FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 40, No 1 (2025), 113-121
https://doi.org/10.22190/FUMI240120009V
Original Scientific Paper

ON RICCI SEMI-SYMMETRIC MIXED QUASI-EINSTEIN HERMITIAN MANIFOLD

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Abstract. The object of the present paper is to study Bochner Ricci semi-symmetric mixed quasi-Einstein Hermitian manifold and holomorphically projective Ricci semi-symmetric mixed quasi-Einstein Hermitian manifold.
Keywords: Generalized quasi-Einstein manifold, mixed quasi-Einstein manifold, Bochner curvature tensor, holomorphically projective curvature tensor.

1. Introduction

A manifold M^n of dimension n $(n = 2m)$, m is positive integer is called Hermitian manifold if a complex structure $\mathcal{J}$ of type $(1,1)$ and the pseudo-Riemannian metric g satisfy

$$\mathcal{J}^2 = -I \tag{1.1}$$

and

$$g(\mathcal{J}V_1, \mathcal{J}V_2) = g(V_1, V_2), \tag{1.2}$$

for all vector fields V_1 and V_2. A Riemannian manifold $\mathcal{M}$ is named an Einstein manifold [1] if the Ricci tensor $Ric(\neq 0)$ of type $(0,2)$ satisfies: $Ric = \frac{scal}{n} g$, where $scal$ represents the scalar curvature. Since Einstein manifolds have important differential geometric properties and have significant physical applications, therefore,

Received: January 20, 2024, revised: August 24, 2024, accepted: August 24, 2024
Communicated by Ljubica Velimirović
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2020 *Mathematics Subject Classification.* Primary 53C25; Secondary 53B35

they are studied by geometers in a broad perspective. Also, Einstein manifolds play a key role in Riemannian geometry, general theory of relativity as well as in mathematical physics.

A Riemannian manifold $\mathcal{M}$ is said to be a quasi-Einstein (QE) manifold [8, 9] if its $Ric(\neq 0)$ satisfies:

$$Ric(V_1, V_2) = ag(V_1, V_2) + bA(V_1)A(V_2), \tag{1.3}$$

where a, $b(\neq 0) \in \mathbb{R}$ and $A(\neq 0)$ is 1-form such that

$$g(V_1, \rho) = A(V_1), \tag{1.4}$$

for all vector field V_1 and a unit vector field ρ named the generator of QE manifolds. Also, the 1-form A is called the associated 1-form. From (1.3) it is clear that for $b = 0$, QE manifolds reduces to an Einstein manifold.

Now contraction of (1.3) over V_1 and V_2 gives

$$scal = an + b. \tag{1.5}$$

From (1.2) and (1.3), we have

$$\begin{aligned} &Ric(V_1, \rho) = (a + b)A(V_1),\ Ric(\rho, \rho) = (a + b), \\ &g(\mathcal{J}\rho, \rho) = 0,\ Ric(\mathcal{J}\rho, \rho) = 0. \end{aligned} \tag{1.6}$$

A Riemannian manifold $\mathcal{M}$ is said to be a generalized quasi-Einstein (GQE) manifold [10, 11, 12] if its $Ric(\neq 0)$ satisfies:

$$Ric(V_1, V_2) = ag(V_1, V_2) + bA(V_1)A(V_2) + cC(V_1)C(V_2), \tag{1.7}$$

for all vector fields V_1 and V_2, where a, $b(\neq 0)$, $c(\neq 0) \in \mathbb{R}$ and $A(\neq 0)$, $C(\neq 0)$ are 1-forms such that

$$g(V_1, \rho) = A(V_1),\ g(V_1, \sigma) = C(V_1),\ g(\rho, \rho) = 1,\ g(\sigma, \sigma) = 1. \tag{1.8}$$

After the contraction of (1.7) over V_1 and V_2, we get

$$scal = an + b + c. \tag{1.9}$$

From (1.2), (1.7) and (1.8), we have

$$\begin{aligned} &Ric(V_1, \rho) = (a + b)A(V_1),\ Ric(V_1, \sigma) = (a + c)C(V_1), \\ &g(\mathcal{J}\rho, \rho) = g(\mathcal{J}\sigma, \sigma) = 0,\ Ric(\mathcal{J}\rho, \rho) = Ric(\mathcal{J}\sigma, \sigma) = 0, \\ &Ric(\sigma, \sigma) = a + c,\ Ric(\rho, \rho) = a + b. \end{aligned} \tag{1.10}$$

A Riemannian manifold $\mathcal{M}$ is said to be a nearly quasi-Einstein (NQE) manifold [16] if its $Ric(\neq 0)$ satisfies:

$$Ric(V_1, V_2) = ag(V_1, V_2) + bE(V_1, V_2), \tag{1.11}$$

where a, $b(\neq 0) \in \mathbb{R}$ and $E(\neq 0)$ is symmetric tensor of type $(0,2)$. There are many author works have been done on this manifold like [4, 13].

In [13] R. N. Singh, R. K. Pandey and D. Gautam have defined a new type of nearly quasi-Einstein manifold by choosing the tensor E as follows:

$$E(V_1, V_2) = A(V_1)C(V_2) + A(V_2)C(V_1), \tag{1.12}$$

then, from (1.11) and (1.12), we have

$$Ric(V_1, V_2) = ag(V_1, V_2) + b[A(V_1)C(V_2) + A(V_2)C(V_1)], \tag{1.13}$$

where a, $b(\neq 0) \in \mathbb{R}$ and $A(\neq 0)$, $C(\neq 0)$ are 1-forms such that

$$g(V_1, \rho) = A(V_1),\ g(V_1, \sigma) = C(V_1),\ g(\rho, \rho) = 1,\ g(\sigma, \sigma) = 1. \tag{1.14}$$

After the contraction of (1.13) over V_1 and V_2 gives

$$scal = na. \tag{1.15}$$

From (1.2), (1.13) and (1.14), we have

$$\begin{aligned} &Ric(V_1, \rho) = aA(V_1) + bC(V_1),\ Ric(V_1, \sigma) = aC(V_1) + bA(V_1), \\ &g(\mathcal{J}\rho, \rho) = g(\mathcal{J}\sigma, \sigma) = 0,\ Ric(\mathcal{J}\rho, \rho) = Ric(\mathcal{J}\sigma, \sigma) = 0, \\ &Ric(\sigma, \sigma) = Ric(\rho, \rho) = a. \end{aligned} \tag{1.16}$$

In 2010, H. G. Nagaraja introduced and studied the notion of a mixed quasi-Einstein (MQE) manifold [5]. They have also studied some properties of a MQE manifold [17].

Thus we can write the following proposition:

Proposition 1.1. *A MQE manifold is a special type of NQE manifold.*

Definition 1.1. An n-dimensional Riemannian manifold $\mathcal{M}$ is said to be Ricci semi-symmetric if satisfies:

$$\begin{aligned} (K(V_1, V_2).Ric)(V_3, V_4) &= -Ric(K(V_1, V_2)V_3, V_4) \\ &- Ric(V_3, K(V_1, V_2)V_4) \\ &= 0, \end{aligned} \tag{1.17}$$

for all vector fields V_1, V_2, V_3, V_4.

Now we introduce the following:

Definition 1.2. An even dimensional Herimitian manifold $\mathcal{M}^n$ is said to be Bochner Ricci semi-symmetric Hermitian manifold if the Bochner curvature tensor satisfies $B.Ric = 0$, that is,

$$(B(V_1, V_2).Ric)(V_3, V_4) = -Ric(B(V_1, V_2)V_3, V_4) - Ric(V_3, B(V_1, V_2)V_4) \tag{1.18}$$

for all vector fields V_1, V_2, V_3, V_4.

In 2012, U. C. De [18] studied some geometric and global properties of MQE manifolds. Also, the studies of the existence of a MQE manifold have been proved by two non-trivial examples. In 2017, Mallick, Yildiz and De [15] studied some geometric properties of MQE manifolds and also discussed MQE spacetime with space-matter tensor and some properties related to it. In 2018, Suh, Majhi and De [19] proved that every Z Ricci pseud osymmetric MQE spacetimes is a Z Ricci semisymmetric spacetime; after that they studied Z flat spacetimes. Zhiming, Huazhen and Weijun [6] continued the study of MQE manifolds satisfying some geometric properties such as Ricci semi-symmetric, concircular Ricci pseudo symmetric and Wi Ricci pseudo symmetric. In 2018, Gupta, Chaturvedi and Lone [2] studied Ricci semi-symmetric mixed super quasi-Einstein Hermitian manifold. In 2023, Chaturvedi and Gupta [3] continued the study on Bochner Rici semi-symmetric super quasi-Einstein Hermitian manifold. Motivated from the above studies, authors continued the study of Ricci semi-symmetric MQE Hermitian manifolds.

2. Bochner Ricci semi-symmetric mixed quasi-Einstein Hermitian (MQEH) manifold

The notion of Bochner curvature tensor B was introduced by S. Bochner [14] and is defined by

$$\begin{aligned} B(V_1,V_2,V_3,V_4) &= \overline{K}(V_1,V_2,V_3,V_4) - \frac{1}{2(n+2)}[Ric(V_1,V_4)g(V_2,V_3) \\ &- Ric(V_1,V_3)g(V_2,V_4) + g(V_1,V_4)Ric(V_2,V_3) \\ &- g(V_1,V_3)Ric(V_1,V_4) + Ric(\mathcal{J}V_1,V_4)g(\mathcal{J}V_2,V_3) \\ &- Ric(\mathcal{J}V_1,V_3)g(\mathcal{J}V_2,V_4) + Ric(\mathcal{J}V_2,V_3)g(\mathcal{J}V_1,V_4) \\ &- g(\mathcal{J}V_1,V_3)Ric(\mathcal{J}V_1,V_4) - 2Ric(\mathcal{J}V_1,V_2)g(\mathcal{J}V_3,V_4) \\ &- 2g(\mathcal{J}V_1,V_2)Ric(\mathcal{J}V_3,V_4)] + \frac{scal}{(2n+2)(2n+4)} \\ &[g(V_2,V_3)g(V_1,V_4) - g(V_1,V_3)g(V_2,V_4) + g(\mathcal{J}V_2,V_3)g(\mathcal{J}V_1,V_4) \\ &- g(\mathcal{J}V_1,V_3)g(\mathcal{J}V_2,V_4) - 2g(\mathcal{J}V_1,V_2)g(\mathcal{J}V_3,V_4)], \end{aligned} \tag{2.1}$$

where $scal$ is scalar curvature of the manifold and $\overline{K}$ is curvature tensor of type $(0,4)$.

In a Hermitian manifold Bochner curvature tensor satisfies:

$$B(V_1,V_2,V_3,V_4) = -B(V_1,V_2,V_4,V_3). \tag{2.2}$$

Now, we introduce the following:

Definition 2.1. A Hermitian manifold is said to be a MQEH manifold if it satisfies the condition (1.13).

Definition 2.2. A MQEH manifold is said to be a Bochner Ricci semi-symmetric MQEH manifold if it satisfies the condition (1.18).

If we take a Bochner Ricci semi-symmetric MQEH manifold, then using (1.13) in (1.18), we get

$$\begin{aligned} & a[g(B(V_1,V_2)V_3,V_4)+g(V_3,B(V_1,V_2)V_4)] \\ + \quad & b[A(B(V_1,V_2)V_3)C(V_4)+A(V_4)C(B(V_1,V_2)V_3) \\ + \quad & A(B(V_1,V_2)V_4)C(V_3)+A(V_3)C(B(V_1,V_2)V_4)]=0, \end{aligned} \tag{2.3}$$

since $g(B(V_1,V_2)V_3,V_4)=B(V_1,V_2,V_3,V_4)$ and using (2.2) in (2.3), we get

$$\begin{aligned} & b[A(B(V_1,V_2)V_3)C(V_4)+A(4)C(B(V_1,V_2)V_3) \\ + \quad & A(B(V_1,V_2)V_4)C(V_3)+A(V_3)C(B(V_1,V_2)V_4)]=0. \end{aligned} \tag{2.4}$$

Putting $V_3=V_4=\rho$ in (2.4) and using (1.14), we have

$$bC(B(V_1,V_2)\rho)=0, \tag{2.5}$$

this implies that either $b=0$ or $C(B(V_1,V_2)\rho)=B(V_1,V_2,\rho,\sigma)=0$. If $b=0$, then form (1.13), we have

$$Ric(V_1,V_2)=ag(V_1,V_2), \tag{2.6}$$

which is an Einstein manifold.

This leads to the following:

Theorem 2.1. *A Bochner Ricci semi-symmetric MQEH manifold is either a Bochner Ricci semi-symmetric Einstein Hermitian manifold or* $B(V_1,V_2,\rho,\sigma)=0$.

Consider the manifold of Bochner curvature tensor is flat. Then, we have

$$B(V_1,V_2,V_3,V_4)=0. \tag{2.7}$$

Using (2.7) in (2.1), we have

$$\begin{aligned} \overline{K}(V_1,V_2,V_3,V_4) &= \frac{1}{2(n+2)}[Ric(V_1,V_4)g(V_2,V_3)-Ric(V_1,V_3)g(V_2,V_4) \\ &+ g(V_1,V_4)Ric(V_2,V_3)g(V_1,V_3)-Ric(V_2,V_4) \\ &+ Ric(\mathcal{J}V_1,V_4)g(\mathcal{J}V_2,V_3)-Ric(\mathcal{J}V_1,V_3)g(\mathcal{J}V_2,V_4) \\ &+ Ric(\mathcal{J}V_2,V_3)g(\mathcal{J}V_1,V_4)-g(\mathcal{J}V_1,V_3)Ric(\mathcal{J}V_2,V_4) \\ &- 2Ric(\mathcal{J}V_1,V_2)g(\mathcal{J}V_3,V_4)-2g(\mathcal{J}V_1,V_2)Ric(\mathcal{J}V_3,V_4)] \\ &- \frac{scal}{(2n+2)(2n+4)}[g(V_2,V_3)g(V_1,V_4) \\ &- g(V_1,V_3)g(V_2,V_4)+g(\mathcal{J}V_2,V_3)g(\mathcal{J}V_1,V_4) \\ &- g(\mathcal{J}V_1,V_3)g(\mathcal{J}V_2,V_4)-2g(\mathcal{J}V_1,V_2)g(\mathcal{J}V_3,V_4)]. \end{aligned} \tag{2.8}$$

From (1.17) and (2.8), we have

$$\begin{aligned} & [Ric(QV_1, V_4)g(V_2, V_3) - g(V_1, V_3)Ric(QV_2, V_4) \\ + & Ric(Q\mathcal{J}V_1, V_4)g(\mathcal{J}V_2, V_3) - g(\mathcal{J}V_1, V_3)Ric(\mathcal{J}QV_2, V_4) \\ + & g(V_2, V_4)Ric(QV_1, V_3) - g(V_1, V_4)Ric(QV_2, V_3) \\ + & Ric(Q\mathcal{J}V_1, V_3)g(\mathcal{J}V_2, V_4) - g(\mathcal{J}V_1, V_4)Ric(\mathcal{J}QV_2, V_3) \\ - & 2g(\mathcal{J}V_1, V_2)Ric(\mathcal{J}QV_3, V_4) - 2g(\mathcal{J}V_1, V_2)Ric(\mathcal{J}QV_4, V_3)] \\ - & \frac{scal}{2n+2}[g(V_2, V_3)Ric(V_1, V_4) - g(V_1, V_3)Ric(V_2, V_4) \\ + & g(\mathcal{J}V_2, V_3)Ric(\mathcal{J}V_1, V_4) - g(\mathcal{J}V_1, V_3)Ric(\mathcal{J}V_2, V_4) \\ + & g(V_2, V_4)Ric(V_1, V_3) - g(V_1, V_4)Ric(V_2, V_3) \\ + & g(\mathcal{J}V_2, V_4)Ric(\mathcal{J}V_1, V_3) - g(\mathcal{J}V_1, V_4)Ric(\mathcal{J}V_2, V_3)] \\ = & 0. \end{aligned} \tag{2.9}$$

If we take λ be an eigenvalue of Q and $\mathcal{J}Q$ corresponding to the eigenvectors V_1 and $\mathcal{J}V_1$, then $QV_1 = \lambda V_1$ and $Q\mathcal{J}V_1 = \lambda\mathcal{J}V_1$, i.e., $Ric(V_1, V_3) = \lambda g(V_1, V_3)$ (where the manifold is not Einstein), and therefore

$$Ric(QV_1, V_3) = \lambda Ric(V_1, V_3), \;\; Ric(Q\mathcal{J}V_1, V_3) = \lambda Ric(\mathcal{J}V_1, V_3). \tag{2.10}$$

Using (2.10) in (2.9), we have

$$\begin{aligned} & (\lambda - \frac{scal}{2n+2})\Big[g(V_2, V_3)Ric(V_1, V_4) - g(V_1, V_3)Ric(V_2, V_4) \\ + & g(V_2, V_2)Ric(V_1, V_3) - g(V_1, V_4)Ric(V_2, V_3) \\ + & g(\mathcal{J}V_2, V_3)Ric(\mathcal{J}V_1, V_4) - g(\mathcal{J}V_1, V_3)Ric(\mathcal{J}V_2, V_4) \\ + & g(\mathcal{J}V_2, V_4)Ric(\mathcal{J}V_1, V_3) - g(\mathcal{J}V_1, V_4)Ric(\mathcal{J}V_2, V_3)\Big] = 0. \end{aligned} \tag{2.11}$$

If we take $\lambda \neq \frac{scal}{2n+2}$, then from (2.11), we have

$$\begin{aligned} & g(V_2, V_3)Ric(V_1, V_4) - g(V_1, V_3)Ric(V_2, V_4) \\ + & g(V_2, V_4)Ric(V_1, V_3) - g(V_1, V_4)Ric(V_2, V_3) \\ + & g(\mathcal{J}V_2, V_3)Ric(\mathcal{J}V_1, V_4) - g(\mathcal{J}V_1, V_3)Ric(\mathcal{J}V_2, V_4) \\ + & g(\mathcal{J}V_2, V_4)Ric(\mathcal{J}V_1, V_3) - g(\mathcal{J}V_1, V_4)Ric(\mathcal{J}V_2, V_3) = 0. \end{aligned} \tag{2.12}$$

Taking $V_4 = V_3 = \rho$ in (2.12), we have

$$\begin{aligned} & g(V_2, \rho)Ric(V_1, \rho) - g(V_1, \rho)Ric(V_2, \rho) \\ + & g(\mathcal{J}V_2, \rho)Ric(\mathcal{J}V_1, \rho) - g(\mathcal{J}V_1, \rho)Ric(\mathcal{J}V_2, \rho) = 0. \end{aligned} \tag{2.13}$$

Making use of (1.14) and (1.16) in (2.13), we have

$$b[A(V_2)C(V_1) - A(V_1)C(V_2) + A(\mathcal{J}V_2)C(\mathcal{J}V_1) - A(\mathcal{J}V_1)C(\mathcal{J}V_2)] = 0,$$

this implies that either $b = 0$ or

$$A(V_2)C(V_1) - A(V_1)C(V_2) + A(\mathcal{J}V_2)C(\mathcal{J}V_2) - A(\mathcal{J}V_2)C(\mathcal{J}V_2) = 0. \tag{2.14}$$

If we take $\lambda \neq \frac{scal}{2n+2}$ and $b \neq 0$, then from (1.2), (1.14) and (2.14), we obtain $g(V_2,\rho)g(V_1,\sigma) - g(V_1,\rho)g(V_2,\sigma) = g(V_1,\mathcal{J}\rho)g(V_2,\sigma) - g(V_2,\mathcal{J}\rho)g(V_1,\mathcal{J}\sigma) = 0$, implies that $g(V_2,\rho)g(V_1,\sigma) = g(V_1,\rho)g(V_2,\sigma)$ if and only if $g(V_1,\mathcal{J}\rho)g(V_2,\sigma) = g(V_2,\mathcal{J}\rho)g(V_2,\mathcal{J}\sigma)$, this shows that the vector field ρ and σ corresponding to the 1-forms A and C respectively are co-directional if and only if the vector field ρ' and σ' corresponding to the 1-forms A and C respectively are co-directional.

Theorem 2.2. *In a Bochner flat Ricci semi-symmetric MQEH manifold, if $\frac{scal}{2n+2}$ is not an eigenvalue of the Ricci operator Q and $\mathcal{J}Q$, and $b(\neq 0)$, then the vector fields ρ and σ corresponding to the 1-forms A and C respectively are co-directional if and only if the vector fields ρ' and σ' corresponding to the 1-forms A and C respectively are co-directional.*

3. Holomorphically projective Ricci semi-symmetric MQEH manifold

The holomorphically projective curvature tensor P is defined by [7]

$$\begin{aligned} P(V_1,V_2,V_3,V_4) &= K(V_1,V_2,V_3,V_4) - \frac{1}{n-2}\Big[Ric(V_2,V_3)g(V_1,V_4) \\ &- Ric(V_1,V_3)g(V_2,V_4) + Ric(\mathcal{J}V_1,V_3)g(\mathcal{J}V_2,V_4) \\ &- Ric(\mathcal{J}V_2,V_3)g(\mathcal{J}V_1,V_4)\Big]. \end{aligned} \tag{3.1}$$

The holomorphically projective curvature tensor satisfying the following properties:

$P(V_1,V_2,V_3,V_4) = -P(V_2,V_1,V_3,V_4)$, $P(\mathcal{J}V_1,\mathcal{J}V_2,V_3,V_4) = P(V_1,V_2,V_3,V_4)$.

Now we introduce the following:

Definition 3.1. An even dimensional Hermitian manifold M^n is said to be a holomorphically projective Ricci semi-symmetric MQEH manifold if the holomorphically projective curvature tensor of the manifold satisfies $P.Ric = 0$, i.e.,

$$(P(V_1,V_2).Ric)(V_3,V_4) = -Ric(P(V_1,V_2)V_3,V_4) - Ric(V_3,P(V_1,V_2)V_4), \tag{3.2}$$

for all vector fields V_1, V_2, V_3, V_4.

Let us consider a holomorphically projective Ricci semi-symmetric MQEH manifold, then from (1.13) and (3.2), we have

$$\begin{aligned} & a[g(P(V_1,V_2)V_3,V_4) + g(V_3,P(V_1,V_2)V_4)] \\ + \; & b[A(P(V_1,V_2)V_3)C(V_4) + A(V_4)C(P(V_1,V_2)V_3) \\ + \; & A(V_3)C(P(V_1,V_2)V_4) + C(V_3)A(P(V_1,V_2)V_4)] = 0. \end{aligned} \tag{3.3}$$

Putting $V_3 = V_4 = \rho$ in (3.3) and using (1.14), we have

$$2aP(V_1, V_2, \rho, \rho) + 2bP(V_1, V_2, \rho, \sigma) = 0. \tag{3.4}$$

Again putting $V_3 = V_4 = \rho$ in (3.1), we have

$$\begin{aligned} P(V_1, V_2, \rho, \rho) &= -\frac{b}{n-2}[A(V_1)C(V_2) - A(V_2)C(V_1) \\ &+ A(\mathcal{J}V_2)C(\mathcal{J}V_1) - A(\mathcal{J}V_1)C(\mathcal{J}V_2)]. \end{aligned} \tag{3.5}$$

Similarly, putting $V_3 = \rho$ and $V_4 = \sigma$ in (3.1), we have

$$\begin{aligned} P(V_1, V_2, \rho, \sigma) &= K(V_1, V_2, \rho, \sigma) - \frac{a}{n-2}[A(V_2)C(V_1) - C(V_2)A(V_1) \\ &+ A(\mathcal{J}V_1)C(\mathcal{J}V_2) - A(\mathcal{J}V_2)C(\mathcal{J}V_1)], \end{aligned} \tag{3.6}$$

which in view of (3.5) and (3.6) the relation (3.4) reduces to

$$bK(V_1, V_2, \rho, \sigma) = 0, \tag{3.7}$$

this implies that either $b = 0$ or $K(V_1, V_2, \rho, \sigma) = 0$. If $b = 0$, then form (1.13), we have

$$Ric(V_1, V_2) = ag(V_1, V_2), \tag{3.8}$$

which is an Einstein manifold.

Thus we are in the position to state the following:

Theorem 3.1. *A holomorphically projective Ricci semi-symmetric MQEH manifold is either a holomorphically projective Ricci semi-symmetric Einstein Hermitian manifold or $K(V_1, V_2, \rho, \sigma) = 0$.*

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