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COEFFICIENT ESTIMATE OF BI-BAZILEVIČ FUNCTIONS OF SAKAGUCHI TYPE BASED ON SRIVASTAVA-ATTIYA OPERATOR

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Abstract. In this paper, we introduce and investigate a new subclass of the function class Σ of bi-univalent functions defined in the open unit disk, which are associated with the Hurwitz-Lerch zeta function, satisfying subordinate conditions. Furthermore, we find estimates on the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in this new subclass. Several (known or new) consequences of the results are also pointed out.

1. Introduction, Definitions and Preliminaries

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk

$$\triangle = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\triangle$. Some of the important and well-investigated subclasses of the univalent function class $\mathcal{S}$ include (for example) the class $\mathcal{S}^*(\alpha)$ of starlike functions of order α in $\triangle$ and the class $\mathcal{K}(\alpha)$ of convex functions of order α in $\triangle$.

The convolution or Hadamard product of two functions $f, h \in \mathcal{A}$ is denoted by $f * h$ and is defined as

$$(f * h)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \tag{1.2}$$

where $f(z)$ is given by (1.1) and $h(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n$.

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We recall a general Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ defined by (see [27])

$$\Phi(z,s,a) := \sum_{k=0}^{\infty} \frac{z^k}{(k+a)^s} \quad (a \in \mathbb{C} \setminus \{0,-1,-2,\ldots\}; s \in \mathbb{C}, \Re(s) > 1 \text{ and } |z| < 1). \tag{1.3}$$

Several interesting properties and characteristics of the Hurwitz-Lerch zeta function $\Phi(z,s,a)$ can be found in the recent investigations by Choi and Srivastava [6],Mustafa et al[17], Lin and Srivastava [14], Lin et al [15] and references stated therein .

For the class $\mathcal{A}$, Srivastava and Attiya [26] (see also Raducanu and Srivastava [21] and Prajapat and Goyal [20]) introduced and investigated the following linear operator:

$$\mathcal{J}_\mu^b : \mathcal{A} \longrightarrow \mathcal{A}$$

defined in terms of the Hadamard product (or convolution) by

$$\mathcal{J}_\mu^b f(z) = (\mathcal{G}_\mu^b * f)(z) \quad (z \in \triangle;\ b \in \mathbb{C} \setminus \{0,-1,-2,\ldots\};\ \mu \in \mathbb{C};\ f \in \mathcal{A}), \tag{1.4}$$

where, for convenience.

$$\mathcal{G}_\mu^b(z) = (1+b)^\mu [\Phi(z,\mu,b) - b^{-\mu}]. \tag{1.5}$$

It is easy to observe from (given earlier by [20], [21]) (1.1), (1.4) and (1.5) that

$$\mathcal{J}_\mu^b f(z) = z + \sum_{k=2}^{\infty} \Theta_k a_k z^k, \tag{1.6}$$

where

$$\Theta_k = \left| \left(\frac{1+b}{k+b} \right)^\mu \right| \tag{1.7}$$

and (throughout this paper unless otherwise mentioned) the parameters μ, b are considered as $\mu \in \mathbb{C}$ and $b \in \mathbb{C} \setminus \{0,-1,-2,\ldots\}$.

We note that

- For $\mu = 1$ and $b = \nu$ $(\nu > -1)$, generalized Libera-Bernardi integral operator [22]

$$\begin{aligned} \mathcal{J}_1^\nu f(z) &= \frac{1+\nu}{z^\nu} \int_0^z t^{\nu-1} f(t)\,dt \\ &= z + \sum_{k=2}^{\infty} \left(\frac{\nu+1}{k+\nu} \right) a_k z^k = \mathcal{L}_\nu f(z). \end{aligned}$$

- For $\mu = \sigma\,(\sigma > 0)$ and $b = 1$, Jung-Kim-Srivastava integral operator [11]

$$\begin{aligned}\mathcal{J}_\sigma^1 f(z) &= \frac{2^\sigma}{z\Gamma(\sigma)}\int_0^z \left(\log\left(\frac{z}{t}\right)\right)^{\sigma-1} f(t)dt \\ &= z + \sum_{k=2}^{\infty}\left(\frac{z}{k+1}\right)^{\sigma} a_k z^k = \mathcal{I}_\sigma f(z)\end{aligned}$$

closely related to some multiplier transformations studied by Flett [8].

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \qquad (z \in \triangle)$$

and

$$f(f^{-1}(w)) = w \qquad \left(|w| < r_0(f);\ r_0(f) \geqq \frac{1}{4}\right),$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \tag{1.8}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\triangle$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\triangle$. Let Σ denote the class of bi-univalent functions in $\triangle$ given by (1.1).

An analytic function f is subordinate to an analytic function g, written $f(z) \prec g(z)$, provided there is an analytic function w defined on $\triangle$ with $w(0) = 0$ and $|w(z)| < 1$ satisfying $f(z) = g(w(z))$. Ma and Minda [16] unified various subclasses of starlike and convex functions for which either of the quantity $\frac{z\,f'(z)}{f(z)}$ or $1 + \frac{z\,f''(z)}{f'(z)}$ is subordinate to a more general superordinate function. For this purpose, they considered an analytic function ϕ with positive real part in the unit disk $\triangle$, $\phi(0) = 1$, $\phi'(0) > 0$, and ϕ maps $\triangle$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. The class of Ma-Minda starlike functions consists of functions $f \in \mathcal{A}$ satisfying the subordination $\frac{z\,f'(z)}{f(z)} \prec \phi(z)$. Similarly, the class of Ma-Minda convex functions of functions $f \in \mathcal{A}$ satisfying the subordination $1 + \frac{z\,f''(z)}{f'(z)} \prec \phi(z)$.

A function f is bi-starlike of Ma-Minda type or bi-convex of Ma-Minda type if both f and f^{-1} are respectively Ma-Minda starlike or convex. These classes are denoted respectively by $\mathcal{S}_\Sigma^*(\phi)$ and $\mathcal{K}_\Sigma(\phi)$. In the sequel, it is assumed that ϕ is an analytic function with positive real part in the unit disk $\triangle$, satisfying

$$\phi(0) = 1 \qquad \text{and} \qquad \phi'(0) > 0,$$

and $\phi(\triangle)$ is symmetric with respect to the real axis. Such a function has a series expansion of the form

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots, \quad (B_1 > 0). \tag{1.9}$$

Recently there has been triggering interest to study bi-univalent function class Σ and obtained non-sharp coefficient estimates on the first two coefficients $|a_2|$ and $|a_3|$ of (1.1). But the coefficient problem for each of the following Taylor-Maclaurin coefficients:

$$|a_n| \qquad (n \in \mathbb{N} \setminus \{1,2,3\}; \;\; \mathbb{N} := \{1,2,3,\cdots\}$$

is still an open problem (see [3, 4, 5, 12, 18, 29]). Many researchers (see [2, 9, 13, 28, 30, 31]) have recently introduced and investigated several interesting subclasses of the bi-univalent function class Σ and they have found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$.

Several authors have discussed various subfamilies of Bazilevič functions of type λ from various perspective. They discussed it from the perspective of convexity, inclusion theorem, radii of starlikeness and convexity boundary rotational problem, subordination just to mention few. The most amazing thing is that, it is difficult to see any of these authors discussing the coefficient inequalities, and coefficient bounds of these subfamilies of Bazilevič function most especially when the parameter λ is greater than 1 ($\lambda \in \mathbb{R}$). Motivated by the earlier work of Sakaguchi [23] on the class of starlike functions with respect to symmetric points denoted by $\mathcal{S}_s$ consisting of functions $f \in \mathcal{A}$ satisfy the condition $\mathrm{Re}\left(\frac{zf'(z)}{f(z)-f(-z)}\right) > 0,\ (z \in \mathbb{U})$and using the techniques of Deniz [7] in the present paper, we introduce new families of Sakaguchi-type Bazilevič functions of complex order [10] of the function class Σ, involving Hurwitz-Lerch zeta function, and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the new subclasses of function class Σ. Several related classes are also considered, and connection to earlier known results are made.

Definition 1.1. A function $f \in \Sigma$ given by (1.1) is said to be in the class $\mathcal{S}^{\mu,b}_{\Sigma,t}(\gamma,\lambda,\phi)$ if the following conditions are satisfied:

$$1 + \frac{1}{\gamma}\left(\frac{[(1-t)z]^{1-\lambda}(\mathcal{J}^b_\mu f(z))'}{[\mathcal{J}^b_\mu f(z) - \mathcal{J}^b_\mu f(tz)]^{1-\lambda}} - 1\right) \prec \phi(z) \tag{1.10}$$

and

$$1 + \frac{1}{\gamma}\left(\frac{[(1-t)w]^{1-\lambda}(\mathcal{J}^b_\mu g(w))'}{[\mathcal{J}^b_\mu g(w) - \mathcal{J}^b_\mu g(tw)]^{1-\lambda}} - 1\right) \prec \phi(w), \tag{1.11}$$

where $|t| \leq 1\ (t \neq 1)$, $\gamma \in \mathbb{C}\setminus\{0\}; \lambda \geq 0; z, w \in \triangle$ and the function g is given by (1.8).

For the sake of brevity throughout this paper we let

$$|t| \leq 1, \quad (t \neq 1)$$

and

$$\gamma \in \mathbb{C}\setminus\{0\}; \lambda \geq 0; z, w \in \triangle$$

unless otherwise stated.

Example 1.1. If we set $\phi(z) = \frac{1+Az}{1+Bz}$, $(-1 \leq B < A \leq 1)$, then the class $\mathcal{S}_{\Sigma,t}^{\mu,b}(\gamma,\lambda,\phi) \equiv \mathcal{S}_{\Sigma,t}^{\mu,b}(\gamma,\lambda,A,B)$ which is defined as $f \in \Sigma$,

$$1 + \frac{1}{\gamma}\left(\frac{[(1-t)z]^{1-\lambda}(\mathcal{J}_\mu^b f(z))'}{[\mathcal{J}_\mu^b f(z) - \mathcal{J}_\mu^b f(tz)]^{1-\lambda}} - 1\right) \prec \frac{1+Az}{1+Bz}$$

and

$$1 + \frac{1}{\gamma}\left(\frac{[(1-t)w]^{1-\lambda}(\mathcal{J}_\mu^b g(w))'}{[\mathcal{J}_\mu^b g(w) - \mathcal{J}_\mu^b g(tw)]^{1-\lambda}} - 1\right) \prec \frac{1+Aw}{1+Bw}.$$

Example 1.2. If we set $\phi(z) = \frac{1+(1-2\alpha)z}{1-z}$,$(\, 0 \leq \alpha < 1)$ then the class $\mathcal{S}_{\Sigma,t}^{\mu,b}(\gamma,\lambda,\phi) \equiv \mathcal{S}_{\Sigma,t}^{\mu,b}(\gamma,\lambda,\alpha)$ which is defined as $f \in \Sigma$,

$$\mathrm{Re}\left[1 + \frac{1}{\gamma}\left(\frac{[(1-t)z]^{1-\lambda}(\mathcal{J}_\mu^b f(z))'}{[\mathcal{J}_\mu^b f(z) - \mathcal{J}_\mu^b f(tz)]^{1-\lambda}} - 1\right)\right] > \alpha$$

and

$$\mathrm{Re}\left[1 + \frac{1}{\gamma}\left(\frac{[(1-t)w]^{1-\lambda}(\mathcal{J}_\mu^b g(w))'}{[\mathcal{J}_\mu^b g(w) - \mathcal{J}_\mu^b g(tw)]^{1-\lambda}} - 1\right)\right] > \alpha.$$

On specializing the parameters λ one can state the various new subclasses of Σ as illustrated in the following examples.

Example 1.3. For $\lambda = 0$ and a function $f \in \Sigma$, given by (1.1), is said to be in the class $\mathcal{S}_{\Sigma,t}^{\mu,b}(\gamma,\phi)$ if the following conditions are satisfied:

$$1 + \frac{1}{\gamma}\left(\frac{[(1-t)z](\mathcal{J}_\mu^b f(z))'}{\mathcal{J}_\mu^b f(z) - \mathcal{J}_\mu^b f(tz)}\right) \prec \phi(z) \tag{1.12}$$

and

$$1 + \frac{1}{\gamma}\left(\frac{[(1-t)w](\mathcal{J}_\mu^b g(w))'}{\mathcal{J}_\mu^b g(w) - \mathcal{J}_\mu^b g(tw)}\right) \prec \phi(w) \tag{1.13}$$

where $\gamma \in \mathbb{C} \setminus \{0\}; z, w \in \triangle$ and the function g is given by(1.8).

Example 1.4. For $\lambda = 1$ and a function $f \in \Sigma$, given by (1.1) is said to be in the class $\mathcal{H}_{\Sigma}^{\mu,b}(\gamma,\phi)$ if the following conditions are satisfied:

$$1 + \frac{1}{\gamma}\left((\mathcal{J}_\mu^b f(z))' - 1\right) \prec \phi(z) \tag{1.14}$$

and

$$1 + \frac{1}{\gamma}\left((\mathcal{J}_\mu^b g(w))' - 1\right) \prec \phi(w), \tag{1.15}$$

where $\gamma \in \mathbb{C} \setminus \{0\}; z, w \in \triangle$ and the function g is given by (1.8).

It is of interest to note that for $\gamma = 1$ the class $\mathcal{S}_{\Sigma,t}^{\mu,b}(\gamma,\lambda,\phi)$ reduces to the following new subclass $\mathcal{B}_{\Sigma,t}^{\mu,b}(\lambda,\phi)$.

Definition 1.2. A function $f \in \Sigma$ given by (1.1) is said to be in the class $\mathcal{B}_{\Sigma,t}^{\mu,b}(\lambda,\phi)$ if the following conditions are satisfied:

$$\left(\frac{[(1-t)z]^{1-\lambda}(\mathcal{J}_\mu^b f(z))'}{[\mathcal{J}_\mu^b f(z) - \mathcal{J}_\mu^b f(tz)]^{1-\lambda}}\right) \prec \phi(z) \tag{1.16}$$

and

$$\left(\frac{[(1-t)w]^{1-\lambda}(\mathcal{J}_\mu^b g(w))'}{[\mathcal{J}_\mu^b g(w) - \mathcal{J}_\mu^b g(tw)]^{1-\lambda}}\right) \prec \phi(w), \tag{1.17}$$

where $\lambda \geq 0$; $z, w \in \triangle$ and the function g is given by (1.8).

For $\gamma = 1$ and particular values of λ, we have the following subclasses as illustrated below.

Example 1.5. For $\lambda = 0$ and a function $f \in \Sigma$, given by (1.1) is said to be in the class $\mathcal{B}_{\Sigma,t}^{\mu,b}(0,\phi) \equiv \mathcal{S}_{\Sigma,t}^{*,\mu,b}(\phi)$ if the following conditions are satisfied:

$$\left(\frac{[(1-t)z](\mathcal{J}_\mu^b f(z))'}{\mathcal{J}_\mu^b f(z) - \mathcal{J}_\mu^b f(tz)}\right) \prec \phi(z) \tag{1.18}$$

and

$$\left(\frac{[(1-t)w](\mathcal{J}_\mu^b g(w))'}{\mathcal{J}_\mu^b g(w) - \mathcal{J}_\mu^b g(tw)}\right) \prec \phi(w), \tag{1.19}$$

where $z, w \in \triangle$ and the function g is given by (1.8).

Example 1.6. For $\lambda = 1$ and a function $f \in \Sigma$, given by (1.1) is said to be in the class $\mathcal{B}_{\Sigma,t}^{\mu,b}(1,\phi) \equiv \mathcal{H}_{\Sigma}^{\mu,b}(\phi)$ if the following conditions are satisfied:

$$(\mathcal{J}_\mu^b f(z))' \prec \phi(z) \tag{1.20}$$

and

$$(\mathcal{J}_\mu^b g(w))' \prec \phi(w) \tag{1.21}$$

where $z, w \in \triangle$ and the function g is given by(1.8).

In the following section we find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the above-defined subclasses $\mathcal{S}_{\Sigma,t}^{\mu,b}(\gamma,\lambda,\phi)$ of the function class Σ.

In order to derive our main results, we shall need the following lemma:

Lemma 1.1. (see [19]). *If $p \in \mathcal{P}$, then $|p_k| \leq 2$ for each k, where $\mathcal{P}$ is the family of all functions p analytic in $\triangle$ for which $\Re\,(p(z)) > 0$, where $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ for $z \in \triangle$.*

2. Coefficient Bounds for the Function Class $\mathcal{S}_{\Sigma,t}^{\mu,b}(\gamma,\lambda,\phi)$

We begin by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $\mathcal{S}_{\Sigma,t}^{\mu,b}(\gamma,\lambda,\phi)$.

Theorem 2.1. *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{S}_{\Sigma,t}^{\mu,b}(\gamma,\lambda,\phi)$. Then*

$$|a_2| \leqq \frac{|\gamma|B_1\sqrt{2B_1}}{\sqrt{|\{\gamma B_1^2\Lambda(\lambda,t)[(\lambda-2)(1+t)+4]-2(B_2-B_1)[\Lambda(\lambda,t)+2]^2\}\Theta_2^2+2\gamma B_1^2\{(\lambda-1)(1+t+t^2)+3\}\Theta_3|}} \tag{2.1}$$

and

$$|a_3| \leq \frac{|\gamma B_1|}{[(\lambda-1)(1+t+t^2)+3]\Theta_3} + \left(\frac{|\gamma B_1|}{[\Lambda(\lambda,t)+2]\Theta_2}\right)^2. \tag{2.2}$$

were $\Lambda(\lambda,t)=(\lambda-1)(1+t)$.

Proof. Let $f\in\mathcal{S}_{\Sigma,t}^{\mu,b}(\gamma,\lambda,\phi)$ and $g=f^{-1}$. Then there are analytic functions $u,v:\triangle\longrightarrow\triangle$ with $u(0)=0=v(0)$, satisfying

$$1+\frac{1}{\gamma}\left(\frac{[(1-t)z]^{1-\lambda}(\mathcal{J}_\mu^b f(z))'}{[\mathcal{J}_\mu^b f(z)-\mathcal{J}_\mu^b f(tz)]^{1-\lambda}}-1\right)=\phi(u(z)) \tag{2.3}$$

and

$$1+\frac{1}{\gamma}\left(\frac{[(1-t)w]^{1-\lambda}(\mathcal{J}_\mu^b g(w))'}{[\mathcal{J}_\mu^b g(w)-\mathcal{J}_\mu^b g(tw)]^{1-\lambda}}-1\right)=\phi(v(w)). \tag{2.4}$$

Define the functions $p(z)$ and $q(z)$ by

$$p(z):=\frac{1+u(z)}{1-u(z)}=1+p_1z+p_2z^2+\cdots$$

and

$$q(z):=\frac{1+v(z)}{1-v(z)}=1+q_1z+q_2z^2+\cdots$$

or, equivalently,

$$u(z):=\frac{p(z)-1}{p(z)+1}=\frac{1}{2}\left[p_1z+\left(p_2-\frac{p_1^2}{2}\right)z^2+\cdots\right] \tag{2.5}$$

and

$$v(z):=\frac{q(z)-1}{q(z)+1}=\frac{1}{2}\left[q_1z+\left(q_2-\frac{q_1^2}{2}\right)z^2+\cdots\right]. \tag{2.6}$$

Then $p(z)$ and $q(z)$ are analytic in $\triangle$ with $p(0) = 1 = q(0)$. Since $u, v : \triangle \to \triangle$, the functions $p(z)$ and $q(z)$ have a positive real part in $\triangle$, and $|p_i| \leq 2$ and $|q_i| \leq 2$. Using (2.5) and (2.6) in (2.3) and (2.4) respectively, we have

$$1 + \frac{1}{\gamma}\left(\frac{[(1-t)z]^{1-\lambda}(\mathcal{J}_\mu^b f(z))'}{[\mathcal{J}_\mu^b f(z) - \mathcal{J}_\mu^b f(tz)]^{1-\lambda}} - 1\right) = \varphi\left(\frac{1}{2}\left[p_1 z + \left(p_2 - \frac{p_1^2}{2}\right)z^2 + \cdots\right]\right) \tag{2.7}$$

and

$$1 + \frac{1}{\gamma}\left(\frac{[(1-t)w]^{1-\lambda}(\mathcal{J}_\mu^b g(w))'}{[\mathcal{J}_\mu^b g(w) - \mathcal{J}_\mu^b g(tw)]^{1-\lambda}} - 1\right) = \varphi\left(\frac{1}{2}\left[q_1 w + \left(q_2 - \frac{q_1^2}{2}\right)w^2 + \cdots\right]\right). \tag{2.8}$$

In light of (1.1), (1.8), (1.9), from (2.7) and (2.8), it is evident that

$$\begin{aligned} 1 + \tfrac{1}{\gamma}\{(\lambda-1)(1+t)+2\}\Theta_2 a_2 z + \tfrac{1}{\gamma}\{[(\lambda-1)(1+t+t^2)+3]\Theta_3 a_3 \\ + \tfrac{1}{2}(\lambda-1)(1+t)[(\lambda-2)(1+t)+4]\Theta_2^2 a_2^2\}z^2 + \cdots \\ = 1 + \tfrac{1}{2}B_1 p_1 z + [\tfrac{1}{2}B_1(p_2 - \tfrac{p_1^2}{2}) + \tfrac{1}{4}B_2 p_1^2]z^2 + \cdots \end{aligned}$$

and

$$\begin{aligned} 1 - \tfrac{1}{\gamma}\{(\lambda-1)(1+t)+2\}\Theta_2 a_2 w + \tfrac{1}{\gamma}\{\tfrac{1}{2}(\lambda-1)(1+t)[(\lambda-2)(1+t)+4]\Theta_2^2 a_2^2 \\ + [(\lambda-1)(1+t+t^2)+3](2a_2^2 - a_3)\Theta_3\}w^2 + \cdots \\ = 1 + \tfrac{1}{2}B_1 q_1 w + [\tfrac{1}{2}B_1(q_2 - \tfrac{q_1^2}{2}) + \tfrac{1}{4}B_2 q_1^2]w^2 + \cdots \end{aligned}$$

which yield the following relations :

$$\{(\lambda-1)(1+t)+2\}\Theta_2 a_2 = \frac{\gamma}{2}B_1 p_1 \tag{2.9}$$

$$\left[\{(\lambda-1)(1+t+t^2)+3\}\Theta_3 a_3 + \frac{1}{2}(\lambda-1)(1+t)\{(\lambda-2)(1+t)+4\}\Theta_2^2 a_2^2\right] = \frac{\gamma}{2}B_1(p_2 - \frac{p_1^2}{2}) + \frac{\gamma}{4}B_2 p_1^2 \tag{2.10}$$

$$-\{(\lambda-1)(1+t)+2\}\Theta_2 a_2 = \frac{\gamma}{2}B_1 q_1 \tag{2.11}$$

and

$$\begin{aligned} &\left[\frac{1}{2}(\lambda-1)(1+t)\{(\lambda-2)(1+t)+4\}\Theta_2^2 a_2^2 + [(\lambda-1)(1+t+t^2)+3](2a_2^2 - a_3)\Theta_3\right] \\ &= \frac{\gamma}{2}B_1(q_2 - \frac{q_1^2}{2}) + \frac{\gamma}{4}B_2 q_1^2. \end{aligned} \tag{2.12}$$

From (2.9) and (2.11) , it follows that

$$p_1 = -q_1 \tag{2.13}$$

and

$$8[(\lambda-1)(1+t)+2]^2\Theta_2^2 a_2^2 = \gamma^2 B_1^2(p_1^2+q_1^2). \tag{2.14}$$

By taking $\Lambda(\lambda,t)=(\lambda-1)(1+t)$ for the sake of brevity, adding (2.10) and(2.12), we obtain

$$\begin{aligned}&\Big[\Lambda(\lambda,t)\{(\lambda-2)(1+t)+4\}\Theta_2^2+2\{(\lambda-1)(1+t+t^2)+3\}\Theta_3\Big]a_2^2\\ &\quad= \tfrac{\gamma B_1}{2}(p_2+q_2)+\tfrac{\gamma}{4}(B_2-B_1)(p_1^2+q_1^2).\end{aligned} \tag{2.15}$$

Using(2.14) in (2.15), we get
(2.16)

$$a_2^2=\frac{\gamma^2 B_1^3(p_2+q_2)}{2\gamma B_1^2\Big[\Lambda(\lambda,t)\{(\lambda-2)(1+t)+4\}\Theta_2^2+2\{(\lambda-1)(1+t+t^2)+3\}\Theta_3\Big]-4(B_2-B_1)[\Lambda(\lambda,t)+2]^2\Theta_2^2}.$$

Applying Lemma 1.1 to the coefficients p_2 and q_2, we immediately have

$$|a_2|^2\le\frac{2|\gamma|^2B_1^3}{|\gamma B_1^2\Big[\Lambda(\lambda,t)\{(\lambda-2)(1+t)+4\}\Theta_2^2+2\{(\lambda-1)(1+t+t^2)+3\}\Theta_3\Big]-2(B_2-B_1)[\Lambda(\lambda,t)+2]^2\Theta_2^2|}$$

which gives the bound on $|a_2|$ as asserted in (2.1).

Next, in order to find the bound on $|a_3|$, by subtracting (2.12) from (2.10), we get
(2.17)

$$2[(\lambda-1)(1+t+t^2)+3]\Theta_3 a_3-2[(\lambda-1)(1+t+t^2)+3]\Theta_3 a_2^2=\frac{\gamma B_1}{2}\left[(p_2-q_2)-\frac{1}{2}(p_1^2-q_1^2)\right]+\frac{\gamma B_2}{4}(p_1^2-q_1^2).$$

Using (2.13) and(2.14) in(2.17), we get

$$a_3=\frac{\gamma B_1(p_2-q_2)}{4[(\lambda-1)(1+t+t^2)+3]\Theta_3}+\frac{\gamma^2 B_1^2(p_1^2+q_1^2)}{8[(\lambda-1)(1+t)+2]^2\Theta_2^2}.$$

Applying Lemma 1.1 once again to the coefficients p_1, q_1, p_2 and q_2, we readily get (2.2). This completes the proof of Theorem 2.1.

□

Putting $\lambda=0$ in Theorem 2.1, we have the following corollary.

Corollary 2.1. *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{S}_{\Sigma,t}^{\mu,b}(\gamma,\phi)$. Then*

$$|a_2|\leqq\frac{|\gamma|B_1\sqrt{B_1}}{\sqrt{|\gamma B_1^2\Big[(2-t-t^2)\Theta_3-(1-t^2)\Theta_2^2\Big]-(B_2-B_1)(1-t)^2\Theta_2^2|}} \tag{2.18}$$

and

$$|a_3|\le\frac{|\gamma B_1|}{(2-t-t^2)\Theta_3}+\left(\frac{|\gamma B_1|}{(1-t)\Theta_2}\right)^2. \tag{2.19}$$

Putting $\lambda = 1$ in Theorem 2.1, we have the following corollary.

Corollary 2.2. *Let the function* $f(z)$ *given by* (1.1) *be in the class* $\mathcal{H}_{\Sigma}^{\mu,b}(\gamma,\phi)$. *Then*

$$|a_2| \leqq \frac{|\gamma|B_1\sqrt{B_1}}{\sqrt{|3\gamma B_1^2\Theta_3 - 4(B_2 - B_1)\Theta_2^2|}} \tag{2.20}$$

and

$$|a_3| \leqq \frac{|\gamma|B_1}{3\Theta_3} + \left(\frac{|\gamma|B_1}{2\Theta_2}\right)^2. \tag{2.21}$$

If $\mathcal{J}_{\mu}^{b}$ is the identity map, from Corollary 2.1 and 2.2, we get the following corollaries.

Corollary 2.3. *Let the function* $f(z)$ *given by* (1.1) *be in the class* $\mathcal{S}_{\Sigma,t}^{*}(\gamma,\phi)$. *Then*

$$|a_2| \leqq \frac{|\gamma|B_1\sqrt{B_1}}{\sqrt{|(1-t)[\gamma B_1^2 - (1-t)(B_2 - B_1)]|}} \tag{2.22}$$

and

$$|a_3| \leqq \frac{|\gamma|B_1}{(2-t-t^2)} + \left(\frac{|\gamma|B_1}{1-t}\right)^2. \tag{2.23}$$

Remark 2.1. If $\mathcal{J}_{\mu}^{b}$ is the identity map and $\gamma = 1$, in Corollary 2.4 our result coincides with the result given in [25].

Corollary 2.4. *Let the function* $f(z)$ *given by* (1.1) *be in the class* $\mathcal{H}_{\Sigma}(\gamma,\phi)$. *Then*

$$|a_2| \leqq \frac{|\gamma|B_1\sqrt{B_1}}{\sqrt{|3\gamma B_1^2 - 4(B_2 - B_1)|}} \tag{2.24}$$

and

$$|a_3| \leqq \frac{|\gamma|B_1}{3} + \left(\frac{|\gamma|B_1}{2}\right)^2. \tag{2.25}$$

Remark 2.2. If $\mathcal{J}_{\mu}^{b}$ is the identity map and $\gamma = 1, t = 0$ Theorem 2.1 reduces to Theorem 2.8 of Deniz [7].

Concluding Remarks:

(i) By setting $\phi(z) = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$, we have

$$B_1 = (A - B) \quad \text{and} \quad (B_2 - B_1) = (B - A)(B + 1)$$

(ii) By setting $\phi(z) = \frac{1+(1-2\beta)z}{1-z}$, $0 \leq \beta < 1$, we have

$$B_1 = B_2 = 2(1 - \beta).$$

Hence,we can deduce interesting results analogous to Theorem 2.1. Further, appropriately specializing the parameter μ and b various other interesting consequences of our general results involving Libera-Bernardi integral operator [22] andJung-Kim-Srivastava integral operator [11] (which are asserted by Theorems 2.1 and Corollaries above) can be derived easily. The details involved may be left as an exercise for the interested reader.

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