

Replacing $\mathcal{E}_3$ by ζ in (3.18) and using (2.2) and (2.8), we have

$$\widetilde{\mathfrak{R}}(\mathcal{E}_1,\mathcal{E}_2)\zeta=0. \tag{3.23}$$

Hence, we have the following:

Theorem 3.4. *Every $(2n+1)$-dimensional β-Kenmotsu manifold admitting the GTWC $\widetilde{\nabla}$ is irregular.*

Taking $\widetilde{\mathfrak{R}}(\mathcal{E}_1,\mathcal{E}_2)\mathcal{E}_3=0$ in (3.18), we have

$$\mathfrak{R}(\mathcal{E}_1,\mathcal{E}_2)\mathcal{E}_3=-\beta^2[g(\mathcal{E}_2,\mathcal{E}_3)\mathcal{E}_1-g(\mathcal{E}_1,\mathcal{E}_3)\mathcal{E}_2]. \tag{3.24}$$

Taking inner product with $\mathcal{U}$ in (3.24), we have

$$\mathfrak{R}(\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3,\mathcal{U})=-\beta^2[g(\mathcal{E}_2,\mathcal{E}_3)g(\mathcal{E}_1,\mathcal{U})-g(\mathcal{E}_1,\mathcal{E}_3)g(\mathcal{E}_2,\mathcal{U})]. \tag{3.25}$$

Let $\zeta^{\perp}$ denotes the $(2n+1)$-dimensional distribution orthogonal to ζ in a β-Kenmotsu manifold admitting the GTWC $\widetilde{\nabla}$ whose curvature tensor vanishes. Then $\forall\ \mathcal{E}_1 \in \zeta^{\perp}$, $g(\mathcal{E}_1, \zeta) = 0$ or $\eta(\mathcal{E}_1) = 0$. Now, we shall determine the sectional curvature $'\mathfrak{R}$ of the plane determine by the vectors $\mathcal{E}_1, \mathcal{E}_2 \in \zeta^{\perp}$.
Taking $\mathcal{E}_3 = \mathcal{E}_2$ and $\mathcal{U} = \mathcal{E}_1$ in (3.25), we have

$$\widetilde{\mathfrak{R}}(\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_2, \mathcal{E}_1) = -\beta^2[g(\mathcal{E}_1, \mathcal{E}_1)g(\mathcal{E}_2, \mathcal{E}_2) - g(\mathcal{E}_1, \mathcal{E}_2)^2]. \tag{3.26}$$

Now

$$\begin{aligned} '\mathfrak{R}(\mathcal{E}_1, \mathcal{E}_2) &= \frac{\widetilde{\mathfrak{R}}(\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_2, \mathcal{E}_1)}{[g(\mathcal{E}_1, \mathcal{E}_1)g(\mathcal{E}_2, \mathcal{E}_2) - g(\mathcal{E}_1, \mathcal{E}_2)^2]} \\ &= -\beta^2. \end{aligned} \tag{3.27}$$

Hence, we can state the following:

Theorem 3.5. *If the curvature tensor of a β-Kenmotsu manifold admitting the GTWC $\widetilde{\nabla}$ vanishes, then the sectional curvature of the plane determined by two vectors $\mathcal{E}_1, \mathcal{E}_2 \in \zeta^{\perp}$ is $-\beta^2$.*

Now, the projective curvature tensor [22] $\widetilde{\mathcal{P}}$ with respect to the GTWC $\widetilde{\nabla}$ is defined by

$$\widetilde{\mathcal{P}}(\mathcal{E}_1, \mathcal{E}_2)\mathcal{E}_3 = \widetilde{\mathfrak{R}}(\mathcal{E}_1, \mathcal{E}_2)\mathcal{E}_3 - \frac{1}{2n}[\widetilde{\mathcal{S}}(\mathcal{E}_2, \mathcal{E}_3)\mathcal{E}_1 - \widetilde{\mathcal{S}}(\mathcal{E}_1, \mathcal{E}_3)\mathcal{E}_2]. \tag{3.28}$$

If the projective curvature tensor $\widetilde{\mathcal{P}}$ with respect to the GTWC $\widetilde{\nabla}$ vanishes, then (3.28) takes the form

$$\widetilde{\mathfrak{R}}(\mathcal{E}_1, \mathcal{E}_2)\mathcal{E}_3 = \frac{1}{2n}[\widetilde{\mathcal{S}}(\mathcal{E}_2, \mathcal{E}_3)\mathcal{E}_1 - \widetilde{\mathcal{S}}(\mathcal{E}_1, \mathcal{E}_3)\mathcal{E}_2]. \tag{3.29}$$

By virtue of (3.18) and (3.20), (3.29) takes the form

$$\mathfrak{R}(\mathcal{E}_1, \mathcal{E}_2)\mathcal{E}_3 = \frac{1}{2n}[\mathcal{S}(\mathcal{E}_2, \mathcal{E}_3)\mathcal{E}_1 - \mathcal{S}(\mathcal{E}_1, \mathcal{E}_3)\mathcal{E}_2]. \tag{3.30}$$

Taking inner product with $\mathcal{W}$ in (3.30), we have

$$g(\mathfrak{R}(\mathcal{E}_1, \mathcal{E}_2)\mathcal{E}_3, \mathcal{W}) = \frac{1}{2n}[\mathcal{S}(\mathcal{E}_2, \mathcal{E}_3)g(\mathcal{E}_1, \mathcal{W}) - \mathcal{S}(\mathcal{E}_1, \mathcal{E}_3)g(\mathcal{E}_2, \mathcal{W})]. \tag{3.31}$$

Replacing $\mathcal{W}$ by ζ in (3.31) and using (2.2) and (2.7), we have

$$\mathcal{S}(\mathcal{E}_2, \mathcal{E}_3)\eta(\mathcal{E}_1) - \mathcal{S}(\mathcal{E}_1, \mathcal{E}_3)\eta(\mathcal{E}_2) = 2n\beta^2[g(\mathcal{E}_1, \mathcal{E}_3)\eta(\mathcal{E}_2) - g(\mathcal{E}_2, \mathcal{E}_3)\eta(\mathcal{E}_1)]. \tag{3.32}$$

Taking $\mathcal{E}_1 = \zeta$ in (3.32) and using (2.1), (2.2) and (2.12), we have

$$\mathcal{S}(\mathcal{E}_2, \mathcal{E}_3) = -2n\beta^2 g(\mathcal{E}_2, \mathcal{E}_3). \tag{3.33}$$

Contracting (3.33), we have

$$\mathfrak{r} = -2n(2n+1)\beta^2. \tag{3.34}$$

Using (3.33) in (3.20), we have

$$\widetilde{\mathcal{S}}(\mathcal{E}_2, \mathcal{E}_3) = 0. \tag{3.35}$$

By virtue of (3.29) and (3.35), we have

$$\widetilde{\mathfrak{R}}(\mathcal{E}_1, \mathcal{E}_2)\mathcal{E}_3 = 0. \tag{3.36}$$

Hence, we have the following:

Theorem 3.6. *In a β-Kenmotsu manifold $\mathfrak{M}^{2n+1}$ admitting the GTWC $\widetilde{\nabla}$, vanishing of projective curvature tensor $\widetilde{\mathcal{P}}$ with respect to the GTWC $\widetilde{\nabla}$ leads to vanishing of curvature tensor $\widetilde{\mathfrak{R}}$ with respect to the GTWC $\widetilde{\nabla}$.*

using (3.36) in (3.18), we have

$$\mathfrak{R}(\mathcal{E}_1, \mathcal{E}_2)\mathcal{E}_3 = -\beta^2[g(\mathcal{E}_2, \mathcal{E}_3)\mathcal{E}_1 - g(\mathcal{E}_1, \mathcal{E}_3)\mathcal{E}_2]. \tag{3.37}$$

Taking inner product with $\mathcal{W}$ in (3.37), we have

$$\mathfrak{R}(\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{W}) = -\beta^2[g(\mathcal{E}_2, \mathcal{E}_3)g(\mathcal{E}_1, \mathcal{W}) - g(\mathcal{E}_1, \mathcal{E}_3)g(\mathcal{E}_2, \mathcal{W})]. \tag{3.38}$$

Hence, we have the following:

Theorem 3.7. *In a β-Kenmotsu manifold $\mathfrak{M}^{2n+1}$ admitting the GTWC $\widetilde{\nabla}$, the curvature tensor $\widetilde{\mathfrak{R}}$ with $\widetilde{\nabla}$ vanishes iff the manifold $\mathfrak{M}^{2n+1}$ is isomorphic to the hyperbolic space $\mathcal{H}^{2n+1}(-\beta^2)$.*

Replacing $\mathcal{E}_3$ by ζ in (3.28) and using (3.23) and (3.20), we have

$$\widetilde{\mathcal{P}}(\mathcal{E}_1, \mathcal{E}_2)\zeta = 0. \tag{3.39}$$

Hence, we have the following:

Theorem 3.8. *A β-Kenmotsu manifold $\mathfrak{M}^{2n+1}$ is ζ-projectively flat with respect to the GTWC $\widetilde{\nabla}$.*

Using (3.18) and (3.20) in (3.28), we have

$$\widetilde{\mathcal{P}}(\mathcal{E}_1, \mathcal{E}_2)\mathcal{E}_3 = \mathcal{P}(\mathcal{E}_1, \mathcal{E}_2)\mathcal{E}_3. \tag{3.40}$$

where

$$\mathcal{P}(\mathcal{E}_1, \mathcal{E}_2)\mathcal{E}_3 = \mathfrak{R}(\mathcal{E}_1, \mathcal{E}_2)\mathcal{E}_3 - \frac{1}{2n}[\mathcal{S}(\mathcal{E}_2, \mathcal{E}_3)\mathcal{E}_1 - \mathcal{S}(\mathcal{E}_1, \mathcal{E}_3)\mathcal{E}_2]. \tag{3.41}$$

is the projective curvature tensor with respect to the connection ∇. Hence, we can state the following:

Theorem 3.9. *The projective curvature tensor of a β-Kenmotsu manifold $\mathfrak{M}^{2n+1}$ with respect to the connections $\widetilde{\nabla}$ and ∇ are equivalent.*

Replacing $\mathcal{E}_3$ by ζ in (3.41), we have

$$\widetilde{\mathcal{P}}(\mathcal{E}_1, \mathcal{E}_2)\zeta = \mathcal{P}(\mathcal{E}_1, \mathcal{E}_2)\zeta. \tag{3.42}$$

Hence, we have the following:

Theorem 3.10. *A $(2n+1)$-dimensional β-Kenmotsu manifold is ζ-projectively flat with respect to the GTWC iff the manifold $\mathfrak{M}^{2n+1}$ is ζ-projectively flat with respect to the connection ∇.*

4. Recurrent, conharmonic curvature tensor and Weyl conformal curvature tensor in β-Kenmotsu manifold $(\mathfrak{M}^{2n+1}, \varphi, \zeta, \eta, g)$ admitting the GTWC $\widetilde{\nabla}$

Definition 4.1. A β-Kenmotsu manifold $\mathfrak{M}^{2n+1}$ admitting the GTWC $\widetilde{\nabla}$ is called recurrent if its curvature tensor $\widetilde{\mathfrak{R}}$ satisfies the condition

$$(\widetilde{\nabla}_{\mathcal{E}_1}\widetilde{\mathfrak{R}})(\mathcal{E}_2, \mathcal{E}_3)\mathcal{W} = \mathcal{A}(\mathcal{E}_1)\widetilde{\mathfrak{R}}(\mathcal{E}_2, \mathcal{E}_3)\mathcal{W} \tag{4.1}$$

$\forall\, \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{W} \in \mathfrak{X}(\mathfrak{M})$, where $\widetilde{\mathfrak{R}}$ is the curvature tensor with respect to the GTWC $\widetilde{\nabla}$ and $\mathcal{A}$ is 1-form. By virtue of (4.1), we have

$$\begin{aligned} &\widetilde{\nabla}_{\mathcal{E}_1}\widetilde{\mathfrak{R}}(\mathcal{E}_2, \mathcal{E}_3)\mathcal{W} - \widetilde{\mathfrak{R}}(\widetilde{\nabla}_{\mathcal{E}_1}\mathcal{E}_2, \mathcal{E}_3)\mathcal{W} \\ &-\widetilde{\mathfrak{R}}(\mathcal{E}_2, \widetilde{\nabla}_{\mathcal{E}_1}\mathcal{E}_3)\mathcal{W} - \widetilde{\mathfrak{R}}(\mathcal{E}_2, \mathcal{E}_3)\widetilde{\nabla}_{\mathcal{E}_1}\mathcal{W} = \mathcal{A}(\mathcal{E}_1)\widetilde{\mathfrak{R}}(\mathcal{E}_2, \mathcal{E}_3)\mathcal{W}. \end{aligned} \tag{4.2}$$

Using (3.2) and (3.18) in (4.2), we have

$$\begin{aligned} &\beta g(\mathcal{E}_1, \mathfrak{R}(\mathcal{E}_2, \mathcal{E}_3)\mathcal{W})\zeta + \beta^3 g(\mathcal{E}_1, \mathcal{E}_2)g(\mathcal{E}_3, \mathcal{W})\zeta - \beta^3 g(\mathcal{E}_1, \mathcal{E}_3)g(\mathcal{E}_2, \mathcal{W})\zeta \\ &-\beta^3 g(\mathcal{E}_1, \mathcal{W})\eta(\mathcal{E}_2)\mathcal{E}_3 + \beta^3 g(\mathcal{E}_1, \mathcal{W})\eta(\mathcal{E}_3)\mathcal{E}_2 + \beta^3 g(\mathcal{E}_1, \mathcal{E}_3)\eta(\mathcal{W})\mathcal{E}_2 \\ &-\beta^3 g(\mathcal{E}_1, \mathcal{E}_2)\eta(\mathcal{W})\mathcal{E}_3 - \beta\eta(\mathfrak{R}(\mathcal{E}_2, \mathcal{E}_3)\mathcal{W})\mathcal{E}_1 + \beta\mathfrak{R}(\mathcal{E}_1, \mathcal{E}_3)\eta(\mathcal{E}_2)\mathcal{W} \\ &+\mathfrak{R}(\varphi\mathcal{E}_2, \mathcal{E}_3)\eta(\mathcal{E}_1)\mathcal{W} + \beta\mathfrak{R}(\mathcal{E}_2, \mathcal{E}_1)\eta(\mathcal{E}_3)\mathcal{W} + \mathfrak{R}(\mathcal{E}_2, \varphi\mathcal{E}_3)\eta(\mathcal{E}_1)\mathcal{W} \\ &+\beta\mathfrak{R}(\mathcal{E}_2, \mathcal{E}_3)\eta(\mathcal{W})\mathcal{E}_1 = \beta^2\mathcal{A}(\mathcal{E}_1)[g(\mathcal{E}_3, \mathcal{W})\mathcal{E}_2 - g(\mathcal{E}_2, \mathcal{W})\mathcal{E}_3]. \end{aligned} \tag{4.3}$$

Replacing $\mathcal{W}$ by ζ in (4.3) and using (2.1), (2.2), (2.7) and (2.8), we have

$$\begin{aligned} &\beta^3 g(\mathcal{E}_1, \mathcal{E}_3)\mathcal{E}_2 - \beta^3 g(\mathcal{E}_1, \mathcal{E}_2)\mathcal{E}_3 - \beta^2\eta(\mathcal{E}_1)\eta(\mathcal{E}_3)\varphi\mathcal{E}_2 \\ &+\beta^2\eta(\mathcal{E}_1)\eta(\mathcal{E}_2)\varphi\mathcal{E}_3 + \beta\mathfrak{R}(\mathcal{E}_2, \mathcal{E}_3)\mathcal{E}_1 = \beta^2\mathcal{A}(\mathcal{E}_1)[\eta(\mathcal{E}_3)\mathcal{E}_2 - \eta(\mathcal{E}_2)\mathcal{E}_3]. \end{aligned} \tag{4.4}$$

Taking inner product with $\mathcal{U}$ in (4.4), we have

$$\begin{aligned}&\beta^3 g(\mathcal{E}_1,\mathcal{E}_3)g(\mathcal{E}_2,\mathcal{U})-\beta^3 g(\mathcal{E}_1,\mathcal{E}_2)g(\mathcal{E}_3,\mathcal{U})\\&-\beta^2\eta(\mathcal{E}_1)\eta(\mathcal{E}_3)g(\varphi\mathcal{E}_2,\mathcal{U})+\beta^2\eta(\mathcal{E}_1)\eta(\mathcal{E}_2)g(\varphi\mathcal{E}_3,\mathcal{U})\\&+\beta g(\mathfrak{R}(\mathcal{E}_2,\mathcal{E}_3)\mathcal{E}_1,\mathcal{U})=\beta^2\mathcal{A}(\mathcal{E}_1)[\eta(\mathcal{E}_3)g(\mathcal{E}_2,\mathcal{U})-\eta(\mathcal{E}_2)g(\mathcal{E}_3,\mathcal{U})].\end{aligned}\tag{4.5}$$

Let $\{\varsigma_1,\varsigma_2,\varsigma_3,......,\varsigma_{2n+1}\}$ be a local orthonormal basis of vector fields in $\mathfrak{M}$. Then by putting $\mathcal{E}_2=\mathcal{U}=\varsigma_i$ in (4.5) and summing up over $i\in[1,2n+1]$, we have

$$\mathcal{S}(\mathcal{E}_3,\mathcal{E}_1)=-2n\beta^2 g(\mathcal{E}_1,\mathcal{E}_3)+2n\beta\mathcal{A}(\mathcal{E}_1)\eta(\mathcal{E}_3).\tag{4.6}$$

Suppose the associated 1-form $\mathcal{A}$ is equal to the associated 1-form η, then from (4.6), we have

$$\mathcal{S}(\mathcal{E}_3,\mathcal{E}_1)=-2n\beta^2 g(\mathcal{E}_1,\mathcal{E}_3)+2n\beta\eta(\mathcal{E}_1)\eta(\mathcal{E}_3).\tag{4.7}$$

Hence, we have the following:

Theorem 4.1. *If a β-Kenmotsu manifold $\mathfrak{M}^{2n+1}$ admitting the GTWC $\widetilde{\nabla}$ is recurrent and the associated 1-form $\mathcal{A}$ is equal to the associated 1-form η, then the manifold $\mathfrak{M}^{2n+1}$ is an η-Einstein manifold.*

The conharmonic curvature tensor [5] $\widetilde{\mathcal{K}}$ admitting the GTWC $\widetilde{\nabla}$ is defined by

$$\begin{aligned}\widetilde{\mathcal{K}}(\mathcal{E}_1,\mathcal{E}_2)\mathcal{E}_3 &= \widetilde{\mathfrak{R}}(\mathcal{E}_1,\mathcal{E}_2)\mathcal{E}_3-\frac{1}{2n-1}[\widetilde{\mathcal{S}}(\mathcal{E}_2,\mathcal{E}_3)\mathcal{E}_1-\widetilde{\mathcal{S}}(\mathcal{E}_1,\mathcal{E}_3)\mathcal{E}_2\\&\quad+g(\mathcal{E}_2,\mathcal{E}_3)\widetilde{\mathcal{Q}}\mathcal{E}_1-g(\mathcal{E}_1,\mathcal{E}_3)\widetilde{\mathcal{Q}}\mathcal{E}_2].\end{aligned}\tag{4.8}$$

If $\widetilde{\mathcal{K}}(\mathcal{E}_1,\mathcal{E}_2)\mathcal{E}_3$ with respect to the GTWC $\widetilde{\nabla}$ vanishes, then from (4.8), we have

$$\begin{aligned}\widetilde{\mathfrak{R}}(\mathcal{E}_1,\mathcal{E}_2)\mathcal{E}_3 &= \frac{1}{2n-1}[\widetilde{\mathcal{S}}(\mathcal{E}_2,\mathcal{E}_3)\mathcal{E}_1-\widetilde{\mathcal{S}}(\mathcal{E}_1,\mathcal{E}_3)\mathcal{E}_2\\&\quad+g(\mathcal{E}_2,\mathcal{E}_3)\widetilde{\mathcal{Q}}\mathcal{E}_1-g(\mathcal{E}_1,\mathcal{E}_3)\widetilde{\mathcal{Q}}\mathcal{E}_2].\end{aligned}\tag{4.9}$$

Using (3.18), (3.20) and (3.21) in (4.9), we have

$$\begin{aligned}\mathfrak{R}(\mathcal{E}_1,\mathcal{E}_2)\mathcal{E}_3 &= \frac{1}{2n-1}[\mathcal{S}(\mathcal{E}_2,\mathcal{E}_3)\mathcal{E}_1-\mathcal{S}(\mathcal{E}_1,\mathcal{E}_3)\mathcal{E}_2\\&\quad+4n\beta^2 g(\mathcal{E}_2,\mathcal{E}_3)\mathcal{E}_1-4n\beta^2 g(\mathcal{E}_1,\mathcal{E}_3)\mathcal{E}_2\\&\quad+g(\mathcal{E}_2,\mathcal{E}_3)\mathcal{Q}\mathcal{E}_1-g(\mathcal{E}_1,\mathcal{E}_3)\mathcal{Q}\mathcal{E}_2]\\&\quad-\beta^2[g(\mathcal{E}_2,\mathcal{E}_3)\mathcal{E}_1-g(\mathcal{E}_1,\mathcal{E}_3)\mathcal{E}_2].\end{aligned}\tag{4.10}$$

Taking inner product with $\mathcal{U}$ in (4.10) and using (2.11), we have

$$\begin{aligned}g(\mathfrak{R}(\mathcal{E}_1,\mathcal{E}_2)\mathcal{E}_3,\mathcal{U}) &= \frac{1}{2n-1}[\{\mathcal{S}(\mathcal{E}_2,\mathcal{E}_3)+4n\beta^2 g(\mathcal{E}_2,\mathcal{E}_3)\}g(\mathcal{E}_1,\mathcal{U})\\&\quad-\{\mathcal{S}(\mathcal{E}_1,\mathcal{E}_3)+4n\beta^2 g(\mathcal{E}_1,\mathcal{E}_3)\}g(\mathcal{E}_2,\mathcal{U})\\&\quad+\mathcal{S}(\mathcal{E}_1,\mathcal{U})g(\mathcal{E}_2,\mathcal{E}_3)-\mathcal{S}(\mathcal{E}_2,\mathcal{U})g(\mathcal{E}_1,\mathcal{E}_3)]\\&\quad-\beta^2[g(\mathcal{E}_2,\mathcal{E}_3)g(\mathcal{E}_1,\mathcal{U})-g(\mathcal{E}_1,\mathcal{E}_3)g(\mathcal{E}_2,\mathcal{U})].\end{aligned}\tag{4.11}$$

Replacing $\mathcal{U}$ by ζ in (4.11) and using (2.2), (2.7) and (2.12), we have

$$\mathcal{S}(\mathcal{E}_2,\mathcal{E}_3)\eta(\mathcal{E}_1)-\mathcal{S}(\mathcal{E}_1,\mathcal{E}_3)\eta(\mathcal{E}_2)=2n\beta^2\{g(\mathcal{E}_1,\mathcal{E}_3)\eta(\mathcal{E}_2)-g(\mathcal{E}_2,\mathcal{E}_3)\eta(\mathcal{E}_1)\}. \tag{4.12}$$

Taking $\mathcal{E}_1=\zeta$ in (4.12) and using (2.1), (2.2) and (2.12), we have

$$\mathcal{S}(\mathcal{E}_2,\mathcal{E}_3)=-2n\beta^2 g(\mathcal{E}_2,\mathcal{E}_3). \tag{4.13}$$

Contracting (4.13), we have

$$\mathfrak{r}=-2n\beta^2(2n+1). \tag{4.14}$$

Using (2.13), (3.20), (3.21) and (4.13) in (4.9), we have

$$\widetilde{\mathfrak{R}}(\mathcal{E}_1,\mathcal{E}_2)\mathcal{E}_3=0. \tag{4.15}$$

Hence, we have the following:

Theorem 4.2. *In a β-Kenmotsu manifold $\mathfrak{M}^{2n+1}$, vanishing of conharmonic curvature tensor admitting the GTWC $\widetilde{\nabla}$ leads to vanishing of curvature tensor admitting the GTWC $\widetilde{\nabla}$ and the manifold $\mathfrak{M}^{2n+1}$ is an Einstein manifold.*

In a Riemannian manifold Weyl conformal curvature tensor $\widetilde{\mathfrak{C}}$ admitting the GTWC $\widetilde{\nabla}$ is defined by

$$\begin{aligned}\widetilde{\mathfrak{C}}(\mathcal{E}_1,\mathcal{E}_2)\mathcal{E}_3 &= \widetilde{\mathfrak{R}}(\mathcal{E}_1,\mathcal{E}_2)\mathcal{E}_3-\frac{1}{2n-1}[\widetilde{\mathcal{S}}(\mathcal{E}_2,\mathcal{E}_3)\mathcal{E}_1-\widetilde{\mathcal{S}}(\mathcal{E}_1,\mathcal{E}_3)\mathcal{E}_2\\ &\quad +g(\mathcal{E}_2,\mathcal{E}_3)\widetilde{\mathcal{Q}}\mathcal{E}_1-g(\mathcal{E}_1,\mathcal{E}_3)\widetilde{\mathcal{Q}}\mathcal{E}_2]\\ &\quad +\frac{\widetilde{\mathfrak{r}}}{2n(2n-1)}[g(\mathcal{E}_2,\mathcal{E}_3)\mathcal{E}_1-g(\mathcal{E}_1,\mathcal{E}_3)\mathcal{E}_2].\end{aligned} \tag{4.16}$$

Using (3.18), (3.20), (3.21) and (3.22) in (4.16), we have

$$\begin{aligned}\widetilde{\mathfrak{C}}(\mathcal{E}_1,\mathcal{E}_2)\mathcal{E}_3 &= \mathfrak{R}(\mathcal{E}_1,\mathcal{E}_2)\mathcal{E}_3-\frac{1}{2n-1}[\mathcal{S}(\mathcal{E}_2,\mathcal{E}_3)\mathcal{E}_1-\mathcal{S}(\mathcal{E}_1,\mathcal{E}_3)\mathcal{E}_2\\ &\quad +g(\mathcal{E}_2,\mathcal{E}_3)\mathcal{Q}\mathcal{E}_1-g(\mathcal{E}_1,\mathcal{E}_3)\mathcal{Q}\mathcal{E}_2]\\ &\quad +\frac{\mathfrak{r}}{2n(2n-1)}[g(\mathcal{E}_2,\mathcal{E}_3)\mathcal{E}_1-g(\mathcal{E}_1,\mathcal{E}_3)\mathcal{E}_2].\end{aligned} \tag{4.17}$$

By virtue of (4.17), we have

$$\widetilde{\mathfrak{C}}(\mathcal{E}_1,\mathcal{E}_2)\mathcal{E}_3=\mathfrak{C}(\mathcal{E}_1,\mathcal{E}_2)\mathcal{E}_3. \tag{4.18}$$

where

$$\begin{aligned}\mathfrak{C}(\mathcal{E}_1,\mathcal{E}_2)\mathcal{E}_3 &= \mathfrak{R}(\mathcal{E}_1,\mathcal{E}_2)\mathcal{E}_3-\frac{1}{2n-1}[\mathcal{S}(\mathcal{E}_2,\mathcal{E}_3)\mathcal{E}_1-\mathcal{S}(\mathcal{E}_1,\mathcal{E}_3)\mathcal{E}_2\\ &\quad +g(\mathcal{E}_2,\mathcal{E}_3)\mathcal{Q}\mathcal{E}_1-g(\mathcal{E}_1,\mathcal{E}_3)\mathcal{Q}\mathcal{E}_2]\\ &\quad +\frac{\mathfrak{r}}{2n(2n-1)}[g(\mathcal{E}_2,\mathcal{E}_3)\mathcal{E}_1-g(\mathcal{E}_1,\mathcal{E}_3)\mathcal{E}_2].\end{aligned} \tag{4.19}$$

Hence, we have the following:

Theorem 4.3. *The Weyl conformal curvature tensor of a β-Kenmotsu manifold $\mathfrak{M}^{2n+1}$ with respect to the connections $\widetilde{\nabla}$ and ∇ are equivalent.*

5. Ricci pseudo-symmetric, quasi-concircularly flat and ζ-quasi-concircularly flat β-Kenmotsu manifold admitting the GTWC $\widetilde{\nabla}$

Definition 5.1. A β-Kenmotsu manifold $\mathfrak{M}^{2n+1}$ is said to be Ricci pseudo-symmetric iff the relation [6, 7]

$$\mathfrak{R}\cdot\mathcal{S}=f\mathcal{Q}(g,\mathcal{S}), \tag{5.1}$$

holds on the set $\mathcal{U}_{\mathcal{S}}=\{x\in\mathfrak{M}:\mathcal{S}\neq 0 \text{ at } x\}$, where f is a function on $\mathcal{U}_{\mathcal{S}}$, $\mathfrak{R}\cdot\mathcal{S}$ and $\mathcal{Q}(g,\mathcal{S})$ are respectively defined as

$$(\mathfrak{R}(\mathcal{E}_1,\mathcal{E}_2)\cdot\mathcal{S})(\mathcal{U},\mathcal{V})=-\mathcal{S}(\mathfrak{R}(\mathcal{E}_1,\mathcal{E}_2)\mathcal{U},\mathcal{V})-\mathcal{S}(\mathcal{U},\mathfrak{R}(\mathcal{E}_1,\mathcal{E}_2)\mathcal{V}), \tag{5.2}$$

and

$$\mathcal{Q}(g,\mathcal{S})=((\mathcal{E}_1\wedge_g\mathcal{E}_2)\cdot\mathcal{S})(\mathcal{U},\mathcal{V}), \tag{5.3}$$

where

$$(\mathcal{E}_1\wedge_g\mathcal{E}_2)\mathcal{E}_3=g(\mathcal{E}_2,\mathcal{E}_3)\mathcal{E}_1-g(\mathcal{E}_1,\mathcal{E}_3)\mathcal{E}_2 \tag{5.4}$$

$\forall\ \mathcal{E}_1,\mathcal{E}_2,\mathcal{U},\mathcal{V}\in\mathfrak{X}(\mathfrak{M})$.

Suppose that the manifold $\mathfrak{M}^{2n+1}$ is a Ricci pseudo-symmetric β-Kenmotsu manifold admitting the GTWC $\widetilde{\nabla}$, then we have

$$(\widetilde{\mathfrak{R}}(\mathcal{E}_1,\mathcal{E}_2)\cdot\widetilde{\mathcal{S}})(\mathcal{U},\mathcal{V})=f\widetilde{\mathcal{Q}}(g,\widetilde{\mathcal{S}})(\mathcal{E}_1,\mathcal{E}_2;\mathcal{U},\mathcal{V}) \tag{5.5}$$

$\forall\ \mathcal{E}_1,\mathcal{E}_2,\mathcal{U},\mathcal{V}\in\mathfrak{X}(\mathfrak{M})$. It is equivalent to

$$(\widetilde{\mathfrak{R}}(\mathcal{E}_1,\mathcal{E}_2)\cdot\widetilde{\mathcal{S}})(\mathcal{U},\mathcal{V})=f((\mathcal{E}_1\wedge_g\mathcal{E}_2)\cdot\widetilde{\mathcal{S}})(\mathcal{U},\mathcal{V}), \tag{5.6}$$

By virtue of (5.2) and (5.4), we have

$$\begin{aligned}-\widetilde{\mathcal{S}}(\widetilde{\mathfrak{R}}(\mathcal{E}_1,\mathcal{E}_2)\mathcal{U},\mathcal{V}) &= \widetilde{\mathcal{S}}(\mathcal{U},\widetilde{\mathfrak{R}}(\mathcal{E}_1,\mathcal{E}_2)\mathcal{V})+f[\widetilde{\mathcal{S}}(\mathcal{E}_2,\mathcal{V})g(\mathcal{E}_1,\mathcal{U})\\ &\quad -\widetilde{\mathcal{S}}(\mathcal{E}_1,\mathcal{V})g(\mathcal{E}_2,\mathcal{U})-\widetilde{\mathcal{S}}(\mathcal{U},\mathcal{E}_1)g(\mathcal{E}_2,\mathcal{V})\\ &\quad +\widetilde{\mathcal{S}}(\mathcal{U},\mathcal{E}_2)g(\mathcal{E}_1,\mathcal{V})].\end{aligned} \tag{5.7}$$

Taking $\mathcal{U}=\zeta$ in (5.7) and using (2.2), (3.20) and (3.23), we have

$$f[\eta(\mathcal{E}_1)\widetilde{\mathcal{S}}(\mathcal{E}_2,\mathcal{V})-\eta(\mathcal{E}_2)\widetilde{\mathcal{S}}(\mathcal{E}_1,\mathcal{V})]=0, \tag{5.8}$$

Since $f\neq 0$ therefore from (5.8), we have

$$[\eta(\mathcal{E}_1)\widetilde{\mathcal{S}}(\mathcal{E}_2,\mathcal{V})-\eta(\mathcal{E}_2)\widetilde{\mathcal{S}}(\mathcal{E}_1,\mathcal{V})]=0. \tag{5.9}$$

Taking $\mathcal{E}_2 = \zeta$ in (5.9) and using (2.1) and (3.20), we have

$$\widetilde{\mathcal{S}}(\mathcal{E}_1, \mathcal{V}) = 0. \tag{5.10}$$

Using (3.20) in (5.10), we have

$$\mathcal{S}(\mathcal{E}_1, \mathcal{V}) = -2n\beta^2 g(\mathcal{E}_1, \mathcal{V}). \tag{5.11}$$

Hence, we have the following:

Theorem 5.1. *A $(2n{+}1)$-dimensional Ricci pseudo-symmetric β-Kenmotsu manifold $\mathfrak{M}$ admitting the GTWC $\widetilde{\nabla}$ is an Einstein manifold.*

Analogous to the definition (2.2) the quasi-concircular curvature tensor $\widetilde{\mathcal{C}}$ on $(2n{+}1)$-dimensional β-Kenmotsu manifold admitting the GTWC $\widetilde{\nabla}$ is given by

$$\widetilde{\mathcal{C}}(\mathcal{E}_1, \mathcal{E}_2)\mathcal{E}_3 = a\widetilde{\mathfrak{R}}(\mathcal{E}_1, \mathcal{E}_2)\mathcal{E}_3 + \frac{\widetilde{\mathfrak{r}}}{2n+1}\left(\frac{a}{2n} + 2b\right)[g(\mathcal{E}_2, \mathcal{E}_3)\mathcal{E}_1 - g(\mathcal{E}_1, \mathcal{E}_3)\mathcal{E}_2], \tag{5.12}$$

where a and b are constants such that $a, b \neq 0$. First we suppose that the manifold $\mathfrak{M}^{2n+1}$ admitting the GTWC $\widetilde{\nabla}$ is quasi-concircularly flat, i. e.

$$\widetilde{\mathcal{C}}(\mathcal{E}_1, \mathcal{E}_2)\mathcal{E}_3 = 0. \tag{5.13}$$

By virtue of (5.12), we have

$$a\widetilde{\mathfrak{R}}(\mathcal{E}_1, \mathcal{E}_2)\mathcal{E}_3 + \frac{\widetilde{\mathfrak{r}}}{2n+1}\left(\frac{a}{2n} + 2b\right)[g(\mathcal{E}_2, \mathcal{E}_3)\mathcal{E}_1 - g(\mathcal{E}_1, \mathcal{E}_3)\mathcal{E}_2] = 0. \tag{5.14}$$

Taking inner product with ζ in (5.14) and using (2.2), (2.7) and (3.18), we have

$$\frac{\widetilde{\mathfrak{r}}}{2n+1}\left(\frac{a}{2n} + 2b\right)[g(\mathcal{E}_2, \mathcal{E}_3)\eta(\mathcal{E}_1) - g(\mathcal{E}_1, \mathcal{E}_3)\eta(\mathcal{E}_2)] = 0. \tag{5.15}$$

Thus we have either

$$\widetilde{\mathfrak{r}}\left(\frac{a+4bn}{2n(2n+1)}\right) = 0, \tag{5.16}$$

i.e.,

$$\widetilde{\mathfrak{r}} = 0, \quad \frac{a+4bn}{2n(2n+1)} \neq 0. \tag{5.17}$$

Or

$$g(\mathcal{E}_2, \mathcal{E}_3)\eta(\mathcal{E}_1) - g(\mathcal{E}_1, \mathcal{E}_3)\eta(\mathcal{E}_2) = 0. \tag{5.18}$$

Taking $\mathcal{E}_2 = \zeta$ in (5.18) and using (2.1) and (2.2), we have

$$-g(\mathcal{E}_1, \mathcal{E}_3) + \eta(\mathcal{E}_1)\eta(\mathcal{E}_3) = 0. \tag{5.19}$$

Replacing $\mathcal{E}_1$ by $\widetilde{\mathcal{Q}}\mathcal{E}_1$ in (5.19) and using (2.2), (2.11) and (2.12), we have

$$\mathcal{S}(\mathcal{E}_1, \mathcal{E}_3) = -2n\beta^2 g(\mathcal{E}_1, \mathcal{E}_3). \tag{5.20}$$

Hence, we have the following:

Theorem 5.2. *If a β-Kenmotsu manifold $\mathfrak{M}^{2n+1}$ admitting the GTWC $\widetilde{\nabla}$ is quasi-concircularly flat then either the scalar curvature $\widetilde{\mathfrak{r}}$ is constant or the manifold $\mathfrak{M}^{2n+1}$ is an Einstein manifold.*

Next, we suppose that the manifold $\mathfrak{M}^{2n+1}$ admitting the GTWC $\widetilde{\nabla}$ is ζ-quasi-concircularly flat, i.e.

$$\widetilde{\mathcal{C}}(\mathcal{E}_1, \mathcal{E}_2)\zeta = 0. \tag{5.21}$$

By virtue of (5.12), we have

$$a\widetilde{\mathfrak{R}}(\mathcal{E}_1, \mathcal{E}_2)\zeta + \frac{\widetilde{\mathfrak{r}}}{2n+1}\left(\frac{a}{2n} + 2b\right)[g(\mathcal{E}_2, \zeta)\mathcal{E}_1 - g(\mathcal{E}_1, \zeta)\mathcal{E}_2] = 0. \tag{5.22}$$

Using (2.2) and (3.23) in (5.22), we have

$$\frac{\widetilde{\mathfrak{r}}}{2n+1}\left(\frac{a}{2n} + 2b\right)[\eta(\mathcal{E}_2)\mathcal{E}_1 - \eta(\mathcal{E}_1)\mathcal{E}_2] = 0. \tag{5.23}$$

Since $[\eta(\mathcal{E}_2)\mathcal{E}_1 - \eta(\mathcal{E}_1)\mathcal{E}_2] \neq 0$, therefore we have

$$\widetilde{\mathfrak{r}} = 0, \quad \frac{a + 4bn}{2n(2n+1)} \neq 0. \tag{5.24}$$

Hence, we have the following:

Theorem 5.3. *If a β-Kenmotsu manifold $\mathfrak{M}^{2n+1}$ admitting the GTWC $\widetilde{\nabla}$ is ζ-quasi-concircularly flat, then the scalar curvature $\widetilde{\mathfrak{r}}$ is constant.*

6. Example of a class of β-Kenmotsu manifold

Example 6.1. Let us suppose $\mathfrak{M}^3$=$\{(\mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3) \in \mathbb{R} : \mathfrak{t}_3 > 0\}$ be the 3-dimensional manifold, where $(\mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3)$ are the standard coordinates in $\mathbb{R}^3$. The vector fields [20]

$$\varsigma_1 = \beta\mathfrak{t}_3\frac{\partial}{\partial\mathfrak{t}_1}, \quad \varsigma_2 = \beta\mathfrak{t}_3\frac{\partial}{\partial\mathfrak{t}_2}, \quad \varsigma_3 = \beta\mathfrak{t}_3\frac{\partial}{\partial\mathfrak{t}_3}$$

are linearly independent at each point of the manifold $\mathfrak{M}^3$.

Let g be the Riemannian metric defined by

$$g = \frac{d\mathfrak{t}_1^2 + d\mathfrak{t}_2^2 + d\mathfrak{t}_3^2}{\beta^2\mathfrak{t}_3^2},$$

then we have

$$g(\varsigma_i, \varsigma_j) = 1, \forall i, j(i = j) = 1, 2, 3, \quad g(\varsigma_i, \varsigma_j) = 0, \forall i, j(i \neq j) = 1, 2, 3.$$

Let η be a 1-form defined by $\eta(\mathcal{E}_1) = g(\mathcal{E}_1, \varsigma_3)$ for any vector field $\mathcal{E}_1 \in \mathfrak{X}(\mathfrak{M})$ and φ be the $(1,1)$-tensor field defined as

$$\varphi\varsigma_1 = \varsigma_2, \quad \varphi\varsigma_2 = -\varsigma_1, \quad \varphi\varsigma_3 = 0. \tag{6.1}$$

Using the linearity of φ and g, we have

$$\begin{aligned} &\eta(\varsigma_3) = \eta(\zeta) = 1, \quad \varphi^2(\mathcal{E}_1) = -\mathcal{E}_1 + \eta(\mathcal{E}_1)\varsigma_3, \\ &g(\varphi\mathcal{E}_1, \varphi\mathcal{E}_2) = g(\mathcal{E}_1, \mathcal{E}_2) - \eta(\mathcal{E}_1)\eta(\mathcal{E}_2) \end{aligned} \tag{6.2}$$

$\forall\ \mathcal{E}_1, \mathcal{E}_2 \in \mathfrak{X}(\mathfrak{M}^3)$. Thus for $\varsigma_3 = \zeta$, the structure $(\varphi, \zeta, \eta, g)$ defines an almost contact metric structure on $\mathfrak{M}$.
Let ∇ be the Levi-Civita connection with metric g, then we have

$$[\varsigma_1, \varsigma_2] = 0, \quad [\varsigma_1, \varsigma_3] = -\beta\varsigma_1, \quad [\varsigma_2, \varsigma_3] = -\beta\varsigma_2. \tag{6.3}$$

Koszul's formula for the Riemannian connection ∇ of the metric g is given by

$$\begin{aligned} 2g(\nabla_{\mathcal{E}_1}\mathcal{E}_2, \mathcal{E}_3) \quad = \quad & \mathcal{E}_1 g(\mathcal{E}_2, \mathcal{E}_3) + \mathcal{E}_2 g(\mathcal{E}_1, \mathcal{E}_3) - \mathcal{E}_3 g(\mathcal{E}_1, \mathcal{E}_2) \\ & -g(\mathcal{E}_1, [\mathcal{E}_2, \mathcal{E}_3]) - g(\mathcal{E}_2, [\mathcal{E}_1, \mathcal{E}_3]) + g(\mathcal{E}_3, [\mathcal{E}_1, \mathcal{E}_2]). \end{aligned}$$

Using the above equation, we can easily calculate

$$\begin{aligned} &\nabla_{\varsigma_1}\varsigma_1 = \beta\varsigma_3, \quad \nabla_{\varsigma_1}\varsigma_2 = 0, \qquad \nabla_{\varsigma_1}\varsigma_3 = -\beta\varsigma_1, \\ &\nabla_{\varsigma_2}\varsigma_1 = 0, \qquad \nabla_{\varsigma_2}\varsigma_2 = \beta\varsigma_3, \quad \nabla_{\varsigma_2}\varsigma_3 = -\beta\varsigma_2, \\ &\nabla_{\varsigma_3}\varsigma_1 = 0, \qquad \nabla_{\varsigma_3}\varsigma_2 = 0, \qquad \nabla_{\varsigma_3}\varsigma_3 = 0. \end{aligned} \tag{6.4}$$

From above calculations, it can be easily seen that the manifold $\mathfrak{M}^3(\varphi, \zeta, \eta, g)$ satisfies the condition

$$(\nabla_{\mathcal{E}_1}\varphi)\mathcal{E}_2 = \beta[g(\varphi\mathcal{E}_1, \mathcal{E}_2)\zeta - \eta(\mathcal{E}_2)\varphi\mathcal{E}_1], \quad \nabla_{\mathcal{E}_1}\zeta = \beta[\mathcal{E}_1 - \eta(\mathcal{E}_1)\zeta].$$

Now for $\mathcal{E}_1 = \mathcal{E}_1^1\varsigma_1 + \mathcal{E}_1^2\varsigma_2 + \mathcal{E}_1^3\varsigma_3$, we have

$$\nabla_{\mathcal{E}_1}\zeta = -\beta(\mathcal{E}_1^1\varsigma_1 + \mathcal{E}_1^2\varsigma_2), \tag{6.5}$$

and

$$\beta[\mathcal{E}_1 - \eta(\mathcal{E}_1)\zeta] = \beta(\mathcal{E}_1^1\varsigma_1 + \mathcal{E}_1^2\varsigma_2), \tag{6.6}$$

From (6.5) and (6.6), we have

$$\beta(\mathcal{E}_1^1\varsigma_1 + \mathcal{E}_1^2\varsigma_2) = -\beta(\mathcal{E}_1^1\varsigma_1 + \mathcal{E}_1^2\varsigma_2)$$

i.e.,

$$\beta = 0, \tag{6.7}$$

Thus β is a constant. Hence, the structure $(\varphi, \zeta, \eta, g)$ is a β-Kenmotsu structure and the manifold $\mathfrak{M}^3$ equipped with β-Kenmotsu structure is a β-Kenmotsu manifold. Using (6.4) in (3.2), we obtain

$$\begin{aligned}
&\widetilde{\nabla}_{\varsigma_1}\varsigma_1 = 2\beta\varsigma_3, \quad &&\widetilde{\nabla}_{\varsigma_1}\varsigma_2 = 0, \quad &&\widetilde{\nabla}_{\varsigma_1}\varsigma_3 = -2\beta\varsigma_1\\
&\widetilde{\nabla}_{\varsigma_2}\varsigma_1 = 0, &&\widetilde{\nabla}_{\varsigma_2}\varsigma_2 = 2\beta\varsigma_3, &&\widetilde{\nabla}_{\varsigma_2}\varsigma_3 = -2\beta\varsigma_2\\
&\widetilde{\nabla}_{\varsigma_3}\varsigma_1 = -\varsigma_2, &&\widetilde{\nabla}_{\varsigma_3}\varsigma_2 = \varsigma_1, &&\widetilde{\nabla}_{\varsigma_3}\varsigma_3 = 0.
\end{aligned} \tag{6.8}$$

By virtue of (3.13), the torsion tensor $\widetilde{\mathcal{T}}$ admitting the GTWC $\widetilde{\nabla}$ as follows:

$$\widetilde{\mathcal{T}}(\varsigma_i, \varsigma_i) = 0, \forall i = 1, 2, 3,$$

and

$$\widetilde{\mathcal{T}}(\varsigma_1, \varsigma_2) = 0, \quad \widetilde{\mathcal{T}}(\varsigma_1, \varsigma_3) = -\beta\varsigma_1 + \varsigma_2, \quad \widetilde{\mathcal{T}}(\varsigma_2, \varsigma_3) = -\beta\varsigma_2 - \varsigma_1.$$

Also we have

$$(\widetilde{\nabla}_{\varsigma_1} g)(\varsigma_2, \varsigma_3) = 0, \quad (\widetilde{\nabla}_{\varsigma_2} g)(\varsigma_3, \varsigma_1) = 0, \quad (\widetilde{\nabla}_{\varsigma_3} g)(\varsigma_1, \varsigma_2) = 0.$$

Hence, $\mathfrak{M}^3$ is a 3-dimensional β-Kenmotsu manifold admitting the GTWC $\widetilde{\nabla}$ which is a symmetric connection.
The curvature tensor $\mathfrak{R}(\varsigma_i, \varsigma_j)\varsigma_k$; $i, j, k = 1, 2, 3$ of ∇ can be calculated by using (3.19), (6.3) and (6.4), we have

$$\begin{aligned}
&\mathfrak{R}(\varsigma_1, \varsigma_2)\varsigma_2 = -\beta^2\varsigma_1, \quad &&\mathfrak{R}(\varsigma_1, \varsigma_3)\varsigma_3 = -\beta^2\varsigma_1, \quad &&\mathfrak{R}(\varsigma_2, \varsigma_1)\varsigma_1 = -\beta^2\varsigma_2,\\
&\mathfrak{R}(\varsigma_2, \varsigma_3)\varsigma_3 = -\beta^2\varsigma_2, &&\mathfrak{R}(\varsigma_3, \varsigma_1)\varsigma_1 = -\beta^2\varsigma_3, &&\mathfrak{R}(\varsigma_3, \varsigma_2)\varsigma_2 = -\beta^2\varsigma_3,\\
&\mathfrak{R}(\varsigma_1, \varsigma_2)\varsigma_3 = 0, &&\mathfrak{R}(\varsigma_2, \varsigma_3)\varsigma_2 = -\beta^2\varsigma_3, &&\mathfrak{R}(\varsigma_1, \varsigma_2)\varsigma_1 = -\beta^2\varsigma_2.
\end{aligned} \tag{6.9}$$

Along with $\mathfrak{R}(\varsigma_i, \varsigma_i)\varsigma_i = 0$; $\forall\, i = 1, 2, 3$. In view of above calculations, we can verify (2.8), (2.9) and (2.10).
The Ricci tensor $\mathcal{S}(\varsigma_j, \varsigma_k)$; $j, k = 1, 2, 3$ of ∇ can be calculated by using (6.9), we have

$$\mathcal{S}(\varsigma_j, \varsigma_k) = \sum_{i=1}^{3} g(\mathfrak{R}(\varsigma_i, \varsigma_j)\varsigma_k, \varsigma_i).$$

It follows that

$$\begin{aligned}
&\mathcal{S}(\varsigma_j, \varsigma_k) = -2\beta^2, \forall j, k(j = k) = 1, 2, 3,\\
&\mathcal{S}(\varsigma_j, \varsigma_k) = 0, \forall j, k(j \neq k) = 1, 2, 3.
\end{aligned} \tag{6.10}$$

By virtue of (6.10), we can verify (2.11), (2.12), (2.17), (3.33), (4.13), (5.11) and (5.20). The scalar curvature $\mathfrak{r}$ of ∇ can also be calculated as under:

$$\mathfrak{r} = -6\beta^2. \tag{6.11}$$

In view of (6.11), we can easily verify (3.34) and (4.14).
The curvature tensor $\widetilde{\mathfrak{R}}(\varsigma_i,\varsigma_j)\varsigma_k$; i,j,k=1, 2, 3 with $\widetilde{\nabla}$ can be calculated by using (6.3) and (6.8), we have

$$\begin{aligned}
&\widetilde{\mathfrak{R}}(\varsigma_1,\varsigma_2)\varsigma_2 = -4\beta^2\varsigma_1, \quad \widetilde{\mathfrak{R}}(\varsigma_1,\varsigma_3)\varsigma_3 = -2\beta\varsigma_2 - 2\beta^2\varsigma_1,\\
&\widetilde{\mathfrak{R}}(\varsigma_2,\varsigma_1)\varsigma_1 = -4\beta^2\varsigma_2, \quad \widetilde{\mathfrak{R}}(\varsigma_2,\varsigma_3)\varsigma_3 = 2\beta\varsigma_1 - 2\beta^2\varsigma_2,\\
&\widetilde{\mathfrak{R}}(\varsigma_3,\varsigma_1)\varsigma_1 = -2\beta^2\varsigma_3, \quad \widetilde{\mathfrak{R}}(\varsigma_3,\varsigma_2)\varsigma_2 = -2\beta^2\varsigma_3,\\
&\widetilde{\mathfrak{R}}(\varsigma_1,\varsigma_2)\varsigma_3 = 0, \qquad\qquad \widetilde{\mathfrak{R}}(\varsigma_2,\varsigma_3)\varsigma_2 = 4\beta^2\varsigma_3,\\
&\widetilde{\mathfrak{R}}(\varsigma_1,\varsigma_2)\varsigma_1 = 4\beta^2\varsigma_2.
\end{aligned} \tag{6.12}$$

Along with $\widetilde{\mathfrak{R}}(\varsigma_i,\varsigma_i)\varsigma_i = 0$; $\forall\ i = 1, 2, 3$.
The Ricci tensor $\widetilde{\mathcal{S}}(\varsigma_j,\varsigma_k)$; j,k=1, 2, 3 with $\widetilde{\nabla}$ can be calculated by using (6.12), we have

$$\widetilde{\mathcal{S}}(\varsigma_j,\varsigma_k) = \sum_{i=1}^{3} g(\widetilde{\mathfrak{R}}(\varsigma_i,\varsigma_j)\varsigma_k,\varsigma_i).$$

It follows that

$$\widetilde{\mathcal{S}}(\varsigma_1,\varsigma_1) = -6\beta^2, \quad \widetilde{\mathcal{S}}(\varsigma_2,\varsigma_2) = -6\beta^2, \quad \widetilde{\mathcal{S}}(\varsigma_3,\varsigma_3) = -4\beta^2. \tag{6.13}$$

Along with $\widetilde{\mathcal{S}}(\varsigma_j,\varsigma_k) = 0$; $\forall\ j,k(j\neq k) = 1, 2, 3$.
The scalar curvature $\widetilde{\mathfrak{r}}$ admitting the GTWC $\widetilde{\nabla}$ can also be calculated by using (6.13) as under:

$$\begin{aligned}
\widetilde{\mathfrak{r}} &= \sum_{i=1}^{3} g(\varsigma_i,\varsigma_i)\widetilde{\mathcal{S}}(\varsigma_i,\varsigma_i)\\
&= -6\beta^2 - 6\beta^2 - 4\beta^2\\
&= -16\beta^2.
\end{aligned} \tag{6.14}$$

Using (6.11) in (3.22) and taking $n = 1$, we have

$$\widetilde{\mathfrak{r}} = 0. \tag{6.15}$$

From (6.15), it is clear that the theorems 5.2 and 5.3 are verified by this example.
In a 3-dimensional β-Kenmotsu manifold $\mathfrak{M}^3$, the projective curvature tensor admitting the GTWC $\widetilde{\nabla}$ is given as

$$\widetilde{\mathcal{P}}(\mathcal{E}_1,\mathcal{E}_2)\mathcal{E}_3 = \widetilde{\mathfrak{R}}(\mathcal{E}_1,\mathcal{E}_2)\mathcal{E}_3 - \frac{1}{2}[\widetilde{\mathcal{S}}(\mathcal{E}_2,\mathcal{E}_3)\mathcal{E}_1 - \widetilde{\mathcal{S}}(\mathcal{E}_1,\mathcal{E}_3)\mathcal{E}_2]. \tag{6.16}$$

Replacing $\mathcal{E}_3$ by ς_3 in (6.16), we have

$$\widetilde{\mathcal{P}}(\mathcal{E}_1,\mathcal{E}_2)\varsigma_3 = \widetilde{\mathfrak{R}}(\mathcal{E}_1,\mathcal{E}_2)\varsigma_3 - \frac{1}{2}[\widetilde{\mathcal{S}}(\mathcal{E}_2,\varsigma_3)\mathcal{E}_1 - \widetilde{\mathcal{S}}(\mathcal{E}_1,\varsigma_3)\mathcal{E}_2]. \tag{6.17}$$

Let $\mathcal{E}_1$ and $\mathcal{E}_2$ are any two vector fields as under:

$$\mathcal{E}_1 = \mathcal{E}_1^1\varsigma_1 + \mathcal{E}_1^2\varsigma_2 + \mathcal{E}_1^3\varsigma_3, \quad \mathcal{E}_2 = \mathcal{E}_2^1\varsigma_1 + \mathcal{E}_2^2\varsigma_2 + \mathcal{E}_2^3\varsigma_3, \tag{6.18}$$

where $\mathcal{E}_1^1, \mathcal{E}_1^2, \mathcal{E}_1^3, \mathcal{E}_2^1, \mathcal{E}_2^2$, and $\mathcal{E}_2^3$ are scalars. Using (6.18) in (6.17), we have

$$\widetilde{\mathcal{P}}(\mathcal{E}_1, \mathcal{E}_2)\varsigma_3 = 0. \tag{6.19}$$

Hence, the manifold $\mathfrak{M}^3$ is ζ-projectively flat in a β-Kenmotsu manifold admitting the GTWC $\widetilde{\nabla}$ which verifies theorem 3.5.

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