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ALMOST SCHOUTEN SOLITONS ON N(κ)-CONTACT METRIC MANIFOLDS

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Abstract. In the current paper, we have studied almost Schouten solitons and gradient Schouten solitons on an $N(\kappa)$-contact metric manifold of dimension $(2n+1)$. Besides, we have shown that the almost Schouten soliton does not exist on $N(\kappa)$-contact metric manifold for $\kappa < 1$. In addidtion, it has been proved that the manifold complying with the gradient Schouten solitons is locally isometric to a Lie group. Moreover, we have determined that a 3-dimensional $N(\kappa)$-contact metric manifold admitting a gradient Schouten soliton is either flat or of constant scalar curvature. Finally, an example has been constructed to verify the outcomes.
Keywords: Schouten solitons, metric manifolds, Ricci solitons.

1. Introduction

Ricci solitons are self-similar solutions of the Ricci flow and often appear as singular versions of the solutions. A Schouten soliton is a generalized Ricci soliton which is defined by

$$L_V g + 2S_t + 2bg = 0, \tag{1.1}$$

where S_t is the Schouten tensor given by,

$$S_t = \frac{1}{(n-2)}\big[S - \frac{r}{2(n-1)}g\big], \tag{1.2}$$

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$b \in \mathbb{R}$, L denotes Lie-derivative, S indicates the Ricci tensor, r denotes the scalar curvature and V denotes the potential vector field. The Einsein soliton is a familier example of Schouten soliton. Borges [5] constructed an example of a Schouten soliton on a Reimannian manifold. If b is a smooth function on the manifold, instead of a real number, then the notion of Schouten soliton is generalized to the notion of almost Schouten soliton.

By the concept of Catino and Mazzieri [6] generally a Riemannian manifold (M^n, g) is called a gradient Schouten soliton if for $f \in C^\infty(M)$, a potential function and $b \in \mathbb{R}$

$$S_t + \nabla^2 f = \big[\frac{r}{2(n-1)} + b\big]g, \tag{1.3}$$

holds, where $\nabla^2 f$ is the Hessian of f and r is the scalar curvature.

According to Catino and Mazzieri[6] a Riemannian manifold is a gradient ρ-Einstein solitons for $\rho \in R - \{0\}$ if

$$S + \nabla^2 f = (\rho r + b)g, \tag{1.4}$$

where S is the Ricci tensor and $f \in C^\infty(M)$.
As a consequence Schouten soliton is a ρ-Einstein soliton for $\rho= \frac{1}{2(n-1)}$.

Catino and Mazzieri [6] also studied compact gradient Schouten solitons. According to them every compact gradient soliton is trivial. Moreover, they showed that a complete gradient steady Schouten soliton is trivial and Ricci flat. Pina and Menezes [11] analyzed the complete gradient Schouten solitons. They demonstrated that if a gradient Schouten soliton is both complete and conformal to a Euclidean metric and rotationally symmetric, then it is isomorphic to $\mathbb{R} \times \mathbb{S}^{n-1}$. Borges [5] also proved that a complete gradient Schouten soliton becomes a gradient Ricci soliton if the scalar curvature r vanishes.

In a Riemannian manifold M of dimension $(2n + 1)$ the κ-nulity distribution is defined as

$$\begin{aligned} N(\kappa) : p \to N_p(\kappa) &= [X \in T_p(M) : R(Y,Z)X \\ &= \kappa\{\ g(Y,X)Z - g(Z,X)Y\}\ ], \end{aligned} \tag{1.5}$$

for all $Y, Z \in T_p(M)$, κ is a real number and $T_p(M)$ is the Lie algebra of all vector fields at p.

Let ζ be a characteristic vector field belong to the κ -nulity distribution. Therefore

$$R(Y,Z)\zeta = \kappa[\eta(Z)Y - \eta(Y)Z]. \tag{1.6}$$

A contact metric manifold satisfying the above is known as N(κ)-contact metric manifold of dimension $(2n + 1)$. When $\kappa = 1$, the manifold becomes a Sasakian manifold. N(κ)-contact metric manifolds have been studied by various authors such as Blair, Koufogiorgos and Papantoniou [2], Blair [1], Kar, Majhi and De [9], Mandal [10], and De, Yildiz and Ghosh [7].

In this paper we are encouraged to study some theorems and properties of almost Schouten solitons and gradient Schouten solitons on N(κ)-contact metric manifolds.

We have organized the paper as follows: After the introduction the preliminaries are given in Section-2. After that in Section-3 we have showed that there is no almost Schouten soliton in $N(\kappa)$-contact metric manifold with $\kappa < 1$. When the soliton vector field is pointwise collinear with the potential vector field, the almost Schouten soliton becomes Schouten soliton and also an N(κ)-contact metric manifold does not admit a gradient Schouten soliton. In Section-4 we have studied the 3-dimensional N(κ)-contact metric manifold with gradient Schouten solitons. Finally in Section-5 an example has been given to examine the outcomes.

2. Preliminaries

Let us assume M as a $(2n+1)$ dimensional smooth manifold equipped with an almost contact metric structure (Φ, ζ, η), where Φ, ζ, η are a $(1,1)$ tensor field, a vector field and a 1-form on M respectively, satisfying

$$\Phi^2(Y) = -Y + \eta(Y)\zeta, \quad \eta(\zeta) = 1, \quad \Phi\zeta = 0. \tag{2.1}$$

Introducing a Riemannian metric g in M^{2n+1} we get the consequences as follows:

$$g(Y, \zeta) = \eta(Y), \quad \eta(\Phi Y) = 0, \tag{2.2}$$

$$g(\Phi Y, \Phi Z) = g(Y, Z) - \eta(Y)\eta(Z), \tag{2.3}$$

$$g(\Phi Y, Z) = -g(Y, \Phi Z), \tag{2.4}$$

$$(\nabla_Y \eta)(Z) = g(\nabla_Y \zeta, Z), \tag{2.5}$$

for every vector fields $Y, Z \in \chi(M)$. The manifold M^{2n+1} equiped with the almost contact metric structure is known as an almost contact metric manifold when it is differentiable and is called contact metric manifold when the contact metric structure (Φ, ζ, η, g) satisfies

$$g(Y, \Phi Z) = d\eta(Y, Z), \tag{2.6}$$

for every Y, $Z \in \chi(M)$. In the contact metric manifold M^{2n+1} we assume a symmetric $(1,1)$-tensor field h which is defined by

$$h = \frac{1}{2} L_\zeta \Phi, \tag{2.7}$$

where L is the Lie differentiation operator in the direction of ζ, with the following relations

$$h\zeta = 0, \quad tr(h) = 0, \quad tr(h\Phi) = 0, \quad h\Phi = -\Phi h, \tag{2.8}$$

$$\nabla_Y \zeta = -\Phi Y - \Phi h Y, \quad h^2 = (\kappa - 1)\Phi^2. \tag{2.9}$$

In an $N(\kappa)$-contact metric manifold M^{2n+1} for any vector fields $Y, Z \in \chi(M)$, we have the consequences as follows

$$(\nabla_Y \eta)(Z) = g(Y + hY, \Phi Z), \tag{2.10}$$

$$\begin{aligned} (\nabla_Y h)(Z) &= [(1-\kappa)g(Y, \Phi Z) + g(Y, h\Phi Z)]\zeta \\ &+ \eta(Z)[h(\Phi Y + \Phi hY)], \end{aligned} \tag{2.11}$$

$$(\nabla_Y \Phi)(Z) = g(Y + hY, Z)\zeta - \eta(Z)(Y + hY). \tag{2.12}$$

Let us assume R and S as the Riemannian curvature and Ricci tensor and if r is the scalar curvature of the $N(\kappa)$-contact metric manifold M of dimension $(2n+1)$ respectively, then for every vector fields $Y, Z \in \chi(M)$ the following conditions are hold:

$$R(Y, Z)\zeta = \kappa[\eta(Z)Y - \eta(Y)Z], \tag{2.13}$$

$$R(\zeta, Y)Z = \kappa[g(Y, Z)\zeta - \eta(Z)Y], \tag{2.14}$$

$$\begin{aligned} S(Y, Z) &= 2(n-1)[g(Y, Z) + g(hY, Z)] \\ &+ [2n\kappa - 2(n-1)]\eta(Y)\eta(Z), \end{aligned} \tag{2.15}$$

$$S(Y, \zeta) = 2n\kappa\eta(Y), \tag{2.16}$$

$$r = 2n(2n - 2 + \kappa), \tag{2.17}$$

3. Schouten solitons and gradient Schouten solitons

Theorem 3.1. *There is no almost Schouten soliton on an $N(\kappa)$-contact metric manifold with $\kappa < 1$.*

Proof. From the equations (1.1) and (1.2), we have

$$(L_V g)(Y, Z) + \frac{2}{2n-1}S(Y, Z) + \big[2b - \frac{r}{2n(2n-1)}\big]g(Y, Z) = 0, \tag{3.1}$$

which implies

$$\begin{aligned} g(\nabla_Y V, Z) &+ g(\nabla_Z V, Y) + \frac{2}{2n-1}S(Y, Z) \\ &+ \big[2b - \frac{r}{2n(2n-1)}\big]g(Y, Z) = 0. \end{aligned} \tag{3.2}$$

If we put $V = \zeta$, then the above equation becomes

$$-2g(\Phi hY, Z) + \frac{2}{2n-1}S(Y, Z) + \big[2b - \frac{r}{2n(2n-1)}\big]g(Y, Z) = 0. \tag{3.3}$$

Setting ζ in the place of Y, Z in (3.3), we get

$$r = 8n^2\kappa + 4bn(2n-1). \tag{3.4}$$

Contracting Y and Z in (3.2) and using (2.17), we infer

$$divV + \frac{2n+\kappa-2}{2n-1} + (2n+1)b = \frac{(2n+1)r}{4n(2n-1)}. \tag{3.5}$$

Using (3.4) in the above equation, we obtain

$$divV - \frac{2(n-1)+\kappa-2n\kappa(2n+1)}{2n-1} = 0. \tag{3.6}$$

Integrating (3.6) and using divergence theorem, we conclude

$$\int_M \frac{2(n-1)+\kappa-2n\kappa(2n+1)}{2n-1} dM = 0, \tag{3.7}$$

where dM stands for M's volume. Since $\kappa < 1$ then the equation(3.7) does not hold. Hence the proof. □

Theorem 3.2. *If an $N(\kappa)$-contact metric manifold admits an almost Schouten soliton whose soliton vector field is pointwise colinear with ζ, then it becomes a Schouten soliton and the soliton is shrinking whenever $n > 1$.*

Proof. If we set $V = f\zeta$, f is a smooth function, then the equation (3.2) implies

$$\begin{aligned}(Yf)\eta(Z) + (Zf)\eta(Y) &- 2fg(\Phi hY, Z) + \frac{2}{2n-1}S(Y,Z) \\ &+ \big[2b - \frac{r}{2n(2n-1)}\big]g(Y,Z) = 0.\end{aligned} \tag{3.8}$$

Putting $Y = \Phi Y$ and $Z = \Phi Z$ in the above expression and using (2.9) and (2.15), we get

$$\begin{aligned}-2fg(h\Phi Y, Z) &- \big[2b + \frac{8n(n-1)-r}{2n(2n-1)}\big]\big(\eta(Y)\eta(Z) - g(Y,Z)\big) \\ &- \frac{4(n-1)}{2n-1}g(hY,Z) = 0.\end{aligned} \tag{3.9}$$

Contracting equation (3.9) and with help of (2.8) and (2.17), we infer

$$b = \frac{\kappa}{2(2n-1)} - \frac{2n-2}{2n-1}, \tag{3.10}$$

which shows that b is constant. Also as $\kappa < 1$, the value of b is negative only when $n > 1$ which implies that the soliton is shrinking whenever $n > 1$. □

Theorem 3.3. *An $N(\kappa)$-contact metric manifold with a gradient Schouten soliton is either locally isomorphic to a Lie group G_q equipped with the almost contact metric structure, where $q = \sqrt{-\kappa}$ or, the manifold does not admit a gradient Schouten soliton.*

Proof. From the equation (1.3), we have

$$QY + (2n-1)\nabla_Y Df = \big[\frac{r}{2} + (2n-1)b\big]Y. \tag{3.11}$$

Differentiating (3.11) with respect to Z, we get

$$\begin{aligned}&\nabla_Z(QY) + (2n-1)\nabla_Z\nabla_Y Df\\ &= (2n-1)(Zb)Y + \big[\tfrac{r}{2} + (2n-1)b\big]\nabla_Z Y.\end{aligned} \tag{3.12}$$

Interchanging Y and Z, we have

$$\begin{aligned}&\nabla_Y(QZ) + (2n-1)\nabla_Y\nabla_Z Df\\ &= (2n-1)(Yb)Z + \big[\tfrac{r}{2} + (2n-1)\big]\nabla_Y Z.\end{aligned} \tag{3.13}$$

Again from (3.11), we infer

$$Q[Y,Z] + (2n-1)\nabla_{[Y,Z]}Df = \big[\frac{r}{2} + (2n-1)b\big][Y,Z]. \tag{3.14}$$

Using the above three equations, we find

$$R(Y,Z)Df = \frac{1}{2n-1}\big[(\nabla_Z Q)Y - (\nabla_Y Q)Z + \frac{1}{2}\{\ (Yr)Z - (Zr)Y\}\big]. \tag{3.15}$$

From (2.15), we have

$$QY = 2(n-1)(Y + hY) + [2n\kappa - 2(n-1)]\eta(Y)\zeta. \tag{3.16}$$

Differentiating (3.16) with respect to Z, one obtains

$$\begin{aligned}(\nabla_Z Q)Y &= -[2n\kappa - 2(n-1)]\big(g(Y,\Phi Z)\zeta + g(Y,\Phi hZ)\zeta\\ &+ \eta(Y)(\phi Z + \Phi hZ)\big).\end{aligned} \tag{3.17}$$

Putting the above values in (3.15) and using (2.17), we infer

$$\begin{aligned}R(Y,Z)Df &= \frac{1}{2n-1}\big[4\kappa g(\Phi Y,Z)\zeta + 2n\kappa(\eta(Z)\Phi hY - \eta(Y)\Phi hZ)\\ &- 2\kappa(\eta(Z)\Phi Y - \eta(Y)\Phi Z)\big].\end{aligned} \tag{3.18}$$

After contracting Y above euqation implies

$$S(Z,Df) = 0. \tag{3.19}$$

Substituting Df instead of Y in (2.15) and using the above equation, we infer

$$2(n-1)[g(Z,Df)+g(Z,hDf)]+\{2n\kappa-2(n-1)\}\eta(Df)\eta(Z)=0. \tag{3.20}$$

Replacing Z by ζ in the foregoing equation, we get

$$2n\kappa(\zeta f)=0. \tag{3.21}$$

Therefore, either $\kappa=0$ or $\zeta f=0$. When $\zeta f=0$, (3.20) implies

$$g(Z,Df)=0, \tag{3.22}$$

which infers $(Zf)=0$ for any $Z\in\chi(M)$, that is f is a constant. □

Theorem 3.4. *If an $N(\kappa)$-contact metric manifold admits an almost Schouten soliton whose soliton vector field is pointwise colinear with ζ, then the scalar curvature $r=-4n(2n+1)b$.*

Proof. We set $V=f\zeta$ in (3.2), we have

$$\begin{aligned}(Yf)\eta(Z) &+ (Zf)\eta(Y)-2fg(\Phi hY,Z)+\frac{2}{2n-1}S(Y,Z)\\ &+ \big[2b-\frac{r}{2n(2n-1)}\big]g(Y,Z)=0.\end{aligned} \tag{3.23}$$

Replacing Z, Y by ΦZ, ζ respectively in (3.23), we get

$$g(Df,\Phi Z)=0. \tag{3.24}$$

Putting ΦZ instead of Z in the foregoing equation, we obtain

$$Df=(\zeta f)\zeta. \tag{3.25}$$

Differentiating $V=f\zeta$ and using (2.9), we have

$$g(\nabla_Y V,Z)=\eta(Y)\eta(Z)(\zeta f)-f[g(\Phi Y,Z)+g(\Phi hY,Z)]. \tag{3.26}$$

Contracting the above equation, we infer

$$divV=(\zeta f). \tag{3.27}$$

Integrating (3.27), we have

$$\int_M(\zeta f)dM=\int divV=0, \tag{3.28}$$

where dM indicates M's volume. Hence we have

$$(\zeta f)=0. \tag{3.29}$$

Thus the previous equation (3.25) implies $Df = 0$ which concludes that f is a constant. Now (3.23) gives

$$-2fg(\Phi hY,Z)+\frac{2}{2n-1}S(Y,Z)+\big[2b-\frac{r}{2n(2n-1)}\big]g(Y,Z)=0. \tag{3.30}$$

After contraction, one infers

$$r=-4n(2n+1)b. \tag{3.31}$$

Hence the proof. $\square$

4. Gradient Schouten Solitons in three-Dimensional $N(\kappa)$-Contact Metric Manifolds

A contact metric manifold for which ζ is Killing, is called K-contact manifold from the concept of Blair, Koufogiorgos and Sharma[3]. A 3-dimensional contact metric manifold is a sasakian manifold if

$$R(Y,Z)\zeta=\eta(Z)Y-\eta(Y)Z. \tag{4.1}$$

It can be easily stated that a 3-dimensional contact metric manifold is Sasakian if and only if $h = 0$ where h is the $(1,1)$-type tensor according to Blair and Sharma[4]. Several authors have studied about 3-dimensional $N(\kappa)$-contact metric Manifolds such as [3, 8, 9, 12, 13].

In a 3-dimensional Riemannian manifold the curvature tensor R is given by

$$\begin{aligned} R(Y,Z)W &= S(Z,W)Y-S(Y,W)Z+g(Z,W)QY-g(Y,W)QZ\\ &- \frac{r}{2}[g(Z,W)Y-g(Y,W)Z], \end{aligned} \tag{4.2}$$

for any vector fields Y, Z, $W\in\chi(M)$. Then it is proven by the authors that in a 3-dimensional $N(\kappa)$-contact metric manifold M^{2n+1} the following relations hold:

$$QY=\big(\frac{r}{2}-\kappa\big)Y+\big(3\kappa-\frac{r}{2}\big)\eta(Y)\zeta, \tag{4.3}$$

$$\begin{aligned} R(Y,Z)W &= \big(\frac{r}{2}-2\kappa\big)\big[g(Z,W)Y-g(Y,W)Z\big]\\ &+ \big(3\kappa-\frac{r}{2}\big)\big[g(Z,W)\eta(Y)\zeta-g(Y,W)\eta(Z)\zeta\\ &+ \eta(Z)\eta(W)Y-\eta(Y)\eta(W)Z\big], \end{aligned} \tag{4.4}$$

where Q is the Ricci operator.

Theorem 4.1. *A 3-dimensional $N(\kappa)$-contact metric manifold admitting a gradient Schouten soliton is either flat or the scalar curvature is constant.*

Proof. From (1.3), we have

$$QY + (2n-1)\nabla_Y Df = \big[\frac{r}{2} - (2n-1)b\big]Y, \tag{4.5}$$

which implies the following

$$R(Y,Z)Df = (\nabla_Z Q)Y - (\nabla_Y Q)Z + \frac{1}{2}\big[(Yr)Z - (Zr)Y\big]. \tag{4.6}$$

As the scalar curvature r is constant, from (2.17) and (4.6), we conclude

$$R(Y,Z)Df = (\nabla_Z Q)Y - (\nabla_Y Q)Z. \tag{4.7}$$

For 3-dimensional $N(\kappa)$-contact metric manifold,

$$QY = \big(\frac{r}{2} - \kappa\big)Y + \big(3\kappa - \frac{r}{2}\big)\eta(Y)\zeta. \tag{4.8}$$

Differentiating the above equation, we infer

$$\begin{aligned}(\nabla_Z Q)Y &= -\big(3\kappa - \frac{r}{2}\big)\big[g(Y,\Phi Z)\zeta + g(Y,\Phi hZ)\zeta + \eta(Y)\Phi Z + \eta(Y)\Phi hZ\big]\\ &+ \frac{(Yr)}{2}\big[Y - \eta(Y\zeta)\big].\end{aligned} \tag{4.9}$$

After using this valuein (4.7), we see

$$\begin{aligned}(2n-1)R(Y,Z)Df &= \big(3\kappa - \frac{r}{2}\big)\big[2g(Z,\Phi Y)\zeta + \eta(Y)\nabla_Z\zeta - \eta(Z)\nabla_Y\zeta\big]\\ &+ \frac{1}{2}\big[\eta(Z)(Yr) - \eta(Y)(Zr)\big].\end{aligned} \tag{4.10}$$

Taking inner product and using (2.13), we infer

$$\begin{aligned}-\kappa\big[\eta(Z)(Yf) \quad -\eta(Y)(Zf)\big] &= \big(r - 6\kappa\big)g(\Phi Z, Y)\\ &+\tfrac{1}{2}\big[\eta(Z)(Yr) - \eta(Y)(Zr)\big].\end{aligned} \tag{4.11}$$

Replacing Y, Z by ΦY, ΦZ, respectively, in (4.11), we get

$$r = 6\kappa, \tag{4.12}$$

which shows r is constant. Putting this value of r in (4.11), we have

$$\kappa\big[\eta(Y)(Zf) - \eta(Z)(Yf)\big] = 0, \tag{4.13}$$

which implies either $\kappa = 0$ or $(Zf) = (\zeta f)\eta(Z)$.

If $\kappa \neq 0$, then we have

$$df = (\zeta f)\eta. \tag{4.14}$$

Considering exterior derivative of the forgoing equation and then taking wedge product with η, one obtains $(\zeta f) = 0$. Applying this data in the last equation, we see that $(Zf) = 0$, that is f is a constant.

If $\kappa = 0$, then we infer,

$$R(Y, Z)\zeta = 0. \tag{4.15}$$

Therefore, the manifold is flat.

Hence the proof. □

5. Example

Let us assume the 3-dimensional manifold $M = \{x, y, z \in \mathbb{R}^3 : z \neq 0\}$ where (x, y, z) being the standard co-ordinates in $\mathbb{R}^3$, whose basis vector fields are choosen such that they satisfy the following:

$$[\epsilon_1, \epsilon_2] = 3\epsilon_3, \quad [\epsilon_1, \epsilon_3] = \epsilon_2, \quad [\epsilon_2, \epsilon_3] = 2\epsilon_1.$$

Let the metric tensor g be defined by

$$g(\epsilon_1, \epsilon_1) = g(\epsilon_2, \epsilon_2) = g(\epsilon_3, \epsilon_3) = 1,$$

$$g(\epsilon_1, \epsilon_2) = g(\epsilon_2, \epsilon_3) = g(\epsilon_3, \epsilon_1) = 0.$$

The 1-form η and the $(1, 1)$ tensor field ϕ are, respectively, defined by $\eta(X) = g(X, \epsilon_1)$ for every vector field X on the manifold and

$$\phi\epsilon_1 = 0, \quad \phi\epsilon_2 = \epsilon_3, \quad \phi\epsilon_3 = -\epsilon_2.$$

Then we find that

$$\eta(\epsilon_1) = 1, \quad \phi^2 X = -X + \eta(X)\epsilon_1,$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for every vector fields X, Y on the manifold. Thus $(\phi, \epsilon_1, \eta, g)$ defines an almost contact structure.

By Koszul's formula, one can find

$$\nabla_{\epsilon_1}\epsilon_1 = 0, \quad \nabla_{\epsilon_2}\epsilon_2 = 0, \quad \nabla_{\epsilon_3}\epsilon_3 = 0,$$

$$\nabla_{\epsilon_1}\epsilon_2 = 0, \quad \nabla_{\epsilon_2}\epsilon_1 = -3\epsilon_3, \quad \nabla_{\epsilon_1}\epsilon_3 = 0,$$

$$\nabla_{\epsilon_2}\epsilon_3 = 3\epsilon_1, \quad \nabla_{\epsilon_3}\epsilon_1 = -\epsilon_2, \quad \nabla_{\epsilon_3}\epsilon_2 = \epsilon_1.$$

From the above expressions, we see that $h\epsilon_1 = 0$, $h\epsilon_2 = 2\epsilon_2$ and $h\epsilon_3 = -2\epsilon_3$.

The components of the curvature tensor are given by

$$R(\epsilon_1, \epsilon_2)\epsilon_2 = -3\epsilon_1, \quad R(\epsilon_2, \epsilon_1)\epsilon_1 = -3\epsilon_2, \quad R(\epsilon_1, \epsilon_3)\epsilon_3 = -3\epsilon_1,$$

$$R(\epsilon_2, \epsilon_3)\epsilon_3 = 3\epsilon_2, \quad R(\epsilon_3, \epsilon_1)\epsilon_1 = -3\epsilon_3, \quad R(\epsilon_3, \epsilon_2)\epsilon_2 = 3\epsilon_3,$$

$$R(\epsilon_1, \epsilon_2)\epsilon_3 = 0, \quad R(\epsilon_2, \epsilon_3)\epsilon_1 = 0, \quad R(\epsilon_1, \epsilon_3)\epsilon_2 = 0.$$

Thus the given manifold is an $N(\kappa)$-contact metric manifold with $\kappa = -3$. The non-zero components of the Ricci tensor are

$$\mathcal{S}(\epsilon_1, \epsilon_1) = -6, \quad \mathcal{S}(\epsilon_2, \epsilon_2) = 0, \quad \mathcal{S}(\epsilon_3, \epsilon_3) = 0.$$

Let r be the scalar curvature, then from the above

$$r = \mathcal{S}(\epsilon_1, \epsilon_1) + \mathcal{S}(\epsilon_2, \epsilon_2) + \mathcal{S}(\epsilon_3, \epsilon_3) = -6.$$

Suppose that $f = 2x$ and $b = 3$, then equation (4.5) satisfied. Therefore the $N(-3)$-contact metric manidold defines a gradient Schouten soliton and hence Theorem 3.3 is satisfied.

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