FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 31, No 2 (2016), 373–382

SOME GENERALIZED TRIPLE SEQUENCE SPACES DEFINED BY MODULUS FUNCTION

Shyamal Debnath and Bimal Chandra Das

Abstract. In this paper we introduce some newly defined triple sequence spaces by combining the modulus function and non-negative six dimensional matrix of the form $A = (a_{l,m,n,p,q,r})$ and we study some of their topological properties. We also obtain and prove some inclusion relations.

1. Introduction

A *triple sequence* (real or complex) is a function from $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ to $\mathbb{R}(\mathbb{C})$, where $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote a set of natural numbers, real numbers and complex numbers, respectively. In 2007, Sahiner et. al. [2] introduced the concept of triple sequences and established their *statistical convergence*. Subsequently, Dutta et. al. [3] generalized this concept by using the *Orlicz function*. Later on, Savas and Esi [5] introduced statistical convergence of triple sequences on *probabilistic normed spaces*. Recently, Debnath et. al. [13], Debnath and Das [14] generalized these concepts by using the *difference operator*.

In 1986 Maddox [10] introduced the *strongly Cesaro summable* with respect to a modulus function for the class of sequence. It was further investigated by Connor [11] in 1989 as an extended work for *strong A-summability*, considering $A = (a_{n,k})$ is a non-negative *regular matrix*. Pringsheim gave the definition of the convergence for double sequences in 1900. Since then, this concept has been studied by many authors and rapid development was made on this subject. In 2011, Savas and Patterson [6] introduced the definition for double sequence spaces defined by modulus function and considering the non-negative *four- dimensional matrix* as $A = (a_{m,n,k,l})$. In this paper, we have extended this concept for triple sequence spaces using the non-negative *six-dimensional matrix* $A = (a_{l,m,n,p,q,r})$ defined by

Received September 10, 2015; accepted January 22, 2016
2010 *Mathematics Subject Classification.* Primary 40A99; Secondary 40C05

modulus function and taking w^3, the set of all triple sequence of complex numbers.

Definition 1.1. [2]: A triple sequence (x_{lmn}) is said to be *convergent* to L, in *Pringsheim's sense* if for every $\epsilon > 0$, there exists $\mathbf{N}(\epsilon) \in N$ such that $|x_{lmn} - L| < \epsilon$, whenever $l \geq \mathbf{N}$,$m \geq \mathbf{N}$,$n \geq \mathbf{N}$ and we write $lim_{l,m,n\to\infty} x_{lmn} = L$.

Definition 1.2. [2]: A triple sequence (x_{lmn}) is said to be *bounded* if there exists $M > 0$ such that $|x_{lmn}| < M$ for all $l, m, n \in N$.

***Note*:** A triple sequence convergent in Pringsheim's sense may not be bounded [15].

Definition 1.3. [10]: A function $f : [0, \infty) \to [0, \infty)$ is called a *modulus function* if it satisfies the following four conditions:

1. $f(x) = 0$ if and only if $x = 0$,
2. $f(x + y) \leq f(x) + f(y)$ for all $x \geq 0$ and $y \geq 0$,
3. f is increasing,
4. f is increasing,
5. f is continuous from the right at 0.

Definition 1.4. Let $A = (a_{l,m,n,p,q,r})$ denote the six-dimensional summability method that maps the complex triple sequence x into the triple sequence Ax. Then the lmnth term to Ax will be $(Ax)_{l,m,n} = \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \sum_{r=1}^{\infty} a_{l,m,n,p,q,r} x_{p,q,r}$

Definition 1.5. Let f be a modulus function and $A = (a_{l,m,n,p,q,r})$ be a non negative six- dimensional matrix of real entries with $sup_{l,m,n} \sum_{p=0,q=0,r=0}^{\infty,\infty,\infty} a_{l,m,n,p,q,r} < \infty$
Then

$c_0^3(A, f) = \{x \in w^3 : P - lim_{l,m,n} \sum_{p=0,q=0,r=0}^{\infty,\infty,\infty} a_{l,m,n,p,q,r} f(|x_{p,q,r}|) = 0\}$

$c^3(A, f) = \{x \in w^3 : P - lim_{l,m,n} \sum_{p=0,q=0,r=0}^{\infty,\infty,\infty} a_{l,m,n,p,q,r} f(|x_{p,q,r} - L|) = 0$, for some $L\}$

$l_\infty^3(A, f) = \{x \in w^3 : sup_{l,m,n} \sum_{p=0,q=0,r=0}^{\infty,\infty,\infty} a_{l,m,n,p,q,r} f(|x_{p,q,r}|) < \infty\}$

If $f(x) = x$ then the sequence spaces become:

$c_0^3(A) = \{x \in w^3 : P - lim_{l,m,n} \sum_{p=0,q=0,r=0}^{\infty,\infty,\infty} a_{l,m,n,p,q,r} |x_{p,q,r}| = 0\}$

$c^3(A) = \{x \in w^3 : P - lim_{l,m,n} \sum_{p=0,q=0,r=0}^{\infty,\infty,\infty} a_{l,m,n,p,q,r} |x_{p,q,r} - L| = 0$, for some $L\}$

$l_\infty^3(A) = \{x \in w^3 : sup_{l,m,n} \sum_{p=0,q=0,r=0}^{\infty,\infty,\infty} a_{l,m,n,p,q,r}|x_{p,q,r}| < \infty\}$

The spaces in Definition 1.5 converted to some well-known sequence spaces by specifying A and f. For example, if we consider $A = (C, 1, 1)$ the sequence spaces $c_0^3(f)$, $c^3(f)$ and $l_\infty^3(f)$ will be of the following form:

$c_0^3(f) = \{x \in w^3 : P - lim_{l,m,n} 1/lmn \sum_{p=0,q=0,r=0}^{l-1,m-1,n-1} f(|x_{p,q,r}|) = 0\}$

$c^3(f) = \{x \in w^3 : P - lim_{l,m,n} 1/lmn \sum_{p=0,q=0,r=0}^{l-1,m-1,n-1} f(|x_{p,q,r} - L|) = 0$, for some $L\}$

$l_\infty^3(f) = \{x \in w^3 : sup_{l,m,n} 1/lmn \sum_{p=0,q=0,r=0}^{l-1,m-1,n-1} f(|x_{p,q,r}|) < \infty\}$

Now as a final illustration, if we consider $A = (C, 1, 1)$ and $f(x) = x$, we get the following spaces

$c_0^3 = \{x \in w^3 : P - lim_{l,m,n} 1/lmn \sum_{p=0,q=0,r=0}^{l-1,m-1,n-1} |x_{p,q,r}| = 0\}$

$c^3 = \{x \in w^3 : P - lim_{l,m,n} 1/lmn \sum_{p=0,q=0,r=0}^{l-1,m-1,n-1} |x_{p,q,r} - L| = 0$, for some $L\}$

$l_\infty^3 = \{x \in w^3 : sup_{l,m,n} 1/lmn \sum_{p=0,q=0,r=0}^{l-1,m-1,n-1} |x_{p,q,r}| < \infty\}$.

2. Main Results

In this section, we shall establish the main properties of the sequence spaces in Definition 1.5

Theorem 2.1. *The sequence spaces $c_0^3(A, f)$, $c^3(A, f)$ and $l_\infty^3(A, f)$ all are linear over the complex field C.*

Proof. It is obvious. □

Theorem 2.2. *If $A = (a_{l,m,n,p,q,r})$ is a non-negative six dimensional matrix of real entries with $sup_{l,m,n} \sum_{p=0,q=0,r=0}^{\infty,\infty,\infty} a_{l,m,n,p,q,r} < \infty$, and let f be a modulus function then*

1. $c^3(A, f) \subset l_\infty^3(A, f)$
2. $c_0^3(A, f) \subset l_\infty^3(A, f)$

Proof. Here we shall establish the inclusion (1) only.

Let $x \in c^3(A, f)$. Now using the conditions (2) and (3) of the modulus function (Definition 1.3) we get the following:

$\sum_{p=0,q=0,r=0}^{\infty,\infty,\infty} a_{l,m,n,p,q,r} f(|x_{p,q,r}|)$

$\leq \sum_{p=0,q=0,r=0}^{\infty,\infty,\infty} a_{l,m,n,p,q,r} f(|x_{p,q,r} - L|) + f(|L|) \sum_{p=0,q=0,r=0}^{\infty,\infty,\infty} a_{l,m,n,p,q,r}$

There exists an integer M_1 such that $|L| \leq M_1$. We obtain

$\sum_{p=0,q=0,r=0}^{\infty,\infty,\infty} a_{l,m,n,p,q,r} f(|x_{p,q,r}|)$

$\leq \sum_{p=0,q=0,r=0}^{\infty,\infty,\infty} a_{l,m,n,p,q,r} f(|x_{p,q,r} - L|) + M_1 f(1) \sum_{p=0,q=0,r=0}^{\infty,\infty,\infty} a_{l,m,n,p,q,r}$

As we consider $x \in c^3(A,f)$ and $sup_{l,m,n} \sum_{p=0,q=0,r=0}^{\infty,\infty,\infty} a_{l,m,n,p,q,r} < \infty$ we are conclude that

$x \in l_\infty^3(A,f)$

This completes the proof. □

Theorem 2.3. *If $A = (a_{l,m,n,p,q,r})$ is a non-negative six dimensional matrix of real entries with $sup_{l,m,n} \sum_{p=0,q=0,r=0}^{\infty,\infty,\infty} a_{l,m,n,p,q,r} < \infty$, and let f be a modulus function then the following inclusion holds*

1. $c^3(A) \subset c^3(A,f)$
2. $c_0^3(A) \subset c_0^3(A,f)$
3. $l_\infty^3(A) \subset l_\infty^3(A,f)$.

Proof. Here the inclusions (1) and (2) can be easily proved. Thus we will only establish the inclusion (3).

Let $x \in l_\infty^3(A)$ such that $sup_{l,m,n} \sum_{p=0,q=0,r=0}^{\infty,\infty,\infty} a_{l,m,n,p,q,r} < \infty$. Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f(t) < \epsilon$ for $0 \leq t \leq \delta$. Now we consider the following equality

$\sum_{p=0,q=0,r=0}^{\infty,\infty,\infty} a_{l,m,n,p,q,r} f(|x_{p,q,r}|)$

$$= \sum_{p=0,q=0,r=0_{|x_{p,q,r}| \leq \delta}}^{\infty,\infty,\infty} a_{l,m,n,p,q,r} f(|x_{p,q,r}|) + \sum_{p=0,q=0,r=0_{|x_{p,q,r}| > \delta}}^{\infty,\infty,\infty} a_{l,m,n,p,q,r} f(|x_{p,q,r}|)$$

From the properties of the modulus function we have the following:

$$\sum_{p=0,q=0,r=0_{|x_{p,q,r}| \leq \delta}}^{\infty,\infty,\infty} a_{l,m,n,p,q,r} f(|x_{p,q,r}|) \leq \epsilon \sum_{p=0,q=0,r=0}^{\infty,\infty,\infty} a_{l,m,n,p,q,r} \tag{2.1}$$

For $|x_{p,q,r}| > \delta$ and the fact that

$|x_{p,q,r}| < |x_{p,q,r}|/\delta < [1 + \{|x_{p,q,r}|/\delta\}]$

Where $[t]$ denoted the integer part of t and from the conditions (2) and (3) of the modulus function we can write

$f(|x_{p,q,r}|) < (1 + [|x_{p,q,r}|/\delta])f(1) \leq 2f(1)\{|x_{p,q,r}|/\delta\}$

Now

$$\sum_{p=0,q=0,r=0_{|x_{p,q,r}|>\delta}}^{\infty,\infty,\infty} a_{l,m,n,p,q,r} f(|x_{p,q,r}|) \leq 2f(1)/\delta \sum_{p=0,q=0,r=0}^{\infty,\infty,\infty} a_{l,m,n,p,q,r}|x_{p,q,r}|$$

The last inequality and equation (2.1) gives us the following results

$\sum_{p=0,q=0,r=0}^{\infty,\infty,\infty} a_{l,m,n,p,q,r} f(|x_{p,q,r}|)$

$\leq \epsilon \sum_{p=0,q=0,r=0}^{\infty,\infty,\infty} a_{l,m,n,p,q,r} + 2f(1)/\delta \sum_{p=0,q=0,r=0}^{\infty,\infty,\infty} a_{l,m,n,p,q,r}|x_{p,q,r}|$

Since $sup_{l,m,n} \sum_{p=0,q=0,r=0}^{\infty,\infty,\infty} a_{l,m,n,p,q,r} < \infty$ and $x \in l_\infty^3(A)$

we find that $x \in l_\infty^3(A, f)$.

This completes the proof. □

Theorem 2.4. *If $A = (a_{l,m,n,p,q,r})$ is a non-negative six dimensional matrix of real entries with $sup_{l,m,n} \sum_{p=0,q=0,r=0}^{\infty,\infty,\infty} a_{l,m,n,p,q,r} < \infty$, and let f be a modulus function and $\beta = lim_{t\to\infty} f(t)/t > 0$ then $c^3(A) = c^3(A, f)$.*

Proof. Let $\beta > 0$. By definition of β we have $f(t) \geq \beta t$ for all $t \geq 0$ and since $\beta > 0$ we have $t \leq \{1/\beta\}f(t)$ for all $t \geq 0$.

Now from $x \in c^3(A, f)$ we can write the following inequality

$\sum_{p=0,q=0,r=0}^{\infty,\infty,\infty} a_{l,m,n,p,q,r}|x_{p,q,r} - L| \leq \{1/\beta\} \sum_{p=0,q=0,r=0}^{\infty,\infty,\infty} a_{l,m,n,p,q,r} f(|x_{p,q,r} - L|)$

whence $x \in c^3(A)$. In our previous theorem we have shown that $c^3(A) \subset c^3(A, f)$.

Hence the proof of the theorem is complete. □

Theorem 2.5. *If $A = (a_{l,m,n,p,q,r})$ has only positive entries and $B = (b_{l,m,n,p,q,r})$ is a non-negative six dimensional matrix such that $\{b_{l,m,n,p,q,r}/a_{l,m,n,p,q,r}\}$ is bounded* *then* $l^3_\infty(A,f) \subset l^3_\infty(B,f)$.

Proof. The proof is easy, so omitted. □

Theorem 2.6. *If* $A = (a_{l,m,n,p,q,r})$ *is a non-negative six dimensional matrix of real entries with*

$sup_{l,m,n} \sum_{p=0,q=0,r=0}^{\infty,\infty,\infty} a_{l,m,n,p,q,r} < \infty$,

and let f *be a modulus function then* $c^3_0(A,f)$ *and* $c^3(A,f)$ *are complete linear topological spaces with the paranorm*

$$g(x) = sup_{l,m,n} \sum_{p=0,q=0,r=0}^{\infty,\infty,\infty} a_{l,m,n,p,q,r} f(|x_{p,q,r}|) ,$$

Proof. The space $c^3_0(A,f)$ is a complete linear topological space which is clear from the above statements. Let us consider $c^3(A,f)$. From *Theorem 2.2* for each $x \in c^3(A,f)$, $g(x)$ exists. Clearly $g(\theta) = 0$, $g(-x) = g(x)$ and $g(x+y) \leq g(x)+g(y)$. We shall show now that the scalar multiplication is continuous. First, we observe the following:

$$g(\lambda x) = sup_{l,m,n} \sum_{p=0,q=0,r=0}^{\infty,\infty,\infty} a_{l,m,n,p,q,r} f(|\lambda x_{p,q,r}|) \leq (1 + [|\lambda|])g(x) ,$$

where $[|\lambda|]$ denotes the integer part of $|\lambda|$. In addition, observe that x and $\lambda \to 0$ implies $g(\lambda x) \to 0$. For fixed λ, if x approaches 0 then $g(\lambda x)$ approaches 0. We have to show that for fixed x, λ approaching 0 implies $g(\lambda x)$ approaching 0. Let $x \in c^3(A,f)$, so this implies that

$$P - lim_{l,m,n} \sum_{p=0,q=0,r=0}^{\infty,\infty,\infty} a_{l,m,n,p,q,r} f(|x_{p,q,r} - L|) = 0 ,$$

Let $\epsilon > 0$ and choose N such that

$$\sum_{p=0,q=0,r=0}^{\infty,\infty,\infty} a_{l,m,n,p,q,r} f(|x_{p,q,r} - L|) < \epsilon/4 \qquad (2.2)$$

for $l,m,n > N$. Also for each (l,m,n) with $1 \leq l,m,n \leq N$, since

$$\sum_{p=0,q=0,r=0}^{\infty,\infty,\infty} a_{l,m,n,p,q,r} f(|x_{p,q,r} - L|) < \infty$$

There exists an integer $M_{l,m,n}$ such that

$$\sum_{p,q,r > M_{l,m,n}} a_{l,m,n,p,q,r} f(|x_{p,q,r} - L|) < \epsilon/4$$

Let $M = max_{1\leq l,m,n\leq N}\{M_{l,m,n}\}$

We have for each (l,m,n) with $1 \leq l,m,n \leq N$,

$\sum_{p,q,r>M} a_{l,m,n,p,q,r} f(|x_{p,q,r} - L|) < \epsilon/4$

From the equation (2.2) for $l,m,n > N$ we obtain the following

$\sum_{p,q,r>M} a_{l,m,n,p,q,r} f(|x_{p,q,r} - L|) < \epsilon/4$

Thus M is an integer which is independent of (l,m,n) such that

$$\sum_{p,q,r>M} a_{l,m,n,p,q,r} f(|x_{p,q,r} - L|) < \epsilon/4 \tag{2.3}$$

Further for $|\lambda| < 1$ and for all (l,m,n)

$\sum_{p=0,q=0,r=0}^{\infty,\infty,\infty} a_{l,m,n,p,q,r} f(|\lambda x_{p,q,r}|)$

$= \sum_{p=0,q=0,r=0}^{\infty,\infty,\infty} a_{l,m,n,p,q,r} f(|\lambda x_{p,q,r} - \lambda L + \lambda L|)$

$$\leq \sum_{p=0,q=0,r=0}^{\infty,\infty,\infty} a_{l,m,n,p,q,r} f(|\lambda x_{p,q,r} - \lambda L|) + \sum_{p=0,q=0,r=0}^{\infty,\infty,\infty} a_{l,m,n,p,q,r} f(|\lambda L|) \tag{2.4}$$

$\leq \sum_{p,q,r>M} a_{l,m,n,p,q,r} f(|\lambda x_{p,q,r} - \lambda L|) + \sum_{p,q,r\leq M} a_{l,m,n,p,q,r} f(|\lambda x_{p,q,r} - \lambda L|)$

$+ \sum_{p,q\geq M,r<M} a_{l,m,n,p,q,r} f(|\lambda x_{p,q,r} - \lambda L|) + \sum_{p,q<M,r\geq M} a_{l,m,n,p,q,r} f(|\lambda x_{p,q,r} - \lambda L|)$

$+ \sum_{p,r\geq M,q<M} a_{l,m,n,p,q,r} f(|\lambda x_{p,q,r} - \lambda L|) + \sum_{p,r<M,q\geq M} a_{l,m,n,p,q,r} f(|\lambda x_{p,q,r} - \lambda L|)$

$+ \sum_{q,r\geq M,p<M} a_{l,m,n,p,q,r} f(|\lambda x_{p,q,r} - \lambda L|) + \sum_{q,r<M,p\geq M} a_{l,m,n,p,q,r} f(|\lambda x_{p,q,r} - \lambda L|)$

$+ f(|\lambda L|) \sum_{p=0,q=0,r=0}^{\infty,\infty,\infty} a_{l,m,n,p,q,r}$

For each (l,m,n) and by the continuity of modulus functions as $\lambda \to \infty$ implies

$\sum_{p,q,r\leq M} a_{l,m,n,p,q,r} f(|\lambda x_{p,q,r} - \lambda L|) + f(|\lambda L|) \sum_{p=0,q=0,r=0}^{\infty,\infty,\infty} a_{l,m,n,p,q,r} \to 0$

Using Pringshiem sense. We choose $\delta < 1$ such that $|\lambda| < \delta$ implies

$$\sum_{p,q,r\leq M} a_{l,m,n,p,q,r} f(|\lambda x_{p,q,r} - \lambda L|) + f(|\lambda L|) \sum_{p=0,q=0,r=0}^{\infty,\infty,\infty} a_{l,m,n,p,q,r} < \epsilon/4 \quad (2.5)$$

In a similar way, we can conclude that

$$\sum_{p,q\geq M,r<M} a_{l,m,n,p,q,r} f(|\lambda x_{p,q,r} - \lambda L|) < \epsilon/4 \quad (2.6)$$

$$\sum_{p,q<M,r\geq M} a_{l,m,n,p,q,r} f(|\lambda x_{p,q,r} - \lambda L|) < \epsilon/4 \quad (2.7)$$

$$\sum_{p,r\geq M,q<M} a_{l,m,n,p,q,r} f(|\lambda x_{p,q,r} - \lambda L|) < \epsilon/4 \quad (2.8)$$

$$\sum_{p,r<M,q\geq M} a_{l,m,n,p,q,r} f(|\lambda x_{p,q,r} - \lambda L|) < \epsilon/4 \quad (2.9)$$

$$\sum_{q,r\geq M,p<M} a_{l,m,n,p,q,r} f(|\lambda x_{p,q,r} - \lambda L|) < \epsilon/4 \quad (2.10)$$

$$\sum_{q,r<M,p\geq M} a_{l,m,n,p,q,r} f(|\lambda x_{p,q,r} - \lambda L|) < \epsilon/4 \quad (2.11)$$

It follows from (2.3) through (2.11) that

$$\sum_{p=0,q=0,r=0}^{\infty,\infty,\infty} a_{l,m,n,p,q,r} f(|\lambda x_{p,q,r}|) < \epsilon \text{ for all } (l,m,n)$$

Thus $g(\lambda x)$ approaches 0 as λ approaches 0. Therefore $c_0^3(A,f)$ is a paranormed linear topological space. Now we have to show that $c_0^3(A,f)$ is complete with respect to its paranorm topologies. Let $(x_{p,q,r}^s)$ be a Cauchy sequence in $c_0^3(A,f)$.

Then we can write $g(x^s - x^t) \to 0$ as $s,t \to \infty$ for all (l,m,n)

$$\sum_{p=0,q=0,r=0}^{\infty,\infty,\infty} a_{l,m,n,p,q,r} f(|x_{p,q,r}^s - x_{p,q,r}^t|) \to 0 \quad (2.12)$$

Since $A = (a_{l,m,n,p,q,r})$ is non-negative, we conclude that $f(|x_{p,q,r}^s - x_{p,q,r}^t|) \to 0$ as $s,t \to \infty$, for each fixed p,q,r and by continuity of modulus function, $(x_{p,q,r}^s)$ is a Cauchy sequence in C for each fixed p,q,r. Since C is complete as $s \to \infty$ we have $x_{p,q,r}^s \to x_{p,q,r}$ for each (p,q,r). Now from (2.12) we get for for each fixed $\epsilon > 0$, there exists a natural number N such that

$$\sum_{p,q,r=0,s,t>N}^{\infty,\infty,\infty} a_{l,m,n,p,q,r} f(|x_{p,q,r}^s - x_{p,q,r}^t|) < \epsilon \quad (2.13)$$

For all (l,m,n), Since for any fixed natural number M we have from (2.13)

$$\sum_{p,q,r\leq M,s,t>N}^{\infty,\infty,\infty} a_{l,m,n,p,q,r} f(|x_{p,q,r}^s - x_{p,q,r}^t|) < \epsilon$$

From the above inequality and supposing $t \to \infty$, for all (l, m, n), we obtain

$\sum_{p,q,r \leq M, s>N}^{\infty,\infty,\infty} a_{l,m,n,p,q,r} f(|x^s_{p,q,r} - x_{p,q,r}|) < \epsilon$

Since M is arbitrary, letting $M \to \infty$, we get (x^s) being a sequence in

$\sum_{p=o,q=0,r=0}^{\infty,\infty,\infty} a_{l,m,n,p,q,r} f(|x^s_{p,q,r} - x_{p,q,r}|) < \epsilon$

for all (l, m, n). Thus $g(x^s - x) \to 0$ as $s \to \infty$. Also for $c^3(A, f)$, we have by definition of $c^3(A, f)$ for each s that there exists L^s with

$\sum_{p=o,q=0,r=0}^{\infty,\infty,\infty} a_{l,m,n,p,q,r} f(|x^s_{p,q,r} - L^s|) \to 0$

As $(l, m, n) \to \infty$ and $sup_{l,m,n} \sum_{p=o,q=0,r=0}^{\infty,\infty,\infty} a_{l,m,n,p,q,r} < \infty$ from the condition (2) of modulus function, we have $f(|L^s - L^t|) \to 0$ as $s, t \to \infty$ and thus L^s converges to L. Hence

$\sum_{p=o,q=0,r=0}^{\infty,\infty,\infty} a_{l,m,n,p,q,r} f(|x_{p,q,r} - L|) \to 0$

As $(l, m, n) \to \infty$, thus $x \in c^3(A, f)$ and this completes the proof. □

Acknowledgement: The authors are grateful to the reviewers for their valuable suggestions which led to the improvement of the paper.

REFERENCES

1. A. Pringsheim: *Zur theorie der zweifach unendlichen zahlenfolgen*. Math. Ann. **53** (1900), 289–321.
2. A. Sahiner, M. Gurdal and K. Duden: *Triple sequences and their statistical convergence*. Selcuk. J. Appl. Math. **8(2)** (2007), 49–55.
3. A. J. Datta, A. Esi and B. C. Tripathy: *Statistically convergent triple sequence spaces defined by Orlicz function*. J. Math. Anal. **4(2)** (2013), 16–22.
4. B. C. Tripathy and A. Esi : *A new type of difference sequence spaces*. Int. J. Sci. Tech. **1(1)** (2006), 11–14.
5. E. Savas and A. Esi : *Statistical convergence of triple sequences on probabilistic normed space*. Annals. Univ. Craiova, Math. and Comp. Sci. Series. **39(2)** (2012), 226–236.
6. E. Savas and R. F. Patterson : *Double Sequece Spaces Defined by a Modulus*. Math. Slovaca. **61(2)** (2011), 245–256.
7. G. M. Robinson: *Divergent double sequences and series*. Trans. Amer. Math. Soc. **28** (1926), 50–53.
8. H. J. Hamilton: *Transformations of multiple sequences*. Duke Math. Jour. **2** (1936), 29–60.

9. H. Kizmaz: *On certain sequence spaces*. Canad. Math. Bull. **24(2)** (1981), 169–176.
10. I. J. Maddox: *Sequece Spaces Defined by a Modulus*. Math. Proc. Cambridge Philos. Soc. **100** (1986), 161–166.
11. J. Connor: *On Strong Matrix Summability with Respect to a Modulus and Statistical Convergence*. Canad. Math. Bull. **32** (1989), 194–198.
12. S. Debnath and B. C. Das : *Some New Type of Difference Triple Sequence Spaces*. Palestine J.Math. **4(2)** (2015), 284–290.
13. S. Debnath and B. C. Das : *Regular Matrix Transformation on Triple Sequence Spaces*. Bol. Soc. Paran. Mat. **34(2)** (2016), 1–12.
14. S. Debnath, B. Sarma and B. C. Das : *Some Generalized Triple Sequence Spaces of Real Numbers*. J. Nonlinear Anal. Opti. **6(1)** (2015), 71–79.
15. S. Debnath, B. Sarma and S. Saha : *Some Sequence Spaces of Interval Vectors*. Afri. Mathematika. **26(5)** (2015), 673–678.
16. S. Debnath and S. Saha : *Some Newly Defined Sequence Spaces using Regular Matrix Of Fibonacci Numbers*. AKU-J. Sci. and Eng. **14** (2014), 011301 (1–3).
17. V. N. Mishra: *Some Problems on Approximations of Functions in Banach Spaces*. Ph. D. Thesis, IIT Roorkee–247667, India, 2007.
18. V. N. Mishra, K. Khatri and L. N. Mishra : *Strong Cesaro Summability of Triple Fourier Integrals*. Fasciculi Mathematici. **53** (2014), 95–112.
19. V. N. Mishra, K. Khatri and L. N. Mishra : *Product Summability Transform of Conjugate Series of Fourier series*. Int. J. Math. Mathematical Sci. **2012** (2012), Article ID 298923.
20. V. N. Mishra, K. Khatri and L. N. Mishra : *Using Linear Operators to Approximate Signals of* $Lip(\alpha, p), (p \geq 1)$*-Class*. Filomat. **27(2)** (2013), 355–365.
21. V. N. Mishra, K. Khatri and L. N. Mishra : *Product* $(N, p_n).(C, 1)$ *summability of a sequence of Fourier Coefficients*. Mathematical Sciences. **6** (2012), 38–.
22. V. N. Mishra and L. N. Mishra : *Trigonometric Approximation of Signals (Functions) in* $Lp(p \geq l)$ *norm*. Int. J. Contemp. Math. Sci. **19(7)** (2012), 909–918.

Shyamal Debnath
Department of Mathematics
Tripura University
Pin:799022
Suryamaninagar,Tripura,India
Email:debnathshyamal@tripurauniv.in

Bimal chandra Das
Department of Mathematics
Govt. Degree College, Kamalpur
Pin:799285
Kamalpur,Tripura,India
Email: bcdas3744@gmail.com