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ON CURVATURE TENSORS OBTAINED BY TWO NON-SYMMETRIC LINEAR CONNECTIONS

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Abstract. In the cited papers [1, 2, 8, 9, 10, 12, 14], curvature tensors are considered by polylinear mappings, using non-symmetric connections. In the rest of the works from the References, the curvature tensors are obtained by help of Riccy-type identities in local coordinates. In this paper, the problem is considered more generally using polylinear mappings, after which eight curvature tensor fields are obtained. Further, it is proved that among these fields, five of them are independent, while the rest are linear combinations of the cited five fields.
Keywords: non-symmetric connection, curvature tensors, independent curvature tensors.

1. Introduction

Consider an N-dimensional differentialble manifold $\mathcal{M}_N$ on which a non-symmetric linear connection $\overset{1}{\nabla}$ is defined. If $\mathfrak{X}(\mathcal{M}_N)$ is Lie algebra (see [3]) of smooth vector fields and $X, Y \in \mathfrak{X}(\mathcal{M}_N)$, then the mapping (see [12])

$$\overset{2}{\nabla} : \mathfrak{X}(\mathcal{M}_N) \times \mathfrak{X}(\mathcal{M}_N) \to (\mathcal{M}_N),$$

given by

$$\overset{2}{\nabla}_X Y = \overset{1}{\nabla}_Y X + [X, Y], \tag{1.1}$$

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defines an other non-symmetric connection $\overset{2}{\nabla}$ on $\mathcal{M}_N$. That means that we have

$$\text{(1.2)}\qquad \begin{aligned} &a)\ \overset{\theta}{\nabla}_{Y_1+Y_2}X = \overset{\theta}{\nabla}_{Y_1}X + \overset{\theta}{\nabla}_{Y_2}X, &&b)\ \overset{\theta}{\nabla}_{fY}X = f\overset{\theta}{\nabla}_Y X,\\ &c)\ \overset{\theta}{\nabla}_Y(X_1+X_2) = \overset{\theta}{\nabla}_Y X_1 + \overset{\theta}{\nabla}_Y X_2, &&d)\ \overset{\theta}{\nabla}_Y(fX) = Yf\cdot X + f\overset{\theta}{\nabla}_Y X,\end{aligned}$$

for $\theta = 1, 2$ and $X, Y, X_1, X_2, Y_1, Y_2 \in \mathfrak{X}(\mathcal{M}_N)$, $f \in \mathcal{F}(\mathcal{M}_N)$, where $\mathcal{F}(\mathcal{M}_N)$ is an algebra of smooth real functions on $\mathcal{M}_N$. In that case, we write $L_N = (\mathcal{M}_N, \overset{1}{\nabla}, \overset{2}{\nabla})$ and L_N call a space on non-symmetric linear connections $\overset{1}{\nabla}$, $\overset{2}{\nabla}$.

If we introduce local coordinates $x^1, \ldots, x^N$, and put

$$\text{(1.3)}\qquad \partial/\partial x^i = \partial_i,$$

with respect of (1.1), it will be

$$\text{(1.4)}\qquad \overset{2}{\nabla}_{\partial_j}\partial_k = \overset{1}{\nabla}_{\partial_k}\partial_j.$$

Denoting coefficients of the connection $\overset{1}{\nabla}$ in the base $\partial_1, \ldots, \partial_N$ with L^i_{jk}, we have

$$\text{(1.5)}\qquad a)\ \overset{1}{\nabla}_{\partial_k}\partial_j L^i_{jk}\partial_i,\quad b)\ \overset{2}{\nabla}_{\partial_k}\partial_j \underset{(1.4)}{=} \overset{1}{\nabla}_{\partial_j}\partial_k = L^i_{kj}\partial_i,$$

where " $\underset{(1.4)}{=}$ " denotes "equal with respect to (1.4)".

2. Tensors $\overset{1}{R}, \ldots, \overset{4}{R}$

Firstly, it is easily to prove the following two propositions.

Proposition 2.1. *Torsion tensors of connections $\overset{1}{\nabla}$ and $\overset{2}{\nabla}$ have opposite signs, i.e.*

$$\text{(2.1)}\qquad \overset{2}{T}(X,Y) = -\overset{1}{T}(X,Y).$$

from where it follows that $\overset{1}{\nabla}$ and $\overset{2}{\nabla}$ are in the same time symmetric or non-symmetric.

Proof. Starting from the equation

$$\text{(2.2)}\qquad \overset{1}{T}(X,Y) = \overset{1}{\nabla}_Y X - \overset{1}{\nabla}_X Y + [X,Y],$$

we obtain from (1.1) that

$$\overset{2}{T}(X,Y) = \overset{2}{\nabla}_Y X - \overset{2}{\nabla}_X Y + [X,Y]$$
$$\underset{(1.1)}{=} \overset{1}{\nabla}_X Y + [Y,X] - \overset{1}{\nabla}_Y X - [X,Y] + [X,Y] \underset{(2.1)}{=} -\overset{1}{T}(X,Y).$$

□

Proposition 2.2. *Necessary and sufficient condition for the connection* $\overset{1}{\nabla}$ *and* $\overset{2}{\nabla}$ *to be identical is* $\overset{2}{T}(X,Y) = \overset{1}{T}(X,Y) = 0$, *i.e. we have*

$$\overset{2}{T}(X,Y) = \overset{1}{T}(X,Y) = 0 \Leftrightarrow \overset{1}{\nabla} = \overset{2}{\nabla} = \nabla. \tag{2.3}$$

Proof. From (1.1) and (2.2), it follows

$$\overset{2}{\nabla}_X Y - \overset{1}{\nabla}_X Y = \overset{1}{\nabla}_Y X + [X,Y] - \overset{1}{\nabla}_X Y$$
$$= \overset{1}{T}(X,Y) = -\overset{2}{T}(X,Y).$$

This proves (2.3). □

Let us consider mappings ρ defined in the points of L_N:

$$\rho : \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}, \tag{2.4}$$

so that

$$\rho(X;Y,Z) \equiv \rho(Z,Y)X = \overset{a}{\nabla}_Z \overset{b}{\nabla}_Y X - \overset{c}{\nabla}_Y \overset{d}{\nabla}_Z X + \overset{\lambda}{\nabla}_{[Y,Z]} X, \tag{2.5}$$

for

$$a, b, c, d, \lambda \in \{1, 2\}. \tag{2.6}$$

We will investigate conditions with (2.5) to be defined a curvature tensor field in L_N, i.e. conditions to be in force $\mathcal{F}(L_N)$-linearity on X, Y, Z.

a) $\mathcal{F}(L_N)$ linearity wrt X:

From (2.5), it is

$$\rho(X_1 + X_2; Y, Z) = \rho(X_1; Y, Z) + \rho(X_2; Y, Z). \tag{2.7}$$

The following equalities are satisfied

$$\begin{aligned}\rho(fX;Y,Z) &\underset{(2.5)}{=} \overset{a}{\nabla}_Z\overset{b}{\nabla}_Y(fX)-\overset{c}{\nabla}_Y\overset{d}{\nabla}_Z(fX)+\overset{\lambda}{\nabla}_{[Y,Z]}(fX)\\ &=\overset{a}{\nabla}_Z(Yf\cdot X+f\cdot\overset{b}{\nabla}_YX)-\overset{c}{\nabla}_Y(Zf\cdot X+f\cdot\overset{d}{\nabla}_ZX)+\overset{\lambda}{\nabla}_{[Y,Z]}(fX)\\ &=Z(Yf)\cdot X+Yf\cdot\overset{a}{\nabla}_ZX+Zf\cdot\overset{b}{\nabla}_YX+f\cdot\overset{a}{\nabla}_Z\overset{b}{\nabla}_YX\\ &-Y(Zf)\cdot X-Zf\cdot\overset{c}{\nabla}_YX-Yf\cdot\overset{d}{\nabla}_ZX-f\cdot\overset{c}{\nabla}_Y\overset{d}{\nabla}_ZX+\overset{\lambda}{\nabla}_{[Y,Z]}(fX).\end{aligned}$$

Using the relation

$$\begin{aligned}\overset{\lambda}{\nabla}_{[Y,Z]}(fX)&=\big([Y,Z]f\big)\cdot X+f\cdot\overset{\lambda}{\nabla}_{[Y,Z]}X\\ &=\big[Y(Zf)-Z(Yf)\big]\cdot X+\overset{\lambda}{\nabla}_{[Y,Z]}X,\end{aligned}\tag{2.8}$$

by substitution into previous equation, one obtains

$$\rho(fX;Y,Z)=f\cdot\rho(X;Y,Z)+Yf\cdot(\overset{a}{\nabla}_ZX-\overset{d}{\nabla}_ZX)+Zf\cdot(\overset{b}{\nabla}_YX-\overset{c}{\nabla}_YX),\tag{2.9}$$

fromwhere, we see that generally a linearity n X is not valid.

b) $\mathcal{F}(L_N)$ linearity wrt Y

We have

$$\rho(X;Y_1+Y_2,Z)=\underset{(2.5)}{=}\rho(X;Y_1,Z)+\rho(X;Y_2,Z),\tag{2.10}$$

$$\rho(X;fY,Z)=\underset{(2.5)}{=}\overset{a}{\nabla}_Z\overset{b}{\nabla}_{fY}X-\overset{c}{\nabla}_{fY}\overset{d}{\nabla}_ZX+\overset{\lambda}{\nabla}_{[fY,Z]}X.\tag{2.11}$$

The following equations are also satisfied

$$\overset{\lambda}{\nabla}_{[fY,Z]}X=-\overset{\lambda}{\nabla}_{(Zf)\cdot Y+f\cdot[Z,Y]}X=f\cdot\overset{\lambda}{\nabla}_{[Y,Z]}X-Zf\cdot\overset{\lambda}{\nabla}_YX,\tag{2.12}$$

$$\overset{a}{\nabla}_Z\overset{b}{\nabla}_{fY}X=\overset{a}{\nabla}_Z(f\cdot\overset{b}{\nabla}_YX)=Zf\cdot\overset{b}{\nabla}_YX+f\cdot\overset{a}{\nabla}_Z\overset{b}{\nabla}_YX.\tag{2.13}$$

By substituting (2.12), (2.13) into (2.11), we obtain

$$\rho(X;fY,Z)=f\cdot\rho(X;Y,Z)+Zf(\overset{b}{\nabla}_YX-\overset{\lambda}{\nabla}_YX),\tag{2.14}$$

i.e. in a general case $\mathcal{F}(L_N)$-linearity is not valid in relation to Y.

c) Further, by the same procedure, one obtains

$$\rho(X;Y,Z_1+Z_2) \underset{(2.5)}{=} \rho(X;Y,Z_1)+\rho(X;Y,Z_2), \tag{2.15}$$

$$\rho(X;Y,fZ) = \underset{(2.5)}{=} f\rho(X;Y,Z)+Yf\cdot(\overset{\lambda}{\nabla}_Z X-\overset{d}{\nabla}_Z X), \tag{2.16}$$

i.e. in a general case $\mathcal{F}(L_N)$-linearity is not valid in relation to Z.

From (2.9), (2.14), (2.16), we conclude that with (2.5) curvature tensor fields are defined if and only if $(a=d)\wedge(b=c)\wedge(b=\lambda)\wedge(\lambda=d)$, and next theorem is proved.

Theorem 2.1. *Necessary and sufficient condition with (2.5) to be defined curvature tensor fields in the space L_N with non-symmetric linear connections $\overset{1}{\nabla}$, $\overset{2}{\nabla}$ is*

$$a=b=c=d=\lambda\in\{1,2\}, \tag{2.17}$$

i.e. with (2.2) two curvature tensor fields are defined

$$\overset{a}{R}(X;Y,Z)\equiv\overset{a}{R}(Z,Y)X=\overset{a}{\nabla}_Z\overset{a}{\nabla}_Y X-\overset{a}{\nabla}_Y\overset{a}{\nabla}_Z X+\overset{a}{\nabla}_{[Y,Z]}X,\quad a=1,2. \tag{2.18}$$

In the case of symmetric connection, because of (2.3) by (2.5) is defined one curvature tensor field and then with (2.9), (2.14), (2.16) is also expressed $\mathcal{F}(L_N)$-linearity. In that case, L_N is denoted by $\overset{0}{L}_N$.

In [12], two curvature tensor fields are defined:

$$\overset{3}{R}(X;Y,Z)\equiv\overset{3}{R}(Z,Y)X=\overset{2}{\nabla}_Z\overset{1}{\nabla}_Y X-\overset{1}{\nabla}_Y\overset{2}{\nabla}_Z X+\overset{2}{\nabla}_{\overset{1}{\nabla}_Y Z}X-\overset{1}{\nabla}_{\overset{2}{\nabla}_Z Y}X, \tag{2.19}$$

Author of the present paper has obtained in 1975. the first (in coordinates) the tensor $\overset{4}{R}$, that is affirmed in [12]:

$$\overset{4}{R}(X;Y,Z)\equiv\overset{4}{R}(Z,Y)X=\overset{2}{\nabla}_Z\overset{1}{\nabla}_Y X-\overset{1}{\nabla}_Y\overset{2}{\nabla}_Z X+\overset{2}{\nabla}_{\overset{2}{\nabla}_Y Z}X-\overset{1}{\nabla}_{\overset{1}{\nabla}_Z Y}X. \tag{2.20}$$

Also, in [7] and [8], the present author has obtained a new curvature tensor $\overset{5}{R}$.

We will demonstrate in the following section how new curvature tensors can be obtained with the help of polylinear mappings.

Substituting into (2.18)–(2.20)

$$X=\partial/\partial x^j\equiv\partial_j,\quad Y=\partial_k,\quad Z=\partial_l, \tag{2.21}$$

we get

$$\overset{b}{R}(\partial_j;\partial_k\partial_l)\equiv\overset{b}{R}(\partial_l\partial_k)\partial_j=\overset{b}{R}{}^i_{jkl}\partial_i, b=1,2,3,4, \tag{2.22}$$

where

$$\overset{1}{R}{}^i_{jkl}=L^i_{jk,l}-L^i_{jl,k}+L^p_{jk}L^i_{pl}-L^p_{jl}L^i_{pk}, \tag{2.23}$$

$$\overset{2}{R}{}^i_{jkl}=L^i_{kj,l}-L^i_{lj,k}+L^p_{kj}L^i_{lp}-L^p_{lj}L^i_{kp}, \tag{2.24}$$

$$\overset{3}{R}{}^i_{jkl}=L^i_{jk,l}-L^i_{lj,k}+L^p_{jk}L^i_{lp}-L^p_{lj}L^i_{pk}+L^p_{lk}(L^i_{pj}-L^i_{jp}), \tag{2.25}$$

$$\overset{4}{R}{}^i_{jkl}=L^i_{jk,l}-L^i_{lj,k}+L^p_{jk}L^i_{lp}-L^p_{lj}L^i_{pk}+L^p_{kl}(L^i_{pj}-L^i_{jp}), \tag{2.26}$$

and in that manner we express curvature tensors in coordinates.

3. Some new curvature tensors

Consider the mapping $\tilde\rho:\mathfrak{X}(\mathcal{M}_N)\times\mathfrak{X}(\mathcal{M}_N)\times\mathfrak{X}(\mathcal{M}_N)\to\mathfrak{X}(\mathcal{M}_N)$ defined by

$$\tilde\rho(X;Y,Z)=\frac{1}{2}\big(\overset{a}{\nabla}_Z\overset{b}{\nabla}_YX-\overset{c}{\nabla}_Y\overset{d}{\nabla}_ZX+\overset{e}{\nabla}_Z\overset{f}{\nabla}_YX-\overset{g}{\nabla}_Y\overset{h}{\nabla}_ZX+\overset{\lambda}{\nabla}_{[Y,Z]}X+\overset{\mu}{\nabla}_{[Y,Z]}X, \tag{3.1}$$

for $a,b,c,d,e,f,g,h,\lambda,\mu\in\{1,2\}$.

Let us check conditions in (3.1)for defining a curvature tensor field.

First of all, as line in previous cases, it is easy to check the additivity in relation with all arguments X, Y, Z. Further, we have

$$\begin{aligned}\tilde\rho(\phi X;Y,Z)\underset{(2.9)}{=}&\ \phi\cdot\tilde\rho(X;Y,Z)+\frac{1}{2}\big\{Y\phi\cdot(\overset{a}{\nabla}_ZX-\overset{d}{\nabla}_ZX+\overset{e}{\nabla}_ZX-\overset{h}{\nabla}_ZX)\\&+Z\phi\cdot(\overset{b}{\nabla}_YX-\overset{c}{\nabla}_YX+\overset{f}{\nabla}_YX+\overset{g}{\nabla}_YX)\big\},\end{aligned} \tag{3.2}$$

$$\tilde\rho(X;\phi Y,Z)\underset{(2.14)}{=}\phi\cdot\tilde\rho(X;Y,Z)+\frac{1}{2}Z\phi\cdot(\overset{b}{\nabla}_YX-\overset{\lambda}{\nabla}_YX+\overset{f}{\nabla}_YX-\overset{\mu}{\nabla}_YX), \tag{3.3}$$

(3.4)

$$\tilde\rho(X;Y,\phi Z)\underset{(2.16)}{=}\phi\cdot\tilde\rho(X;Y,Z)+\frac{1}{2}Y\phi\cdot(\overset{\lambda}{\nabla}_ZX-\overset{d}{\nabla}_ZX+\overset{\mu}{\nabla}_ZX-\overset{h}{\nabla}_ZX).$$

Because (3.2, 3.3, 3.4) can be proved in similar manner, we give proof only for (3.3).

$$\begin{aligned}
\widetilde{\rho}(X;\phi Y,Z) &= \frac{1}{2}\big(\overset{a}{\nabla}_Z\overset{b}{\nabla}_{\phi Y}X - \overset{c}{\nabla}_{\phi Y}\overset{d}{\nabla}_Z X + \overset{e}{\nabla}_Z\overset{f}{\nabla}_{\phi Y}X - \overset{g}{\nabla}_{\phi Y}\overset{h}{\nabla}_Z X + \overset{\lambda}{\nabla}_{[\phi Y,Z]}X \\
&\quad + \overset{\mu}{\nabla}_{[\phi Y,Z]}X\big) \\
&= \frac{1}{2}\Big\{\overset{a}{\nabla}_Z(\phi\overset{b}{\nabla}_Y X) - \phi\overset{c}{\nabla}_Y\overset{d}{\nabla}_Z X + \overset{e}{\nabla}_Z(\phi\overset{f}{\nabla}_Y X) - \phi\overset{g}{\nabla}_Y\overset{h}{\nabla}_Z X \\
&\quad + \overset{\lambda}{\nabla}_{\phi[Y,Z]-Z\phi\cdot Y}X + \overset{\mu}{\nabla}_{\phi[Y,Z]-Z\phi\cdot Y}X\Big\} \\
&= \frac{1}{2}\Big\{Z\phi\cdot\overset{b}{\nabla}_Y X + \phi\overset{a}{\nabla}_Z\overset{b}{\nabla}_Y X - \phi\overset{c}{\nabla}_Y\overset{d}{\nabla}_Z X + Z\phi\cdot\overset{f}{\nabla}_Y X + \phi\overset{c}{\nabla}_Z\overset{f}{\nabla}_Y X \\
&\quad - \phi\overset{g}{\nabla}_Y\overset{h}{\nabla}_Z X + \phi\overset{\lambda}{\nabla}_{[Y,Z]}X - Z\phi\cdot\overset{\lambda}{\nabla}_Y X + \phi\overset{\mu}{\nabla}_{[Y,Z]}X - Z\phi\overset{\mu}{\nabla}_Y X\Big\} \\
&= \frac{1}{2}\phi\big(\overset{a}{\nabla}_Z\overset{b}{\nabla}_Y X - \overset{c}{\nabla}_Y\overset{d}{\nabla}_Z X + \overset{e}{\nabla}_Z\overset{f}{\nabla}_Y X - \overset{g}{\nabla}_Y\overset{h}{\nabla}_Z X + \overset{\lambda}{\nabla}_{[Y,Z]}X \\
&\quad + \overset{\mu}{\nabla}_{[Y,Z]}X\big) + \frac{1}{2}Z\phi\big(\overset{b}{\nabla}_Y X + \overset{f}{\nabla}_Y X - \overset{\lambda}{\nabla}_Y X - \overset{\mu}{\nabla}_Y X\big).
\end{aligned}$$

So, from (3.1) one obtains (3.3), if $b=\lambda$ and $f=\mu$.

Let A, B, ..., D' be statements

$$\begin{aligned}
&\mathbb{A} = \Big((a=d)\wedge(e=h)\Big), \quad \mathbb{B} = \Big((b=c)\wedge(f=g)\Big), \\
&\mathbb{C} = \Big((b=\lambda)\wedge(f=\mu)\Big), \quad \mathbb{D} = \Big((\lambda=d)\wedge(\mu=h)\Big) \\
&\mathbb{A}' = \Big((a=h)\wedge(d=e)\Big), \quad \mathbb{B}' = \Big((b=g)\wedge(c=f)\Big), \\
&\mathbb{C}' = \Big((b=\mu)\wedge(\lambda=f)\Big), \quad \mathbb{D}' = \Big((\lambda=h)\wedge(d=\mu)\Big).
\end{aligned} \tag{3.5}$$

Considering (3.2)-(3.4), we conclude that the next theorem is valid:

Theorem 3.1. *Necessary and sufficient condition for defining curvature tensor field by (3.1) is expressed as logical formula (v(F)=T):*

$$\begin{aligned}
\mathbb{F} =& \big(\mathbb{A}\wedge\mathbb{B}\wedge\mathbb{C}\wedge\mathbb{D}\big)\vee\big(\mathbb{A}\wedge\mathbb{B}\wedge\mathbb{C}\wedge\mathbb{D}'\big)\vee\big(\mathbb{A}\wedge\mathbb{B}\wedge\mathbb{C}'\wedge\mathbb{D}\big)\vee\big(\mathbb{A}\wedge\mathbb{B}\wedge\mathbb{C}'\wedge\mathbb{D}'\big) \\
&\vee\big(\mathbb{A}\wedge\mathbb{B}'\wedge\mathbb{C}\wedge\mathbb{D}\big)\vee\big(\mathbb{A}\wedge\mathbb{B}'\wedge\mathbb{C}\wedge\mathbb{D}'\big)\vee\big(\mathbb{A}\wedge\mathbb{B}'\wedge\mathbb{C}'\wedge\mathbb{D}\big)\vee\big(\mathbb{A}\wedge\mathbb{B}'\wedge\mathbb{C}'\wedge\mathbb{D}'\big) \\
&\vee\big(\mathbb{A}'\wedge\mathbb{B}'\wedge\mathbb{C}'\wedge\mathbb{D}'\big)\vee\big(\mathbb{A}'\wedge\mathbb{B}'\wedge\mathbb{C}'\wedge\mathbb{D}\big)\vee\big(\mathbb{A}'\wedge\mathbb{B}'\wedge\mathbb{C}\wedge\mathbb{D}'\big)\vee\big(\mathbb{A}'\wedge\mathbb{B}'\wedge\mathbb{C}\wedge\mathbb{D}\big) \\
&\vee\big(\mathbb{A}'\wedge\mathbb{B}\wedge\mathbb{C}'\wedge\mathbb{D}'\big)\vee\big(\mathbb{A}'\wedge\mathbb{B}\wedge\mathbb{C}'\wedge\mathbb{D}\big)\vee\big(\mathbb{A}'\wedge\mathbb{B}\wedge\mathbb{C}\wedge\mathbb{D}'\big)\vee\big(\mathbb{A}'\wedge\mathbb{B}\wedge\mathbb{C}\wedge\mathbb{D}\big),
\end{aligned} \tag{3.6}$$

where the statements $\mathbb{A},\mathbb{B},\dots,\mathbb{D}'$ *are given in* (3.5).

Example 3.1. For $v(\mathbb{A}\wedge\mathbb{B}'\wedge\mathbb{C}\wedge\mathbb{D}) = \top$ i.e. if the fifth term in (3.6) is veritable, find corresponding curvature tensor $\widetilde{R}$.

Solution. By virtue of (3.1), (3.5), and (3.6) in considered case is

$$a = b = d = g = \lambda \in \{1, 2\}$$
$$c = e = f = h = \mu \in \{1, 2\}.$$

For $\lambda = 1,\ \mu = 2$ one obtains

(3.7)
$$\overset{5}{R}(X; Y, Z) = \frac{1}{2}(\overset{1}{\nabla}_Z \overset{1}{\nabla}_Y X - \overset{2}{\nabla}_Y \overset{1}{\nabla}_Z X + \overset{2}{\nabla}_Z \overset{2}{\nabla}_Y X - \overset{1}{\nabla}_Y \overset{2}{\nabla}_Z X + \overset{1}{\nabla}_{[Y,Z]} X + \overset{2}{\nabla}_{[Y,Z]} X).$$

Considering the same procedure like in the previous example, we derived new curvature tensors in $\mathbb{L}_N$.

Example 3.2. For $v(\mathbb{A} \wedge \mathbb{B} \wedge \mathbb{C} \wedge \mathbb{D}) = \top$ it is $a = b = c = d = \lambda \in \{1, 2\},\ e = f = g = h = \mu \in \{1, 2\}$, and for $\lambda = 1,\ \mu = 2$ is obtained

$$\overset{6}{R}(X; Y, Z) = \frac{1}{2}(\overset{1}{R} + \overset{2}{R})(X; Y, Z). \tag{3.8}$$

Example 3.3. For $v(\mathbb{A} \wedge \mathbb{B} \wedge \mathbb{C} \wedge \mathbb{D}') = \top$ it is $(a = d = f = g = \mu = 1) \wedge (b = c = e = h = \lambda = 2)$, (the same is obtained for $\lambda = 1,\ \mu = 2$) and we obtain

$$\overset{7}{R}(X; Y, Z) = \frac{1}{2}(\overset{1}{\nabla}_Z \overset{2}{\nabla}_Y X - \overset{2}{\nabla}_Y \overset{1}{\nabla}_Z X + \overset{2}{\nabla}_Z \overset{1}{\nabla}_Y X - \overset{1}{\nabla}_Y \overset{2}{\nabla}_Z X + \overset{1}{\nabla}_{[Y,Z]} X + \overset{2}{\nabla}_{[Y,Z]} X). \tag{3.9}$$

Example 3.4. For $\tau(\mathbb{A} \wedge \mathbb{B}' \wedge \mathbb{C} \wedge \mathbb{D}') = \top$ one obtains

$$\overset{8}{R}(X; Y, Z) = \frac{1}{2}(\overset{2}{\nabla}_Z \overset{1}{\nabla}_Y X - \overset{2}{\nabla} Y \overset{2}{\nabla}_Z X + \overset{1}{\nabla}_Z \overset{2}{\nabla}_Y X - \overset{1}{\nabla}_Y \overset{1}{\nabla}_Z X + \overset{1}{\nabla}_{[Y,Z]} X + \overset{2}{\nabla}_{[Y,Z]} X). \tag{3.10}$$

Other cases from (3.6) are reduced to the same of those cited in (3.7)-(3.10).

The components of the tensors $\overset{1}{R}, \ldots, \overset{8}{R}$ are obtained by substituting in corresponding equations

$$X = \partial_j, \quad Y = \partial_k, \quad Z = \partial_l.$$

For example, for $\overset{5}{R}$ by virtue of (3.7) is

$$\begin{aligned}\overset{5}{R}{}^i_{jkl} \partial_i = \overset{5}{R}(\partial_j; \partial_k, \partial_l) &= \frac{1}{2}(\overset{1}{\nabla}_{\partial_l} \overset{1}{\nabla}_{\partial_k} \partial_j - \overset{2}{\nabla}_{\partial_k} \overset{1}{\nabla}_{\partial_l} \partial_j + \overset{2}{\nabla}_{\partial_l} \overset{2}{\nabla}_{\partial_k} \partial_j - \overset{1}{\nabla}_{\partial_k} \overset{2}{\nabla}_{\partial_l} \partial_j + 0)\\ &= \frac{1}{2}\big[\overset{1}{\nabla}_{\partial_l}(L^i_{jk}\partial_i) - \overset{2}{\nabla}_{\partial_k}(L^i_{jl}\partial_i) + \overset{2}{\nabla}_{\partial_l}(L^i_{kj}\partial_i) - \overset{1}{\nabla}_{\partial_k}(L^i_{lj}\partial_i)\big],\end{aligned}$$

from where

$$\overset{5}{R}{}^i_{jkl} = \frac{1}{2}\big(L^i_{jk,l} + L^i_{kj,l} - L^i_{jl,k} - L^i_{lj,k} + L^p_{jk} L^i_{pl} + L^p_{kj} L^i_{lp} - L^p_{jl} L^i_{kp} - L^p_{lj} L^i_{pk}\big). \tag{3.11}$$

4. Independent curvature tensors in L_N

On the basis of Section 2, we get

(4.1)
$$\overset{1}{\nabla}_Y X - \overset{2}{\nabla}_Y X \underset{(1.1)}{=} \overset{1}{\nabla}_Y X - [Y,X] - \overset{1}{\nabla}_X Y \underset{(2.2)}{=} \overset{1}{T}(X,Y) = -\overset{2}{T}(X,Y) = 2\tau(X,Y),$$

(4.2)
$$a)\, \overset{0}{\nabla}_Y X = \frac{1}{2}(\overset{1}{\nabla}_Y X + \overset{2}{\nabla}_Y X), \quad b)\, \tau(X,Y) = \frac{1}{2}\overset{1}{T}(X,Y) = \frac{1}{2}(\overset{1}{\nabla}_Y X - \overset{2}{\nabla}_Y X),$$

(4.3)
$$a)\, \overset{1}{\nabla}_Y X = \frac{1}{2}(\overset{1}{\nabla}_Y X + \overset{2}{\nabla}_Y X) + \frac{1}{2}(\overset{1}{\nabla}_Y X - \overset{2}{\nabla}_Y X) \underset{(4.2)}{\Rightarrow} \overset{1}{\nabla}_Y X = \overset{0}{\nabla}_Y X + \tau(X,Y),$$
$$b)\, \overset{1}{\nabla}_Y X = \frac{1}{2}(\overset{1}{\nabla}_Y X + \overset{2}{\nabla}_Y X) - \frac{1}{2}(\overset{1}{\nabla}_Y X - \overset{2}{\nabla}_Y X) \underset{(4.2)}{\Rightarrow} \overset{2}{\nabla}_Y X = \overset{0}{\nabla}_Y X - \tau(X,Y),$$

$$\overset{0}{\nabla}_{\partial_k}\partial_j \underset{(4.2\text{a})}{=} \frac{1}{2}(\overset{1}{\nabla}_{\partial_k}\partial_j + \overset{2}{\nabla}_{\partial_k}\partial_j) = \frac{1}{2}(L^i_{jk} + L^i_{kj})\partial_i = \overset{o}{L}{}^i_{jk}\partial_i,$$
$$\tau(\partial_j, \partial_k) \underset{(4.2\text{b})}{=} \frac{1}{2}(L^i_{jk} - L^i_{kj})\partial_i = \tau^i_{jk}\partial_i,$$

whence

(4.4)
$$a)\, \overset{0}{L}{}^i_{jk} = \frac{1}{2}(L^i_{jk} + L^i_{kj}), \quad b)\, \tau^i_{jk} = \frac{1}{2}(L^i_{jk} - L^i_{kj}).$$

We will now expose another presentation of curvature tensors $\overset{\theta}{R}$. By virtue of (2.18), (4.3a) is

(4.5)
$$\overset{1}{R}(X;Y,Z) = \overset{0}{\nabla}_Z \overset{1}{\nabla}_Y X + \tau(\overset{1}{\nabla}_Y X, Z) - \overset{0}{\nabla}_Y \overset{1}{\nabla}_Z X - \tau(\overset{1}{\nabla}_Z X, Y) + \overset{0}{\nabla}_{[Y,Z]} X + \tau\big(X,[Y,Z]\big).$$

Because of

$$\begin{aligned}&\tau(\overset{1}{\nabla}_Y X, Z) - \tau(\overset{1}{\nabla}_Z X, Y)\\ =&\tau\big(\overset{0}{\nabla}_Y X + \tau(X,Y), Z\big) - \tau\big(\overset{0}{\nabla}_Z X + \tau(X,Z), Y\big) + \tau\big(X, \overset{2}{\nabla}_Y Z - \overset{1}{\nabla}_Z Y\big),\end{aligned}$$

it follows that

$$\begin{aligned}\overset{1}{R}(X;Y,Z) &= \overset{0}{\nabla}_Z\overset{0}{\nabla}_Y X - \overset{0}{\nabla}_Z\tau(X,Y) - \overset{0}{\nabla}_Y\overset{0}{\nabla}_Z X - \overset{0}{\nabla}_Y\tau(X,Z)\\ &\quad + \tau(\overset{0}{\nabla}_Y X, Z) + \tau\big(\tau(X,Y),Z\big) - \tau(\overset{0}{\nabla}_Z X, Y) - \tau\big(\tau(X,Z),Y\big) + \overset{0}{\nabla}_{[Y,Z]}X\\ &\quad + \tau\big(X, \overset{0}{\nabla}_Y Z - \tau(Z,Y) - \overset{0}{\nabla}_Z Y - \tau(Y,X)\big)\\ &= \overset{0}{R}(X;Y,Z) + \overset{0}{\nabla}_Z\big[\tau(X,Y)\big] - \tau(\overset{0}{\nabla}_Z X, Y) - \tau(X, \overset{0}{\nabla}_Z Y)\\ &\quad - \overset{0}{\nabla}_Y\big[\tau(X,Z)\big] + \tau(\overset{0}{\nabla}_Y X, Z) + \tau(X, \overset{0}{\nabla}_Y Z) + \tau\big(\tau(X,Y),Z\big) - \tau\big(\tau(X,Z),Y\big).\end{aligned}$$

Finally,

$$\begin{aligned}\overset{1}{R}(X;Y,Z) = \overset{0}{R}(X;Y,Z) + \big(\overset{0}{\nabla}_Z\tau\big)(X,Y) - \big(\overset{0}{\nabla}_Y\tau\big)(X,Z)\\ + \tau\big(\tau(X,Y),Z\big) - \tau\big(\tau(X,Z),Y\big).\end{aligned} \tag{4.6}$$

From (4.6) in local coordinates is

$$\begin{aligned}\overset{1}{R}{}^i_{jkl}\partial_i &= \overset{0}{R}{}^i_{jkl}\partial_i + \big(\overset{0}{\nabla}_{\partial_l}\tau\big)(\partial_j,\partial_k) - \big(\overset{0}{\nabla}_{\partial_k}\tau\big)(\partial_j,\partial_l) + \tau\big(\tau(\partial_j,\partial_k),\partial_l\big) - \tau\big(\tau(\partial_j,\partial_l),\partial_k\big)\\ &= \overset{0}{R}{}^i_{jkl}\partial_i + \overset{0}{\nabla}_{\partial_l}\big[\tau(\partial_j,\partial_k)\big] - \tau(\overset{0}{\nabla}_{\partial_l}\partial_j,\partial_k) - \tau(\partial_j,\overset{0}{\nabla}_{\partial_l}\partial_k)\\ &\quad - \overset{0}{\nabla}_{\partial_k}\big[\tau(\partial_j,\partial_l)\big] + \tau(\overset{0}{\nabla}_{\partial_k}\partial_j,\partial_l) + \tau(\partial_j,\overset{0}{\nabla}_{\partial_k}\partial_l) + \tau(\tau^p_{jk}\partial_p,\partial_l) - \tau(\tau^p_{jl}\partial_p,\partial_k)\\ &= \overset{0}{R}{}^i_{jkl}\partial_i + \overset{0}{\nabla}_{\partial_l}(\tau^i_{jk}\partial_i) - \tau(\overset{o}{L}{}^p_{jl}\partial_p,\partial_k) - \tau(\partial_j,\overset{o}{L}{}^p_{kl}\partial_p) - \overset{0}{\nabla}_{\partial_k}(\tau^i_{jl}\partial_i) + \tau(\overset{o}{L}{}^p_{jk}\partial_p,\partial_l)\\ &\quad + \tau(\partial_j\overset{o}{L}{}^p_{lk}\partial_p) + \tau^p_{jk}\tau^i_{pl}\partial_i - \tau^p_{jl}\tau^i_{pk}\partial_i,\end{aligned}$$

and therefrom

$$\overset{1}{R}{}^i_{jkl} = \overset{0}{R}{}^i_{jkl} + \tau^i_{jk;l} - \tau^i_{jl;k} + \tau^p_{jk}\tau^i_{pl} - \tau^p_{jl}\tau^i_{pk}, \tag{4.6'}$$

where by $(;)$ covariant derivative wrp $\overset{o}{L}$ (i.e. $\overset{0}{\nabla}$) is defined.

By analogical procedure, the following is obtained

$$\begin{aligned}\overset{2}{R}(X;Y,Z) = \overset{0}{R}(X;Y,Z) - \big(\overset{0}{\nabla}_Z\tau\big)(X,Y) + \big(\overset{0}{\nabla}_Y\tau\big)(X,Z)\\ + \tau\big(\tau(X,Y),Z\big) - \tau\big(\tau(X,Z),Y\big),\end{aligned} \tag{4.7}$$

$$\overset{2}{R}{}^i_{jkl} = \overset{0}{R}{}^i_{jkl} - \tau^i_{jk;l} + \tau^i_{jl;k} + \tau^p_{jk}\tau^i_{pl} - \tau^p_{jl}\tau^i_{pk}. \tag{4.7'}$$

Starting from (2.19) for $\overset{3}{R}$, we first find

(4.8)
$$\begin{aligned}\overset{2}{\nabla}_Z\overset{1}{\nabla}_Y X - \overset{1}{\nabla}_Y\overset{2}{\nabla}_Z X &= \overset{2}{\nabla}_Z\big[\overset{0}{\nabla}_Y X+\tau(X,Y)\big] - \overset{1}{\nabla}_Y\big[\overset{0}{\nabla}_Z X - \tau(X,Z)\big]\\ &= \overset{0}{\nabla}_Z\overset{0}{\nabla}_Y X - \tau(\overset{0}{\nabla}_Y X, Z) + \overset{0}{\nabla}_Z\tau(X,Y) - \tau\big(\tau(X,Y),Z\big)\\ &\quad - \overset{0}{\nabla}_Y\overset{0}{\nabla}_Z X - \tau(\overset{0}{\nabla}_Z X, Y) + \overset{0}{\nabla}_Y\tau(X,Z) + \tau\big(\tau(X,Z),Y\big),\end{aligned}$$

(4.9)
$$\begin{aligned}\overset{2}{\nabla}_{\overset{1}{\nabla}_Y Z} X - \overset{1}{\nabla}_{\overset{2}{\nabla}_Z Y} X &= \overset{2}{\nabla}_{\overset{0}{\nabla}_Y Z+\tau(Z,Y)} X - \overset{1}{\nabla}_{\overset{0}{\nabla}_Z Y-\tau(Y,Z)} X\\ &= \overset{2}{\nabla}_{\overset{0}{\nabla}_Y Z} X - \overset{2}{\nabla}_{\tau(Y,Z)} X - \overset{1}{\nabla}_{\overset{0}{\nabla}_Z Y} X + \overset{1}{\nabla}_{\tau(Y,Z)} X\\ &= \overset{0}{\nabla}_{\overset{0}{\nabla}_Y Z} X - \tau(X,\overset{0}{\nabla}_Y Z) - \overset{0}{\nabla}_{\tau(Y,Z)} X + \tau\big(X,\tau(Y,Z)\big)\\ &\quad - \overset{0}{\nabla}_{\overset{0}{\nabla}_Z Y} X - \tau(X,\overset{0}{\nabla}_Z Y) + \overset{0}{\nabla}_{\tau(Y,Z)} X + \tau\big(X,\tau(Y,Z)\big)\\ &= \overset{0}{\nabla}_{\overset{0}{\nabla}_Y Z-\overset{0}{\nabla}_Z Y} X - \tau(X,\overset{0}{\nabla}_Y Z) - \tau(X,\overset{0}{\nabla}_Z Y) + 2\tau\big(X,\tau(Y,Z)\big)\\ &= \overset{0}{\nabla}_{[Y,Z]} X - \tau(X,\overset{0}{\nabla}_Y Z) - \tau(X,\overset{0}{\nabla}_Z Y) - 2\tau\big(X,\tau(Y,Z)\big).\end{aligned}$$

By substitution (4.8), (4.9) into (2.19), we get

$$\begin{aligned}\overset{3}{R}(X;Y,Z) = \overset{0}{R}(X;Y,Z) &- \tau(\overset{0}{\nabla}_Y X, Z) + \overset{0}{\nabla}_Z\tau(X,Y) - \tau\big(\tau(X,Y),Z\big)\\ &- \tau(\overset{0}{\nabla}_Z X, Y) + \overset{0}{\nabla}_Y\tau(X,Z) + \tau\big(\tau(X,Z),Y\big)\\ &- \tau(X,\overset{0}{\nabla}_Y Z) - \tau(X,\overset{0}{\nabla}_Z Y) - 2\tau\big(\tau(Y,Z),X\big),\end{aligned}$$

i.e.

$$\overset{3}{R}(X;Y,Z) = M(X;Y,Z) - 2\tau\big(\tau(Y,Z),X\big), \tag{4.10}$$

where

$$\begin{aligned}M(X;Y,Z) = \overset{0}{R}(X;Y,Z) &+ \big(\overset{0}{\nabla}_Z\big)(X,Y) + \big(\overset{0}{\nabla}_Y\tau\big)(X,Z)\\ &- \tau\big(\tau(X,Y),Z\big) + \tau\big(\tau(X,Z),Y\big).\end{aligned} \tag{4.11}$$

In local coordinates is

$$\overset{3}{R}{}^i_{jkl} = M^i_{jkl} - 2\tau^p_{kl}\tau^i_{pj}, \tag{4.10'}$$

where

$$M^i_{jkl} = \overset{0}{R}{}^i_{jkl} + \tau^i_{jk;l} + \tau^i_{jl;k} - \tau^p_{jk}\tau^i_{pl} + \tau^p_{jl}\tau^i_{pk}. \tag{4.11'}$$

On the base of (2.20), by analogical procedure for $\overset{4}{R}$ is obtained

$$\overset{4}{R}(X;Y,Z) = M(X;Y,Z) + 2\tau\big(\tau(Y,Z),X\big), \tag{4.12}$$

that is

$$\overset{4}{R}{}^i_{jkl} = M^i_{jkl} + 2\tau^p_{kl}\tau^i_{pj}, \tag{4.12'}$$

where M is given in (4.11), respectively in (4.11').

For $\overset{5}{R}$ based on (3.7), one obtains

$$\begin{aligned}\overset{5}{R}(X;Y,Z) = \frac{1}{2}\big[&\overset{0}{\nabla}_Z\overset{1}{\nabla}_Y X + \tau(\overset{1}{\nabla}_Y X, Z) - \overset{0}{\nabla}_Y\overset{1}{\nabla}_Z X + \tau(\overset{1}{\nabla}_Z X, Y)\\ &+ \overset{0}{\nabla}_Z\overset{2}{\nabla}_Y X - \tau(\overset{2}{\nabla}_Y X, Z) - \overset{0}{\nabla}_Y\overset{2}{\nabla}_Z X - \tau(\overset{2}{\nabla}_Z X, Y)\\ &+ 2\overset{0}{\nabla}_{[Y,Z]}X + \tau\big(X,[Y,Z]\big) - \tau\big(X,[Y,Z]\big)\big],\end{aligned}$$

from where, after arranging

$$\overset{5}{R}(X;y,Z) = \overset{0}{R}(X;Y,Z) + \tau\big(\tau(X,Y),Z\big) + \tau\big(\tau(X,Z),Y\big), \tag{4.13}$$

and

$$\overset{5}{R}{}^i_{jkl} = \overset{0}{R}{}^i_{jkl} + \tau^p_{jk}\tau^i_{pl} + \tau^p_{jl}\tau^i_{pk}. \tag{4.13'}$$

From (3.8) and (4.4, 4.6, 4.7) it follows

$$\overset{6}{R}(X;Y,Z) = \overset{0}{R}(X;Y,Z) + \tau\big(\tau(X,Y),Z\big) - \tau\big(\tau(X,Z),Y\big), \tag{4.14}$$

and

$$\overset{6}{R}{}^i_{jkl} = \overset{0}{R}{}^i_{jkl} + \tau^p_{jk}\tau^i_{pl} - \tau^p_{jl}\tau^i_{pk}. \tag{4.14'}$$

Starting from (3.9), we get

$$\overset{7}{R}(X'Y,Z) = \overset{0}{R}(X;Y,Z) - \tau\big(\tau(X,Y),Z\big) + \tau\big(\tau(X,Z),Y\big), \tag{4.15}$$

and

$$\overset{7}{R}{}^i_{jkl} = \overset{0}{R}{}^i_{jkl} - \tau^p_{jk}\tau^i_{pl} + \tau^p_{jl}\tau^i_{pk}. \tag{4.15'}$$

Finally, for $\overset{8}{R}$ by help of (3.10) one obtains

$$\overset{8}{R}(X;Y,Z) = \overset{0}{R}(X;Y,Z) - \tau\big(\tau(X,Y),Z\big) - \tau\big(\tau(X,Z),Y\big), \tag{4.16}$$

and

$$\overset{8}{R}{}^i_{jkl} = \overset{0}{R}{}^i_{jkl} - \tau^p_{jk}\tau^i_{pl} - \tau^p_{jl}\tau^i_{pk}. \tag{4.16'}$$

Omitting the argumens and introducing denotations

$$\begin{aligned}&\mathcal{A} = \big(\overset{0}{\nabla}_Z\tau\big)(X,Y),\ \mathcal{A}' = \big(\overset{0}{\nabla}_Y\tau\big)(X,Z),\ \mathcal{B} = \tau\big(\tau(X,Y),Z\big)\\ &\mathcal{B}' = \tau\big(\tau(X,Z),Y\big),\ \mathcal{C} = \tau\big(\tau(Y,Z),X\big),\end{aligned} \tag{4.17}$$

we write the equations (4.6, 4.10, 4.12)–(4.16) in the form
(4.18)

$$\begin{aligned}&\overset{1}{R} = \overset{0}{R} + \mathcal{A} - \mathcal{A}' + \mathcal{B} - \mathcal{B}', \quad \overset{2}{R} = \overset{0}{R} - \mathcal{A} + \mathcal{A}' + \mathcal{B} - \mathcal{B}'\\ &\overset{3}{R} = \overset{0}{R} + \mathcal{A} + \mathcal{A}' - \mathcal{B} + \mathcal{B}' - 2\mathcal{C}, \quad \overset{4}{R} = \overset{0}{R} + \mathcal{A} + \mathcal{A}' - \mathcal{B} + \mathcal{B}' + 2\mathcal{C},\\ &\overset{5}{R} = \overset{0}{R} + \mathcal{B} + \mathcal{B}', \quad \overset{6}{R} = \overset{0}{R} + \mathcal{B} - \mathcal{B}', \quad \overset{7}{R} = \overset{0}{R} - \mathcal{B} + \mathcal{B}', \quad \overset{8}{R} = \overset{0}{R} - \mathcal{B} - \mathcal{B}'.\end{aligned}$$

We can observe the equation (4.18) as a system of linear algebraic equations on $\mathcal{A}$, $\mathcal{A}'$, $\mathcal{B}$, $\mathcal{B}'$, $\mathcal{C}$. In order to find the rank of this system, its expanded matrix

$$P = \left(\begin{array}{ccccc:c} \mathcal{A} & \mathcal{A}' & \mathcal{B} & \mathcal{B}' & \mathcal{C} & \\ 1 & -1 & 1 & -1 & 0 & \overset{1}{R} - \overset{0}{R}\\ -1 & 1 & 1 & -1 & 0 & \overset{2}{R} - \overset{0}{R}\\ 1 & 1 & -1 & 1 & -2 & \overset{3}{R} - \overset{0}{R}\\ 1 & 1 & -1 & 1 & 2 & \overset{4}{R} - \overset{0}{R}\\ 0 & 0 & 1 & 1 & 0 & \overset{5}{R} - \overset{0}{R}\\ 0 & 0 & 1 & -1 & 0 & \overset{6}{R} - \overset{0}{R}\\ 0 & 0 & -1 & 1 & 0 & \overset{7}{R} - \overset{0}{R}\\ 0 & 0 & -1 & -1 & 0 & \overset{8}{R} - \overset{0}{R} \end{array}\right) \tag{4.19}$$

is reduced to an equivalent form with help of elementary transformations:

$$P = \left(\begin{array}{ccccc:c} \mathcal{A} & \mathcal{A}' & \mathcal{B} & \mathcal{B}' & \mathcal{C} & \\ 1 & -1 & 1 & -1 & 0 & \overset{1}{R} - \overset{0}{R}\\ 0 & 2 & -2 & 2 & -2 & \overset{3}{R} - \overset{1}{R}\\ 0 & 0 & 1 & 1 & 0 & \overset{5}{R} - \overset{0}{R}\\ 0 & 0 & 0 & -2 & 0 & \overset{6}{R} - \overset{5}{R}\\ 0 & 0 & 0 & 0 & 4 & \overset{4}{R} - \overset{3}{R}\\ 0 & 0 & 0 & 0 & 0 & \overset{6}{R} + \overset{7}{R} - 2\overset{0}{R}\\ 0 & 0 & 0 & 0 & 0 & \overset{1}{R} + \overset{2}{R} - 2\overset{6}{R}\\ 0 & 0 & 0 & 0 & 0 & \overset{5}{R} + \overset{8}{R} - 2\overset{0}{R} \end{array}\right) \tag{4.20}$$

By using the values (4.18), we see that

$$\overset{6}{R}+\overset{7}{R}-2\overset{0}{R}\equiv 0,\quad \overset{1}{R}+\overset{2}{R}-2\overset{6}{R}\equiv 0,\quad \overset{5}{R}+\overset{8}{R}-2\overset{0}{R}\equiv 0, \tag{4.21}$$

i.e. rank of the system (4.18) is 5 and it is in accordance.

From (4.20) is

$$\overset{6}{R}=\frac{1}{2}(\overset{1}{R}+\overset{2}{R}),\quad \overset{7}{R}=2\overset{0}{R}-\frac{1}{2}(\overset{1}{R}+\overset{2}{R}),\quad \overset{8}{R}=2\overset{0}{R}-\overset{5}{R}. \tag{4.22}$$

So, we have proved the main theorem:

Theorem 4.1. *Among curvature tensors obtained by virtue of non-symmetric connections* $\overset{1}{\nabla}$, $\overset{2}{\nabla}$ *linked with relation (1.1), there are five independent curvature tensors* $\overset{1}{R}$, …, $\overset{5}{R}$, *given by equations (2.18)–(2.20), (3.7). The remaining tensors can be expressed in terms of these five, along with the curvature tensors* $\overset{0}{R}$ *of corresponding symmetric connection* $\overset{0}{\nabla}$, *as defined in (4.2a).*

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