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SOME GEOMETRIC PROPERTIES OF PARA-KENMOTSU MANIFOLDS ADMITTING $Q\varphi = \varphi Q$

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Abstract. Hereby, some geometric properties such as $\varphi-$ Ricci symmetric, weakly φ-Ricci symmetric and Ricci-Yamabe soliton as well as parallel 2-form of the para-Kenmotsu manifolds admitting $Q\varphi = \varphi Q$ are discussed. This paper also deals with Ricci-Yamabe soliton on para-Kenmotsu manifold admitting the curvature condition $\widetilde{\mathbf{R}}(\xi, \mathcal{X}_1).\mathcal{S} = \mathbf{0}$ and locally $\varphi-$ Ricci symmetric para-Kenmotsu manifolds of dimension three. Together with we have cited some examples of 3-dimensional $\varphi-$ Ricci symmetric and locally $\varphi-$ Ricci symmetric para-Kenmotsu manifold.
Keywords: Ricci-Yamabe soliton, para-Kenmotsu manifold, Einstein manifold.

1. Introduction

Sato [23] established the idea of almost para contact manifolds based on the analogy of almost contact manifolds. An almost paracontact manifold may be of even dimension, whereas an almost contact manifold is invariably of odd dimension. Takahashi [28] defined almost contact manifolds, in particular, Sasakian manifolds equipped with an associated pseudo-Riemannian metric. Subsequently, as a natural odd dimensional counterpart to para Hermitian structure, Kaneyuki and Williams [12] proposed the notion of an almost paracontact pseudo-Riemannian structure. In

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[30], any almost paracontact structure is shown to yield a pseudo-Riemannian metric with the signature $(n+1, n)$ by Zamkovoy. Many authors have researched almost paracontact structure in recent years, especially since the appearance of [30]. The curvature identity for different classes of almost paracontact geometry was obtained in ([4],[30]). Welyczko [29] first proposed the idea of a para-Kenmotsu manifold. The Kenmotsu manifold in paracontact geometry is analogous to this structure [13]. Studies on para-Kenmotsu (also referred as p-Kenmotsu) and special para-Kenmotsu (also referred as sp-Kenmotsu) manifolds have been done by Sinha and Prasad [25], Blaga [2], Sai Prasad and Satyanarayana [22], Prakasha and Vikas [21] and others ([20], [19], [14]).

Ricci-Yamabe flow of type (α, β) is an advanced class of geometric flows defined by Güler and Crasmareanu (2019) [8] as a scalar combination of Ricci and Yamabe flows and is defined as follows:

$$\frac{\partial}{\partial t}g(t) + 2\alpha\mathcal{S}(g(t)) + \beta r(t)g(t) = 0, \qquad g(0) = g_0 \tag{1.1}$$

where α and β are some scalars.

If a solution to the Ricci-Yamabe flow depends solely on one scaling and diffeomorphism parameter group, it is referred to as a Ricci-Yamabe soliton. A manifold M that is Riemannian (or semi-Riemannian) is considered to have a Ricci-Yamabe soliton if

$$\pounds_V g + 2\alpha\mathcal{S} + (2\lambda - \beta r)g = 0 \tag{1.2}$$

where, $\alpha, \beta, \lambda \in \mathbb{R}$(the set of real numbers). It is noted that a Ricci-Yamabe soliton of types $(\alpha, 0)$ and $(0, \beta)$ are known as $\alpha-$Ricci soliton and $\beta-$Yamabe soliton respectively. Also, a Ricci-Yamabe soliton is called as shrinking, steady or expanding if $\lambda <, = or >$, respectively. In short, a Ricci-Yamabe soliton is said to be a

1. Ricci [9] soliton if $\alpha = 1, \beta = 0$,
2. Yamabe soliton [10] if $\alpha = 0, \beta = 1$,
3. Einstein soliton [3] if $\alpha = 1, \beta = -1$.

Moreover, in [27], locally φ-symmetric Sasakian manifolds are a weaker form of the local symmetry of such manifolds and were first proposed by T. Takahashi. Further, U.C. De studied $\varphi-$symmetric Kenmotsu manifolds with several examples in [7]. A class of contact metric manifolds known as the Kenmotsu manifold was subsequently introduced by K. Kenmotsu [13] in 1971 but this manifold is not a Sasakian manifold. The study of Kenmotsu manifolds has been done by a number of researchers, including Pitis [17], Binh, Tamassy, De and Tarafdar [1], De and Pathak [5], Özgür [16], Özgür and De [15] and many other geometricians [11]. The concept of $\varphi-$Ricci symmetric Sasakian manifolds was recently presented by U.C. De in [6] and he also got some noteworthy findings for this manifold.

We are working with the Ricci-Yamabe soliton on para-Kenmotsu manifolds in this paper as a result of the investigations mentioned above. The structure of the current paper is as follows: In section 2, we begin by studying the para-Kenmotsu manifold preliminary data. Following that, in section 3, we talk about $\varphi-$Ricci symmetric para-Kenmotsu manifolds. Moreover, we explore three-dimensional $\varphi-$Ricci symmetric para-Kenmotsu manifolds in section 4. In section 5, we build examples of para-Kenmotsu manifolds in three dimensions, supporting the findings from sections 3 and 4. After that section 6 and 7 deals with weakly φ-Ricci symmetric and Ricci-Yamabe solitons on para-Kenmotsu manifold respectively. Furthermore, Ricci-Yamabe solitons on para-Kenmotsu manifold satisfying $\widetilde{\mathbf{R}}(\xi, \mathcal{X}_\mathbf{1}).\mathcal{S} = \mathbf{0}$ and parallel 2-form in the para-Kenmotsu manifolds is also covered in the paper's last two sections.

2. Preliminaries

A $(2n+1)$-dimensional smooth manifold $\widetilde{M}^{2n+1}$ has an almost para contact structure (φ, ξ, η, g) if it admits a tensor field φ of type $(1,1)$, a vector field ξ, a 1-form η and a semi-Riemannian metric tensor g satisfying the following conditions {[30], [18]};

$$\varphi^2\mathcal{X}_1 = \mathcal{X}_1 - \eta(\mathcal{X}_1)\xi, \quad \eta(\xi) = 1, \quad \varphi\xi = \eta \circ \varphi = 0 \tag{2.1}$$

$$g(\varphi\mathcal{X}_1, \varphi\mathcal{X}_2) = -g(\mathcal{X}_1, \mathcal{X}_2) + \eta(\mathcal{X}_1)\eta(\mathcal{X}_2), \quad \eta(\mathcal{X}_1) = g(\mathcal{X}_1, \xi) \tag{2.2}$$

and

$$d\eta(\mathcal{X}_1, \mathcal{X}_2) = g(\mathcal{X}_1, \varphi\mathcal{X}_2), \tag{2.3}$$

for all vector fields $\mathcal{X}_1, \mathcal{X}_2$ on $\widetilde{M}^{2n+1}$.

An almost para contact metric manifold $\widetilde{M}^{2n+1}(\varphi, \xi, \eta, g)$ is said to be para-Kenmotsu manifold if the Levi-Civita connection $\widetilde{\nabla}$ of g satisfies

$$\left(\widetilde{\nabla}_{\mathcal{X}_1}\varphi\right)\mathcal{X}_2 = g(\varphi\mathcal{X}_1, \mathcal{X}_2)\xi - \eta(\mathcal{X}_2)\varphi\mathcal{X}_1 \tag{2.4}$$

for all $\mathcal{X}_1, \mathcal{X}_2 \in \Gamma(T\widetilde{M})$, where $\Gamma(T\widetilde{M})$ denote the set of all differentiable vector fields on $\widetilde{M}^{2n+1}$[16]. From equations 2.1 and 2.4, we have

$$\widetilde{\nabla}_{\mathcal{X}_1}\xi = \varphi^2\mathcal{X}_1 = \mathcal{X}_1 - \eta(\mathcal{X}_1)\xi. \tag{2.5}$$

In a para-Kenmotsu manifold $\widetilde{M}^{2n+1}(\varphi, \xi, \eta, g)$, we have the following formulas:

$$\widetilde{R}(\mathcal{X}_1, \mathcal{X}_2)\xi = \eta(\mathcal{X}_1)\mathcal{X}_2 - \eta(\mathcal{X}_2)\mathcal{X}_1, \tag{2.6}$$

$$\widetilde{R}(\xi, \mathcal{X}_1)\mathcal{X}_2 = \eta(\mathcal{X}_2)\mathcal{X}_1 - g(\mathcal{X}_1, \mathcal{X}_2)\xi, \tag{2.7}$$

$$\mathcal{S}(\xi, \mathcal{X}_1) = -2n\eta(\mathcal{X}_1), \qquad Q\xi = -2n\xi, \tag{2.8}$$

$$(\widetilde{\nabla}_{\mathcal{X}_1}\eta)\mathcal{X}_2 = g(\mathcal{X}_1, \mathcal{X}_2) - \eta(\mathcal{X}_1)\eta(\mathcal{X}_2) \tag{2.9}$$

for any vector fields $\mathcal{X}_1, \mathcal{X}_2 \in \Gamma(T\widetilde{M})$, where $\widetilde{R}$ and $\mathcal{S}$ denote the Riemannian curvature tensor and Ricci tensor of $\widetilde{M}^{2n+1}$ respectively.

Also, since $\mathcal{S}(\mathcal{X}_1, \mathcal{X}_2) = g(Q\mathcal{X}_1, \mathcal{X}_2)$, we have

$$\mathcal{S}(\varphi\mathcal{X}_1, \varphi\mathcal{X}_2) = g(Q\varphi\mathcal{X}_1, \varphi\mathcal{X}_2),$$

where Q is the Ricci operator.

Using the properties $g(\varphi\mathcal{X}_1, \varphi\mathcal{X}_2) = -g(\mathcal{X}_1, \mathcal{X}_2) + \eta(\mathcal{X}_1)\eta(\mathcal{X}_2)$ and $Q\varphi = \varphi Q$, we obtain

$$\mathcal{S}(\varphi\mathcal{X}_1, \varphi\mathcal{X}_2) = -\mathcal{S}(\mathcal{X}_1, \mathcal{X}_2) - 2n\eta(\mathcal{X}_1)\eta(\mathcal{X}_2). \tag{2.10}$$

3. $\varphi-$Ricci Symmetric Para-Kenmotsu Manifolds

Definition 3.1. A para-Kenmotsu manifold is said to be φ-symmetric if

$$\varphi^2\left(\left(\widetilde{\nabla}_{\mathcal{X}_4}\widetilde{R}\right)(\mathcal{X}_1, \mathcal{X}_2)\mathcal{X}_3\right) = 0,$$

for arbitrary vector fields $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{X}_4$.

If vector fields $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{X}_4$ are orthogonal to ξ, then the manifold is said to be locally φ-symmetric. This notion was introduced for Sasakian manifold by Takahashi [27].

Definition 3.2. A para-Kenmotsu manifold M is said to be locally φ-Ricci symmetric, if

$$\varphi^2(\widetilde{\nabla}_{\mathcal{X}_1}Q)(\mathcal{X}_2) = 0$$

for any vector fields $\mathcal{X}_1, \mathcal{X}_2$ orthogonal to ξ.

Definition 3.3. [6] A para-Kenmotsu manifold M is said to be φ-Ricci symmetric, if the Ricci operator satisfies

$$\varphi^2(\widetilde{\nabla}_{\mathcal{X}_1}Q)(\mathcal{X}_2) = 0$$

for any vector fields $\mathcal{X}_1, \mathcal{X}_2$ on M and $\mathcal{S}(\mathcal{X}_1, \mathcal{X}_2) = g(Q\mathcal{X}_1, \mathcal{X}_2)$.

Definition 3.4. [6] A para-Kenmotsu manifold M is said to be Einstein manifold if its Ricci tensor $\mathcal{S}$ is of the form

$$\mathcal{S}(\mathcal{X}_1, \mathcal{X}_2) = \alpha g(\mathcal{X}_1, \mathcal{X}_2),$$

where α is a constant and $\mathcal{X}_1, \mathcal{X}_2$ are any vector fields on M.

Theorem 3.1. *A $(2n+1)$-dimensional φ-Ricci symmetric para-Kenmotsu manifold is an Einstein manifold.*

Proof. Let us consider that the manifold is φ-Ricci symmetric. Then we have the following condition:

$$\varphi^2(\widetilde{\nabla}_{\mathcal{X}_1} Q)(\mathcal{X}_2) = 0.$$

Using equation 2.1 in the above equation, we have

$$(\widetilde{\nabla}_{\mathcal{X}_1} Q)(\mathcal{X}_2) - \eta((\widetilde{\nabla}_{\mathcal{X}_1} Q)\mathcal{X}_2)\xi = 0. \tag{3.1}$$

From above equation 3.1, it follows that

$$g((\widetilde{\nabla}_{\mathcal{X}_1} Q)(\mathcal{X}_2), \mathcal{X}_3) - \eta((\widetilde{\nabla}_{\mathcal{X}_1} Q)(\mathcal{X}_2))\eta(\mathcal{X}_3) = 0 \tag{3.2}$$

which on simplifying gives

$$g\left(\widetilde{\nabla}_{\mathcal{X}_1}(Q\mathcal{X}_2), \mathcal{X}_3\right) - \mathcal{S}\left(\widetilde{\nabla}_{\mathcal{X}_1}\mathcal{X}_2, \mathcal{X}_3\right) - \eta\left(\left(\widetilde{\nabla}_{\mathcal{X}_1} Q\right)\mathcal{X}_2\right)\eta(\mathcal{X}_3) = 0.$$

Replacing $\mathcal{X}_2$ by ξ in above equation, we get

$$g\left(\widetilde{\nabla}_{\mathcal{X}_1}(Q\xi), \mathcal{X}_3\right) - \mathcal{S}(\widetilde{\nabla}_{\mathcal{X}_1}\xi, \mathcal{X}_3) - \eta((\widetilde{\nabla}_{\mathcal{X}_1} Q)\xi)\eta(\mathcal{X}_3) = 0. \tag{3.3}$$

By using equations 2.5 and 2.7 in above equation 3.3, we obtain

$$2n[g(\mathcal{X}_1, \mathcal{X}_3) - \eta(\mathcal{X}_1)\eta(\mathcal{X}_3)] + \mathcal{S}(\mathcal{X}_1, \mathcal{X}_3) - \eta(\mathcal{X}_1)\mathcal{S}(\xi, \mathcal{X}_3) + \eta((\widetilde{\nabla}_{\mathcal{X}_1} Q)\xi)\eta(\mathcal{X}_3) = 0, \tag{3.4}$$

which further implies

$$2ng(\mathcal{X}_1, \mathcal{X}_3) + \mathcal{S}(\mathcal{X}_1, \mathcal{X}_3) + \eta((\widetilde{\nabla}_{\mathcal{X}_1} Q)\xi)\eta(\mathcal{X}_3) = 0 \tag{3.5}$$

by using equation 2.8.

Now, replacing $\mathcal{X}_1$ by $\varphi\mathcal{X}_1$ and $\mathcal{X}_3$ by $\varphi\mathcal{X}_3$ in equation 3.5, we have

$$\mathcal{S}(\varphi\mathcal{X}_1, \varphi\mathcal{X}_3) = -2ng(\varphi\mathcal{X}_1, \varphi\mathcal{X}_3). \tag{3.6}$$

In account of equations 2.2 and 2.10, 3.6 becomes

$$\mathcal{S}(\mathcal{X}_1, \mathcal{X}_3) = -2ng(\mathcal{X}_1, \mathcal{X}_3) \tag{3.7}$$

which implies that the manifold is an Einstein manifold. □

Moreover, in the view of definitions 3.1 and 3.3, we can observe that a φ-symmetric para-Kenmotsu manifold is also φ-Ricci symmetric, hence we have the following corollary:

Corollary 3.1. *A $(2n+1)$-dimensional φ-symmetric para-Kenmotsu manifold is an Einstein manifold.*

Theorem 3.2. *If a $(2n+1)$-dimensional para-Kenmotsu manifold is an Einstein manifold, then it is φ-Ricci symmetric.*

Proof. Let us suppose that the manifold is an Einstein manifold. Then

$$\mathcal{S}(\mathcal{X}_1, \mathcal{X}_2) = \alpha g(\mathcal{X}_1, \mathcal{X}_2),$$

where $\mathcal{S}(\mathcal{X}_1, \mathcal{X}_2) = g(Q\mathcal{X}_1, \mathcal{X}_2)$ and α is a constant. Hence $Q\mathcal{X}_1 = \alpha\mathcal{X}_1$. Thus, we have

$$\varphi^2 \left(\left(\widetilde{\nabla}_{\mathcal{X}_2} Q \right) (\mathcal{X}_1) \right) = 0.$$

This completes the proof. $\square$

In account of Theorem 3.1 and Theorem 3.2, we have

Theorem 3.3. *A (2n+1)-dimensional para-Kenmotsu manifold is φ-Ricci symmetric if and only if it is an Einstein manifold.*

4. Three-dimensional φ-Ricci symmetric para-Kenmotsu manifolds

Theorem 4.1. *If the scalar curvature r of a 3-dimensional para-Kenmotsu manifold is equal to -6, then the manifold is φ-Ricci symmetric.*

Proof. It is known that for any 3-dimensional pseudo Riemannian manifold, we have the following well known expression:

$$\begin{aligned} \widetilde{R}(\mathcal{X}_1, \mathcal{X}_2)\mathcal{X}_3 &= g(\mathcal{X}_2, \mathcal{X}_3)Q\mathcal{X}_1 - g(\mathcal{X}_1, \mathcal{X}_3)Q\mathcal{X}_2 + \mathcal{S}(\mathcal{X}_2, \mathcal{X}_3)\mathcal{X}_1 \\ &\quad - \mathcal{S}(\mathcal{X}_1, \mathcal{X}_3)\mathcal{X}_2 - \frac{r}{2}\{g(\mathcal{X}_2, \mathcal{X}_3)\mathcal{X}_1 - g(\mathcal{X}_1, \mathcal{X}_3)\mathcal{X}_2\}. \end{aligned} \tag{4.1}$$

Setting $\mathcal{X}_2 = \mathcal{X}_3 = \xi$ in above relation and making use of equations 2.6 and 2.8, we obtain

$$Q\mathcal{X}_1 = (1 + \frac{r}{2})\mathcal{X}_1 - (\frac{r+6}{2})\eta(\mathcal{X}_1)\xi \tag{4.2}$$

which is equivalent to

$$\mathcal{S}(\mathcal{X}_1, \mathcal{X}_2) = g(Q\mathcal{X}_1, \mathcal{X}_2) = (1 + \frac{r}{2})g(\mathcal{X}_1, \mathcal{X}_2) - (\frac{r+6}{2})\eta(\mathcal{X}_1)\eta(\mathcal{X}_2). \tag{4.3}$$

Taking the covariant differentiation of equation 4.2 with respect to $\mathcal{X}_4$, we get

$$\begin{aligned} (\widetilde{\nabla}_{\mathcal{X}_4} Q)\mathcal{X}_1 + Q(\widetilde{\nabla}_{\mathcal{X}_4}\mathcal{X}_1) &= \left(\frac{1}{2}\right) dr(\mathcal{X}_4)\mathcal{X}_1 + (\frac{r+2}{2})(\widetilde{\nabla}_{\mathcal{X}_4}\mathcal{X}_1) - [\left(\frac{1}{2}\right) dr(\mathcal{X}_4)\eta(\mathcal{X}_1)\xi \\ &\quad + (\frac{r+6}{2})\mathcal{X}_4\{\eta(\mathcal{X}_1)\}\xi + (\frac{r+6}{2})\eta(\mathcal{X}_1)\widetilde{\nabla}_{\mathcal{X}_4}\xi] \end{aligned}$$

which gives

$$\left(\widetilde{\nabla}_{\mathcal{X}_4} Q\right)\mathcal{X}_1 = \left(\frac{1}{2}\right) dr(\mathcal{X}_4)\mathcal{X}_1 - \left(\frac{1}{2}\right) dr(\mathcal{X}_4)\eta(\mathcal{X}_1)\xi$$
$$- \left(\frac{r+6}{2}\right) g(\mathcal{X}_1, -\varphi\mathcal{X}_4)\xi - \left(\frac{r+6}{2}\right)\eta(\mathcal{X}_1)\left(\widetilde{\nabla}_{\mathcal{X}_4}\xi\right).$$

Now, applying φ^2 on both sides of above equation and using equation 2.1, we have

(4.4)
$$\varphi^2\left(\left(\widetilde{\nabla}_{\mathcal{X}_4} Q\right)(\mathcal{X}_1)\right) = \left(\frac{1}{2}\right)\left[dr(\mathcal{X}_4)(-\mathcal{X}_1 + \eta(\mathcal{X}_1)\xi) - (r+6)\eta(\mathcal{X}_1)\varphi^2\left(\widetilde{\nabla}_{\mathcal{X}_4}\xi\right)\right].$$

This completes the proof of the theorem. □

Theorem 4.2. *A 3-dimensional para-Kenmotsu manifold is locally φ-Ricci symmetric if and only if the scalar curvature r is constant.*

Proof. Taking $\mathcal{X}_1$ orthogonal to ξ in equation 4.4, we obtain

$$\varphi^2\left(\left(\widetilde{\nabla}_{\mathcal{X}_4} Q\right)(\mathcal{X}_1)\right) = -\frac{1}{2} dr(\mathcal{X}_4)\mathcal{X}_1. \tag{4.5}$$

Hence, the proof follows from above equation and theorem 4.1. □

5. Examples

In this section, we give examples of three-dimensional φ-Ricci symmetric as well as locally φ-Ricci symmetric para-Kenmotsu manifold which verifies theorem 4.1 and 4.2 as well as theorem 3.3.

Example 5.1. We consider three-dimensional manifold $M = \left\{(x,y,z) \in \mathcal{R}^3, z \neq 0\right\}$ with the cartesian coordinates (x,y,z) and the vector fields:

$$\epsilon_1 = \varphi\epsilon_2, \quad \epsilon_2 = \varphi\epsilon_1, \quad \varphi\epsilon_3 = 0,$$

where

$$\epsilon_1 = \frac{\partial}{\partial x}, \quad \epsilon_2 = \frac{\partial}{\partial y}, \quad \epsilon_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$

are linearly independent vectors at each point of the manifold. The 1-form $\eta = dz$ defines an almost para contact structure on M with characteristic vector field $\xi = \epsilon_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$. Let g be a pseudo-Riemannian metric defined by: $g = dx^2 - dy^2 + (1 - x^2 + y^2)dz^2$. Using Koszul's formula,

$$2g(\widetilde{\nabla}_{\mathcal{X}_1}\mathcal{X}_2, \mathcal{X}_3) = \mathcal{X}_1(g(\mathcal{X}_2,\mathcal{X}_3)) + \mathcal{X}_2(g(\mathcal{X}_3,\mathcal{X}_1)) - \mathcal{X}_3(g(\mathcal{X}_1,\mathcal{X}_2)) - g(\mathcal{X}_1,[\mathcal{X}_2,\mathcal{X}_3]) +$$
$$g(\mathcal{X}_2,[\mathcal{X}_3,\mathcal{X}_1]) + g(\mathcal{X}_3,[\mathcal{X}_1,\mathcal{X}_2])$$

we have the followings:

$$\widetilde{\nabla}_{\epsilon_1}\epsilon_1 = -\epsilon_3, \qquad \widetilde{\nabla}_{\epsilon_1}\epsilon_2 = 0, \qquad \widetilde{\nabla}_{\epsilon_1}\epsilon_3 = \epsilon_1,$$
$$\widetilde{\nabla}_{\epsilon_2}\epsilon_1 = 0, \qquad \widetilde{\nabla}_{\epsilon_2}\epsilon_2 = \epsilon_3, \qquad \widetilde{\nabla}_{\epsilon_2}\epsilon_3 = \epsilon_2,$$
$$\widetilde{\nabla}_{\epsilon_3}\epsilon_1 = 0, \qquad \widetilde{\nabla}_{\epsilon_3}\epsilon_2 = 0, \qquad \widetilde{\nabla}_{\epsilon_3}\epsilon_3 = 0.$$

It is not hard to verify that the conditions 2.4 and 2.5 for para-Kenmotsu manifold are satisfied. Hence, the manifold under consideration is a para-Kenmotsu manifold. Now, the components of the curvature tensor are given by:

$$\widetilde{R}\left(\epsilon_1,\epsilon_2\right)\epsilon_1 = \epsilon_2, \qquad \widetilde{R}\left(\epsilon_1,\epsilon_2\right)\epsilon_2 = \epsilon_1, \qquad \widetilde{R}\left(\epsilon_1,\epsilon_2\right)\epsilon_3 = 0,$$
$$\widetilde{R}\left(\epsilon_1,\epsilon_3\right)\epsilon_1 = \epsilon_3, \qquad \widetilde{R}\left(\epsilon_1,\epsilon_3\right)\epsilon_2 = 0, \qquad \widetilde{R}\left(\epsilon_1,\epsilon_3\right)\epsilon_3 = -\epsilon_1,$$
$$\widetilde{R}\left(\epsilon_2,\epsilon_3\right)\epsilon_1 = 0, \qquad \widetilde{R}\left(\epsilon_2,\epsilon_3\right)\epsilon_2 = -\epsilon_3, \qquad \widetilde{R}\left(\epsilon_2,\epsilon_3\right)\epsilon_3 = -\epsilon_2,$$

which further gives

$$\widetilde{R}(\mathcal{X}_1,\mathcal{X}_2)\mathcal{X}_3 = -[g(\mathcal{X}_2,\mathcal{X}_3)\mathcal{X}_1 - g(\mathcal{X}_1,\mathcal{X}_3)\mathcal{X}_2].$$

Now, since

$$\mathcal{S}(\mathcal{X}_1,\mathcal{X}_2) = \sum_{\alpha=1}^{3}\lambda_\alpha g(\widetilde{R}(\epsilon_\alpha,\mathcal{X}_1)\mathcal{X}_2,\epsilon_\alpha),$$

where $\lambda_\alpha = g(\epsilon_\alpha,\epsilon_\alpha)$ and $\alpha = 1,2,3$.
Thus, the components of Ricci tensor are given by:

$$\mathcal{S}\left(\epsilon_1,\epsilon_1\right) = -2, \qquad \mathcal{S}\left(\epsilon_2,\epsilon_2\right) = 2, \qquad \mathcal{S}\left(\epsilon_3,\epsilon_3\right) = -2.$$

Hence, we have constant scalar curvature as follows:

$$r = \mathcal{S}\left(\epsilon_1,\epsilon_1\right) - \mathcal{S}\left(\epsilon_2,\epsilon_2\right) + \mathcal{S}\left(\epsilon_3,\epsilon_3\right) = -6.$$

Since, the scalar curvature and Ricci tensor of the manifold under consideration is given by $r=-6$ and $\mathcal{S}(\mathcal{X}_1,\mathcal{X}_2) = -2g(\mathcal{X}_1,\mathcal{X}_2)$ respectively, where $Q\mathcal{X}_1 = -2\mathcal{X}_1$ which further implies that $\varphi^2\left(\left(\widetilde{\nabla}_{\mathcal{X}_4}Q\right)(\mathcal{X}_1)\right) = 0$. This leads us to the conclusion that M is φ-Ricci symmetric, which proves theorem 4.1. Additionally, it is simple to validate the conclusion of theorem 4.2. In addition to this, M is an Einstein manifold due to the fact that $\mathcal{S}(\mathcal{X}_1,\mathcal{X}_2) = -2g(\mathcal{X}_1,\mathcal{X}_2)$ which validates the theorem 3.3 .

Example 5.2. Consider, a three-dimensional manifold $\mathcal{M}^3 = \left\{(x,y,z)\in\mathbb{R}^3, z\neq 0\right\}$, where (x,y,z) are the standard coordinates in $\mathbb{R}^3$ and the vector fields are given as:

$$\delta_1 = e^z\frac{\partial}{\partial x}, \quad \delta_2 = e^z\frac{\partial}{\partial y}, \quad \delta_3 = -\frac{\partial}{\partial z}$$

are linearly independent vectors at each point of the manifold.

Now, we define

$$\delta_1 = \varphi\delta_2, \quad \delta_2 = \varphi\delta_1, \quad \varphi\delta_3 = 0,$$
$$\xi = -\frac{\partial}{\partial z}, \qquad \eta = -dz,$$

$$g(\delta_1, \delta_1) = 1, \qquad g(\delta_2, \delta_2) = -1, \qquad g(\delta_3, \delta_3) = 1$$

and

$$g(\delta_\alpha, \delta_\beta) = 0, \qquad if \quad \alpha \neq \beta \qquad \alpha, \beta = 1, 2, 3.$$

Then it follows that

$$\eta(\delta_1) = 0, \qquad \eta(\delta_2) = 0, \qquad \eta(\delta_3) = 1.$$

Let $\widetilde{\nabla}$ be the Levi-Civita connection with respect to metric g, then we obtain the followings:

$$[\delta_1, \delta_2] = 0, \qquad [\delta_1, \delta_3] = \delta_1, \qquad [\delta_2, \delta_3] = \delta_2.$$

Now, in view of Koszul's formula

$$\begin{aligned} 2g(\widetilde{\nabla}_{\mathcal{X}_1}\mathcal{X}_2, \mathcal{X}_3) =& \mathcal{X}_1(g(\mathcal{X}_2, \mathcal{X}_3)) + \mathcal{X}_2(g(\mathcal{X}_3, \mathcal{X}_1)) - \mathcal{X}_3(g(\mathcal{X}_1, \mathcal{X}_2)) - g(\mathcal{X}_1, [\mathcal{X}_2, \mathcal{X}_3]) + \\ & g(\mathcal{X}_2, [\mathcal{X}_3, \mathcal{X}_1]) + g(\mathcal{X}_3, [\mathcal{X}_1, \mathcal{X}_2]) \end{aligned}$$

we can deduce the following relations:

$$\begin{array}{lll} \widetilde{\nabla}_{\delta_1}\delta_1 = -\delta_3, & \widetilde{\nabla}_{\delta_1}\delta_2 = 0, & \widetilde{\nabla}_{\delta_1}\delta_3 = \delta_1, \\ \widetilde{\nabla}_{\delta_2}\delta_1 = 0, & \widetilde{\nabla}_{\delta_2}\delta_2 = \delta_3, & \widetilde{\nabla}_{\delta_2}\delta_3 = \delta_2, \\ \widetilde{\nabla}_{\delta_3}\delta_1 = 0, & \widetilde{\nabla}_{\delta_3}\delta_2 = 0, & \widetilde{\nabla}_{\delta_3}\delta_3 = 0. \end{array}$$

With the help of above results, we can see that manifold satisfies

$$\widetilde{\nabla}_{\mathcal{X}_1}\xi = \mathcal{X}_1 - \eta(\mathcal{X}_1)\xi,$$

for $\delta_3 = \xi$. Hence, the manifold $\mathcal{M}^3(\varphi, \xi, \eta, g)$ is a para-Kenmotsu manifold of dimension three.

Now, in view of definition of curvature tensor which is given as follows:

$$\widetilde{R}(\mathcal{X}_1, \mathcal{X}_2)\mathcal{X}_3 = \widetilde{\nabla}_{\mathcal{X}_1}\widetilde{\nabla}_{\mathcal{X}_2}\mathcal{X}_3 - \widetilde{\nabla}_{\mathcal{X}_2}\widetilde{\nabla}_{\mathcal{X}_1}\mathcal{X}_3 - \widetilde{\nabla}_{[\mathcal{X}_1, \mathcal{X}_2]}\mathcal{X}_3,$$

we can easily verify the following observations:

$$\begin{array}{lll} \widetilde{R}(\delta_1, \delta_2)\delta_1 = \delta_2, & \widetilde{R}(\delta_1, \delta_2)\delta_2 = \delta_1, & \widetilde{R}(\delta_1, \delta_2)\delta_3 = 0, \\ \widetilde{R}(\delta_1, \delta_3)\delta_1 = \delta_3, & \widetilde{R}(\delta_1, \delta_3)\delta_2 = 0, & \widetilde{R}(\delta_1, \delta_3)\delta_3 = -\delta_1, \\ \widetilde{R}(\delta_2, \delta_3)\delta_1 = 0, & \widetilde{R}(\delta_2, \delta_3)\delta_2 = -\delta_3, & \widetilde{R}(\delta_2, \delta_3)\delta_3 = -\delta_2, \end{array}$$

which further gives

$$\widetilde{R}(\mathcal{X}_1, \mathcal{X}_2)\mathcal{X}_3 = -[g(\mathcal{X}_2, \mathcal{X}_3)\mathcal{X}_1 - g(\mathcal{X}_1, \mathcal{X}_3)\mathcal{X}_2].$$

From the definition of Ricci tensor in 3-dimensional manifold,

$$\mathcal{S}(\mathcal{X}_1, \mathcal{X}_2) = \sum_{\alpha=1}^{3} \lambda_\alpha g(\widetilde{R}(\delta_\alpha, \mathcal{X}_1)\mathcal{X}_2, \delta_\alpha),$$

where $\lambda_\alpha = g(\delta_\alpha, \delta_\alpha)$ and $\alpha = 1, 2, 3$.

Thus, using the components of curvature tensor, we obtain the following results:

$$\mathcal{S}(\delta_1,\delta_1) = -2, \qquad \mathcal{S}(\delta_2,\delta_2) = 2, \qquad \mathcal{S}(\delta_3,\delta_3) = -2.$$

Hence, we have scalar curvature r of manifold as follows:

$$r = \mathcal{S}(\delta_1,\delta_1) - \mathcal{S}(\delta_2,\delta_2) + \mathcal{S}(\delta_3,\delta_3) = -6.$$

Since, the scalar curvature and Ricci tensor of the manifold under consideration is given by $r = -6$ and $\mathcal{S}(\mathcal{X}_1,\mathcal{X}_2) = -2g(\mathcal{X}_1,\mathcal{X}_2)$ respectively, where $Q\mathcal{X}_1 = -2\mathcal{X}_1$. Thus, we can conclude that $\mathcal{M}^3$ is φ-Ricci symmetric, which verifies the theorem 4.1. Also one can easily verify the theorems 3.3 and 4.2.

6. Weakly φ-Ricci Symmetric Para Kenmotsu Manifolds

Definition 6.1. A para-Kenmotsu manifold M of $2n+1$-dimension with almost para contact structure (φ,ξ,η,g) is said to be weakly φ-Ricci symmetric if the Ricci operator satisfies the following condition:

$$\varphi^2((\nabla_{\mathcal{X}_1}Q)\mathcal{X}_2) = \mathcal{A}(\mathcal{X}_1)\varphi^2(Q(\mathcal{X}_2)) + \mathcal{B}(\mathcal{X}_2)\varphi^2(Q(\mathcal{X}_1)) + \mathcal{S}(\mathcal{X}_2,\mathcal{X}_1)\varphi^2(\rho). \tag{6.1}$$

where, $\mathcal{X}_1,\mathcal{X}_2$ are any vector fields on M. $\mathcal{A},\mathcal{B},\mathcal{D}$ are 1-form and ρ is a vector field associated with 1-form $\mathcal{D}$ by relation $g(\rho,\mathcal{X}) = \mathcal{D}(\mathcal{X})$.

Definition 6.2. A para-Kenmotsu manifold M is said to be η-Einstein if its Ricci tensor $\mathcal{S}$ of type $(0,2)$ is of the form

$$\mathcal{S} = \varsigma g + \varepsilon\eta\otimes\eta \tag{6.2}$$

where ς,ε are smooth functions on M.

If the one-forms $\mathcal{A} = \mathcal{B} = \rho = 0$, then above relation 6.1 reduces to the concept of φ-Ricci symmetric which is given by the following:

$$\varphi^2((\nabla_{\mathcal{X}_1}Q)\mathcal{X}_2) = 0. \tag{6.3}$$

Initially, this concept was introduced by S.S. Shukla and M.K. Shukla [24]. Now, consider weakly φ- Ricci symmetric para-Kenmotsu manifold $M(\varphi,\xi,\eta,g)$ and in the account of equations 2.1 and 6.1, we obtain

$$\begin{aligned}(\nabla_{\mathcal{X}_1}Q)(\mathcal{X}_2) - \eta((\nabla_{\mathcal{X}_1}Q)(\mathcal{X}_2))\xi &= \mathcal{A}(\mathcal{X}_1)[Q\mathcal{X}_2 - \eta(Q\mathcal{X}_2)\xi] \\ &\quad + \mathcal{B}(\mathcal{X}_2)[Q\mathcal{X}_1 - \eta(Q\mathcal{X}_1)\xi] + \mathcal{S}(\mathcal{X}_2,\mathcal{X}_1)[\rho - \eta(\rho)\xi]\end{aligned} \tag{6.4}$$

which further gives

$$\begin{aligned}\nabla_{\mathcal{X}_1}(Q(\mathcal{X}_2)) - g((\nabla_{\mathcal{X}_1}Q)(\mathcal{X}_2),\xi)\xi &= \mathcal{A}(\mathcal{X}_1)Q\mathcal{X}_2 - \mathcal{A}(\mathcal{X}_1)\eta(Q\mathcal{X}_2)\xi + \mathcal{B}(\mathcal{X}_2)Q\mathcal{X}_1 \\ &\quad - \mathcal{B}(\mathcal{X}_2)\eta(Q\mathcal{X}_1)\xi + \mathcal{S}(\mathcal{X}_2,\mathcal{X}_1)\rho - \mathcal{S}(\mathcal{X}_2,\mathcal{X}_1)\eta(\rho)\xi.\end{aligned} \tag{6.5}$$

Now, taking inner product w.r.t. $\mathcal{X}_3$ in above relation, we obtain

$$\begin{aligned} g(\nabla_{\mathcal{X}_1}(Q(\mathcal{X}_2)), \mathcal{X}_3) - g((\nabla_{\mathcal{X}_1}Q)(\mathcal{X}_2), \xi)g(\xi, \mathcal{X}_3) \\ = \mathcal{A}(\mathcal{X}_1)g(Q\mathcal{X}_2, \mathcal{X}_3) - \mathcal{A}(\mathcal{X}_1)\eta(Q\mathcal{X}_2)g(\xi, \mathcal{X}_3) \\ + \mathcal{B}(\mathcal{X}_2)g(Q\mathcal{X}_1, \mathcal{X}_3) - \mathcal{B}(\mathcal{X}_2)\eta(Q\mathcal{X}_1)g(\xi, \mathcal{X}_3) \\ + \mathcal{S}(\mathcal{X}_2, \mathcal{X}_1)g(\rho, \mathcal{X}_3) - \mathcal{S}(\mathcal{X}_2, \mathcal{X}_1)\eta(\rho)g(\xi, \mathcal{X}_3) \end{aligned}$$

and $\mathcal{X}_2 = \xi$ in above equation, we get

$$\begin{aligned} g(\nabla_{\mathcal{X}_1}(Q\xi), \mathcal{X}_3) - g((\nabla_{\mathcal{X}_1}Q)\xi, \xi)\eta(\mathcal{X}_3) = \mathcal{A}(\mathcal{X}_1)g(Q\xi, \mathcal{X}_3) - \mathcal{A}(\mathcal{X}_1)g(Q\xi, \xi)\eta(\mathcal{X}_3) \\ + \mathcal{B}(\xi)g(Q\mathcal{X}_1, \mathcal{X}_3) - \mathcal{B}(\xi)g(Q\mathcal{X}_1, \xi)\eta(\mathcal{X}_3) \\ + \mathcal{S}(\xi, \mathcal{X}_1)g(\rho, \mathcal{X}_3) - \mathcal{S}(\xi, \mathcal{X}_1)g(\rho, \xi)\eta(\mathcal{X}_3). \end{aligned} \tag{6.6}$$

Now, in the account of equations 2.5, 2.8 and 6.6, we have

$$\begin{aligned} -\mathcal{S}(\mathcal{X}_1, \mathcal{X}_3) = 2ng(\mathcal{X}_1, \mathcal{X}_3) + \mathcal{B}(\xi)\mathcal{S}(\mathcal{X}_1, \mathcal{X}_3) + 2n\eta(\mathcal{X}_1)\eta(\mathcal{X}_3)\mathcal{B}(\xi) \\ - 2n\eta(\mathcal{X}_1)\mathcal{D}(\mathcal{X}_3) + 2n\eta(\mathcal{X}_1)\eta(\mathcal{X}_3)\mathcal{D}(\xi) \end{aligned} \tag{6.7}$$

Replacing $\mathcal{X}_1$ by $\varphi\mathcal{X}_1$ and $\mathcal{X}_3$ by $\varphi\mathcal{X}_3$ in 6.7, we obtain

$$[1 + \mathcal{B}(\xi)]\mathcal{S}(\varphi\mathcal{X}_1, \varphi\mathcal{X}_3) = -2ng(\varphi\mathcal{X}_1, \varphi\mathcal{X}_3). \tag{6.8}$$

By virtue of equations 2.2 and 2.10, we get

$$\mathcal{S}(\mathcal{X}_1, \mathcal{X}_3) = \varsigma g(\mathcal{X}_1, \mathcal{X}_3) + \varepsilon\eta(\mathcal{X}_1)\eta(\mathcal{X}_3) \tag{6.9}$$

where,

$$\varsigma = \frac{-2n}{1 + \mathcal{B}(\xi)} \qquad \textit{and} \qquad \varepsilon = \frac{-2n\mathcal{B}(\xi)}{1 + \mathcal{B}(\xi)}$$

provided $1 + \mathcal{B}(\xi) \neq 0$. Therefore, we can state the following theorem:

Theorem 6.1. *A weakly φ-Ricci symmetric para-Kenmotsu manifold is an η-Einstein manifold.*

7. Ricci Yamabe Solitons on Para-Kenmotsu Manifolds

Assume that the Para-Kenmotsu manifold admits a Ricci-Yamabe soliton $(g, \xi, \lambda, \alpha, \beta)$. Then from equation 1.2 following relation holds:

$$(\pounds_\xi g)(\mathcal{X}_1, \mathcal{X}_2) + 2\alpha\mathcal{S}(\mathcal{X}_1, \mathcal{X}_2) + (2\lambda - \beta r)g(\mathcal{X}_1, \mathcal{X}_2) = 0 \tag{7.1}$$

Since,

$$(\pounds_\xi g)(\mathcal{X}_1, \mathcal{X}_2) = g(\mathcal{X}_2, \nabla_{\mathcal{X}_1}\xi) + g(\mathcal{X}_1, \nabla_{\mathcal{X}_2}\xi),$$

so with the help of equation 2.5, we obtain the following

$$(\pounds_\xi g)(\mathcal{X}_1, \mathcal{X}_2) = 2[g(\mathcal{X}_1, \mathcal{X}_2) - \eta(\mathcal{X}_1)\eta(\mathcal{X}_2)]. \tag{7.2}$$

In the account of above equations 7.1 and 7.2, we get

$$\begin{aligned}\mathcal{S}(\mathcal{X}_1,\mathcal{X}_2) &= \frac{1}{2\alpha}[g(\mathcal{X}_1,\mathcal{X}_2)(-2-2\lambda+\beta r)+2\eta(\mathcal{X}_1)\eta(\mathcal{X}_2)] \\ &= [\frac{-2-2\lambda+\beta r}{2\alpha}]g(\mathcal{X}_1,\mathcal{X}_2)+\frac{1}{\alpha}\eta(\mathcal{X}_1)\eta(\mathcal{X}_2) \\ &= \varsigma g(\mathcal{X}_1,\mathcal{X}_2)+\varepsilon\eta(\mathcal{X}_1)\eta(\mathcal{X}_2)\end{aligned} \tag{7.3}$$

where, $\varsigma = \frac{-2-2\lambda+\beta r}{2\alpha}$ and $\varepsilon = \frac{1}{\alpha}$
which results the following theorem:

Theorem 7.1. *Let $\widetilde{M}^{2n+1}$ be a para-Kenmotsu manifold with almost para contact structure (φ,ξ,η,g) and it admits a Ricci-Yamabe soliton then the manifold is η-Einstein manifold.*

8. Ricci Yamabe Solitons on Para-Kenmotsu Manifolds satisfying $\widetilde{\mathbf{R}}(\xi,\mathcal{X}_1).\mathcal{S}=\mathbf{0}$

Let $\widetilde{M}^{2n+1}$ be a para-Kenmotsu manifold admitting Ricci-Yamabe soliton satisfies the condition $\widetilde{R}(\xi,\mathcal{X}_1).\mathcal{S}=0$. Then we have

$$\mathcal{S}(\widetilde{R}(\xi,\mathcal{X}_1)\mathcal{X}_2,\mathcal{X}_3)+\mathcal{S}(\mathcal{X}_2,\widetilde{R}(\xi,\mathcal{X}_1)\mathcal{X}_3)=0.$$

With help of equation 2.7, above equation yields the following:

$$\begin{aligned}&\mathcal{S}(\eta(\mathcal{X}_2)\mathcal{X}_1-g(\mathcal{X}_1,\mathcal{X}_2)\xi,\mathcal{X}_3)+\mathcal{S}(\mathcal{X}_2,\eta(\mathcal{X}_3)\mathcal{X}_1-g(\mathcal{X}_1,\mathcal{X}_3)\xi)=0 \\ &\implies \eta(\mathcal{X}_2)\mathcal{S}(\mathcal{X}_1,\mathcal{X}_3)-g(\mathcal{X}_1,\mathcal{X}_2)\mathcal{S}(\xi,\mathcal{X}_3)+\mathcal{S}(\mathcal{X}_2,\mathcal{X}_1)\eta(\mathcal{X}_3)-g(\mathcal{X}_1,\mathcal{X}_3)\mathcal{S}(\mathcal{X}_2,\xi)=0.\end{aligned} \tag{8.1}$$

Now, putting $\mathcal{X}_3=\xi$, we obtain

$$\eta(\mathcal{X}_2)\mathcal{S}(\mathcal{X}_1,\xi)-g(\mathcal{X}_1,\mathcal{X}_2)\mathcal{S}(\xi,\xi)+\mathcal{S}(\mathcal{X}_2,\mathcal{X}_1)\eta(\xi)-g(\mathcal{X}_1,\xi)\mathcal{S}(\mathcal{X}_2,\xi)=0. \tag{8.2}$$

Since, equation 7.3 gives $\mathcal{S}(\mathcal{X}_1,\xi)=\frac{1}{\alpha}(\frac{\beta r}{2}-\lambda)\eta(\mathcal{X}_1)$. Hence, using equations 2.8 and 7.3 in equation 8.2, one can easily get

$$\mathcal{S}(\mathcal{X}_1,\mathcal{X}_2)=\frac{1}{\alpha}(\frac{\beta r}{2}-\lambda)g(\mathcal{X}_1,\mathcal{X}_2). \tag{8.3}$$

Therefore, we can state an important result as follows:

Theorem 8.1. *Let $\widetilde{M}^{2n+1}$ be a para-Kenmotsu manifold with almost para contact structure (φ,ξ,η,g) and it admits Ricci-Yamabe soliton satisfying $\widetilde{R}(\xi,\mathcal{X}_1).\mathcal{S}=0$, then the manifold is Einstein manifold.*

Corollary 8.1. *Let $\widetilde{M}^{2n+1}$ be a para-Kenmotsu manifold with almost para contact structure (φ,ξ,η,g). If $\widetilde{M}^{2n+1}$ satisfies the curvature condition $\widetilde{R}(\xi,\mathcal{X}_1).\mathcal{S}=0$, then the manifold is φ-Ricci symmetric.*

9. Parallel 2-form in the Para-Kenmotsu manifolds

Definition 9.1. A tensor α of second order is said to be a second order parallel tensor if $\widetilde{\nabla}\alpha = 0$, where $\widetilde{\nabla}$ denotes the operator of covariant differentiation with respect to the metric g [26].

Theorem 9.1. *On a para-Kenmotsu manifold M, there is no non-zero parallel 2-form.*

Proof. We assume α to be a $(0,2)$ type skew symmetric tensor. By definition, it is parallel, if $\widetilde{\nabla}\alpha = 0$. This provides the relation given below

$$\alpha(\widetilde{R}(\mathcal{X}_4, \mathcal{X}_1)\mathcal{X}_2, \mathcal{X}_3) + \alpha(\mathcal{X}_2, \widetilde{R}(\mathcal{X}_4, \mathcal{X}_1)\mathcal{X}_3) = 0 \tag{9.1}$$

for all vector fields $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{X}_4$ on M. Putting $\mathcal{X}_4 = \mathcal{X}_2 = \xi$ in the equation, we obtain

$$\alpha(\widetilde{R}(\xi, \mathcal{X}_1)\xi, \mathcal{X}_3) + \alpha(\xi, \widetilde{R}(\xi, \mathcal{X}_1)\mathcal{X}_3) = 0.$$

Using the equations 2.6 and 2.7, we obtain

$$\alpha(\mathcal{X}_1, \mathcal{X}_3) = \eta(\mathcal{X}_1)\alpha(\xi, \mathcal{X}_3) - \eta(\mathcal{X}_3)\alpha(\xi, \mathcal{X}_1) - g(\mathcal{X}_1, \mathcal{X}_3)\alpha(\xi, \xi). \tag{9.2}$$

Since, α is $(0,2)$ skew-symmetric tensor, which implies that $\alpha(\xi, \xi) = 0$, therefore equation 9.2 reduces to

$$\alpha(\mathcal{X}_1, \mathcal{X}_3) = \eta(\mathcal{X}_1)\alpha(\xi, \mathcal{X}_3) - \eta(\mathcal{X}_3)\alpha(\xi, \mathcal{X}_1). \tag{9.3}$$

Now, let A be $(1,1)$ tensor field, which is metrically equivalent to α i.e. $\alpha(\mathcal{X}_1, \mathcal{X}_2) = g(A\mathcal{X}_1, \mathcal{X}_2)$, then the equation 9.3 becomes

$$g(A\mathcal{X}_1, \mathcal{X}_3) = \eta(\mathcal{X}_1)g(A\xi, \mathcal{X}_3) - \eta(\mathcal{X}_3)g(A\xi, \mathcal{X}_1)$$

which implies that

$$A\mathcal{X}_1 = \eta(\mathcal{X}_1)A\xi - g(A\xi, \mathcal{X}_1)\xi. \tag{9.4}$$

Since, α is parallel, hence A is also parallel and applying $\widetilde{\nabla}_{\mathcal{X}_1}\xi = \varphi^2\mathcal{X}_1 = \mathcal{X}_1 - \eta(\mathcal{X}_1)\xi$, it follows

$$\widetilde{\nabla}_{\mathcal{X}_1}(A\xi) = \left(\widetilde{\nabla}_{\mathcal{X}_1}A\right)\xi + A\left(\widetilde{\nabla}_{\mathcal{X}_1}\xi\right) = A(\mathcal{X}_1 - \eta(\mathcal{X}_1)\xi)$$

or

$$\widetilde{\nabla}_{\mathcal{X}_1}(A\xi) = A\mathcal{X}_1 - \eta(\mathcal{X}_1)A\xi.$$

With the help of the equation 9.4, the above equation reduces to

$$\widetilde{\nabla}_{\mathcal{X}_1}(A\xi) = -g(A\xi, \mathcal{X}_1)\xi.$$

Taking inner product of the above equation with respect to $A\xi$, we have

$$g\left(\widetilde{\nabla}_{\mathcal{X}_1}(A\xi), A\xi\right) = -g(A\xi, \mathcal{X}_1)g(A\xi, \xi).$$

Since $g(A\xi, \xi) = \alpha(\xi, \xi) = 0$, the above equation reduces to

$$g\left(\widetilde{\nabla}_{\mathcal{X}_1}(A\xi), A\xi\right) = 0,$$

for any tangent vector $\mathcal{X}_1$ and consequently $\|A\xi\| =$ constant on M.
From the above equation, we have

$$g\left(\left(\widetilde{\nabla}_{\mathcal{X}_1}A\right)\xi + A\left(\widetilde{\nabla}_{\mathcal{X}_1}\xi\right), A\xi\right) = 0.$$

As we know that A is parallel, thus the first term in the above equation vanishes and we have the following equation

$$g\left(A\left(\widetilde{\nabla}_{\mathcal{X}_1}\xi\right), A\xi\right) = 0.$$

The above equation implies that

$$\alpha\left(\widetilde{\nabla}_{\mathcal{X}_1}\xi, A\xi\right) = 0. \tag{9.5}$$

Since $\alpha(\mathcal{X}_1, \mathcal{X}_2) = -\alpha(\mathcal{X}_2, \mathcal{X}_1)$, hence equation 9.5 reduces to

$$-\alpha\left(A\xi, \widetilde{\nabla}_{\mathcal{X}_1}\xi\right) = 0.$$

which further implies

$$-g\left(A^2\xi, \widetilde{\nabla}_{\mathcal{X}_1}\xi\right) = 0.$$

Using relation $\widetilde{\nabla}_{\mathcal{X}_1}\xi = \mathcal{X}_1 - \eta(\mathcal{X}_1)\xi$ in the above equation, we get

$$g\left(\mathcal{X}_1 - \eta(\mathcal{X}_1)\xi, A^2\xi\right) = 0$$

or

$$g\left(\mathcal{X}_1, A^2\xi\right) - \eta(\mathcal{X}_1)g\left(\xi, A^2\xi\right) = 0$$

or

$$g\left(\mathcal{X}_1, A^2\xi\right) = g\left(\xi, A^2\xi\right)g(\xi, \mathcal{X}_1).$$

which further on simplification gives

$$A^2\xi = -g(A\xi, A\xi)\xi = -\|A\xi\|^2\xi. \tag{9.6}$$

Differentiating above equation covariantly with respect to $\mathcal{X}_1$, we obtain

$$\widetilde{\nabla}_{\mathcal{X}_1}\left(A^2\xi\right) = \left(\widetilde{\nabla}_{\mathcal{X}_1}A^2\right)\xi + A^2\left(\widetilde{\nabla}_{\mathcal{X}_1}\xi\right),$$

or

$$\widetilde{\nabla}_{\mathcal{X}_1}\left(A^2\xi\right) = A^2\mathcal{X}_1 - \eta(\mathcal{X}_1)A^2\xi.$$

In the account of equation 9.6, foregoing equation becomes

$$-\widetilde{\nabla}_{\mathcal{X}_1}\left(\|A\xi\|^2\xi\right) = A^2\mathcal{X}_1 + \eta(\mathcal{X}_1)\|A\xi\|^2\xi,$$

or

$$-\|A\xi\|^2\widetilde{\nabla}_{\mathcal{X}_1}\xi = A^2\mathcal{X}_1 + \eta(\mathcal{X}_1)\|A\xi\|^2\xi,$$

or

$$-\|A\xi\|^2\mathcal{X}_1 + \eta(\mathcal{X}_1)\|A\xi\|^2\xi = A^2\mathcal{X}_1 + \eta(\mathcal{X}_1)\|A\xi\|^2\xi,$$

or

$$A^2\mathcal{X}_1 = -\|A\xi\|^2\mathcal{X}_1. \tag{9.7}$$

If $\|A\xi\| \neq 0$, then from the above equation, we have

$$-\left(\frac{A}{\|A\xi\|}\right)^2\mathcal{X}_1 = \mathcal{X}_1.$$

Consider $F = \frac{A}{\|A\xi\|}$, then we have

$$F^2\mathcal{X}_1 = -\mathcal{X}_1.$$

Therefore F is an almost complex structure on M, then the fundamental 2-form is given by

$$g(F\mathcal{X}_1, \mathcal{X}_2) = g\left(\frac{A\mathcal{X}_1}{\|A\xi\|}, \mathcal{X}_2\right) = \frac{1}{\|A\xi\|}g(A\mathcal{X}_1, \mathcal{X}_2).$$

With the help of $\alpha(\mathcal{X}_1, \mathcal{X}_2) = g(A\mathcal{X}_1, \mathcal{X}_2)$, we obtain

$$\alpha(\mathcal{X}_1, \mathcal{X}_2) = g(A\mathcal{X}_1, \mathcal{X}_2) = \|A\xi\|g(F\mathcal{X}_1, \mathcal{X}_2).$$

But,

$$\alpha(\mathcal{X}_1, \mathcal{X}_3) = \eta(\mathcal{X}_1)\alpha(\xi, \mathcal{X}_3) - \eta(\mathcal{X}_3)\alpha(\xi, \mathcal{X}_1)$$

which shows that α is degenerate which implies that $\alpha = 0$ for all tangent vectors $\mathcal{X}_1$ on M. Thus,

$$\|A\xi\| = 0$$

which is a contradiction. Hence, this completes the proof of the theorem. □

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