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A NOTE ON α-PARA KENMOTSU MANIFOLDS

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Abstract. The object of the present paper is to study 3-dimensional α-Para Kenmotsu manifolds. First we consider ϕ-projectively semi-symmetric 3-dimensional α-Para Kenmotsu manifolds. We also study projectively semi-symmetric and projectively pseudosymmetric 3-dimensional α-para Kenmotsu manifolds. Beside these 3-dimensional α-Para Kenmotsu manifolds satisfying $P.S = 0$ is also considered.
Keywords: α-Para Kenmotsu manifolds, curvature tensor, Euclidian space, Riemannian manifold, Ricci tensor.

1. Introduction

The projective curvature tensor is an important tensor from the differential geometric point of view. Let M be a $(2n + 1)$-dimensional Riemannian manifold. If there exists a one-to-one correspondence between each coordinate neighbourhood of M and a domain in Euclidian space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat. For $n \geq 1$, M is locally projectively flat if and only if the well known projective curvature tensor P vanishes. Here P is defined by [9]

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{2n}[S(Y, Z)X - S(X, Z)Y], \tag{1.1}$$

for all X, Y, $Z \in T(M)$, where R is the curvature tensor and S is the Ricci tensor. In fact M is projectively flat if and only if it is of constant curvature [16]. Thus the projective curvature tensor is the measure of the failure of a Riemannian manifold to be of constant curvature.

Let (M, g) be a Riemannian manifold and let ∇ be the Levi-Civita connection of (M, g). A Riemannian manifold is called locally symmetric [3] if $\nabla R = 0$, where R is the Riemannian curvature tensor of (M, g). A Riemannian manifold M is called semi-symmetric if

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$$R.R = 0 \tag{1.2}$$

holds, where R denotes the curvature tensor of the manifold. It is well known that the class of semi-symmetric manifolds includes the set of locally symmetric manifolds ($\nabla R = 0$) as a proper subset. Semi-symmetric Riemannian manifolds were first studied by E. Cartan, A. Lichnerowich, R. S. Couty and N. S. Sinjukov. A fundamental study on Riemannian semi-symmetric manifolds was made by Z. I. Szabó [10], E. Boeckx et al [2] and O. Kowalski [6]. A Riemannian manifold M is said to be Ricci-semi-symmetric if on M we have

$$R.S = 0, \tag{1.3}$$

where S is the Ricci tensor.

The class of Ricci-semi-symmetric manifolds includes the set of Ricci-symmetric manifolds ($\nabla S = 0$) as a proper subset. Ricci-semi-symmetric manifolds were investigated by several authors.
The present paper is organized as follows:
After in brief introduction in Section 2, we discuss about some preliminaries that will be used in the later sections. In section 3, we consider ϕ-projectively semi-symmetric 3-dimensional α-Para Kenmotsu manifolds. Section 4 is devoted to study projectively semi-symmetric 3-dimensional α-Para Kenmotsu manifolds. In section 5, we consider projectively pseudosymmetric 3-dimensional α-para Kenmotsu manifolds. Finally, 3-dimensional α-Para Kenmotsu manifolds satisfying $P.S = 0$ is also considered.

2. Preliminaries

2.1. Almost Paracontact Metric Manifolds

A smooth manifold M of dimension $2n+1$ is called an almost paracontact manifold ([7],[8]) equipped with the structure (ϕ, ξ, η) where ϕ is a tensor field of type $(1,1)$, a vector field ξ and a 1-form η satisfying

$$\phi^2 = I - \eta \otimes \xi \text{ and } \eta(\xi) = 1. \tag{2.1}$$

From equation (2.1) it can easily deduced that

$$\phi\xi = 0, \ \eta(\phi X) = 0 \text{ and } \text{rank}(\phi) = 2n. \tag{2.2}$$

If an almost paracontact manifold admits a pseudo-Riemannian metric g satisfying

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{2.3}$$

where signature of g is $(n+1, n)$ for any vector fields $X, Y \in \chi(M)$, (where $\chi(M)$ is the set of all differential vector fields on M) then the manifold is called almost paracontact metric manifold.
An almost paracontact structure is said to be a contact structure if $g(X, \phi Y) = d\eta(X, Y)$ with the associated metric g [17]. For an almost paracontact metric manifold, there always exists a special kind of local pseudo orthonormal ϕ basis $\{X_i, \phi X_i, \xi\}$, X_i's and ξ are space-like vector fields and ϕX_i's are time-like. Thus, an almost paracontact metric manifold is an odd dimensional manifold.

2.2. Normal Almost Paracontact Metric Manifolds

An almost paracontact metric manifold is said to be normal if the induced almost paracomplex structure J on the product manifold $M^{2n+1} \times \mathbb{R}$ defined by

$$J(X, f\frac{d}{dt}) = (\phi X + f\xi, \eta(X)\frac{d}{dt}) \tag{2.4}$$

is integrable where X is tangent to M, t is the coordinate of $\mathbb{R}$ and f is a smooth function on $M^{2n+1} \times \mathbb{R}$. The condition for being normal is equivalent to vanishing of the $(1, 2)$-type torsion tensor N_ϕ defined by $N_\phi = [\phi, \phi] - 2d\eta \otimes \xi$, where $[\phi, \phi]$ is the Nijenhuis torsion of ϕ.

Proposition 2.1. *[14] For a 3-dimensional almost paracontact metric manifold M, the following conditions are mutually equivalent*

(a) M is normal,

(b) there exist differential functions α, β on M such that

$$(\nabla_X \phi)Y = \beta\{g(X, Y)\xi - \eta(Y)X\} + \alpha\{g(\phi X, Y)\xi - \eta(Y)\phi X\},$$

(c) there exist differential functions α, β on M such that

$$\nabla_X \xi = \alpha\{X - \eta(X)\xi\} + \beta\phi X,$$

where ∇ is the Levi-Civita connection of the pseudo-Riemannian metric g and α, β are given by

$$2\alpha = Trace\{X \to \nabla_X \xi\}, 2\beta = Trace\{X \to \phi\nabla_X \xi\}.$$

Definition 2.1. A 3-dimensional normal almost paracontact metric manifold M is said to be

1. paracosymplectic if $\alpha = \beta = 0$ [4],
2. α-para Kenmotsu if α is a non-zero constant and $\beta = 0$ [13], in particular para Kenmotsu if $\alpha = 1$ [1],

3. quasi-para Sasakian if and only if $\alpha = 0$ and $\beta \neq 0$ [5],
4. β-para Sasakian if and only if $\alpha = 0$ and β is a non-zero constant, in particular para Sasakian if $\beta = -1$ [17].

In a 3-dimensional α-para Kenmotsu manifold, the following results hold [11]:

$$\begin{aligned} R(X,Y)Z &= (\frac{r}{2} + 2\alpha^2)[g(Y,Z)X - g(X,Z)Y] \\ &\quad -(\frac{r}{2} + 3\alpha^2)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]\xi \\ &\quad +(\frac{r}{2} + 3\alpha^2)[\eta(X)Y - \eta(Y)X]\eta(Z), \end{aligned} \tag{2.5}$$

where r is the scalar curvature of the manifold and g, pseudo-metric.

$$S(X,Y) = (\frac{r}{2} + \alpha^2)g(X,Y) - (\frac{r}{2} + 3\alpha^2)\eta(X)\eta(Y). \tag{2.6}$$

$$S(X,\xi) = -2\alpha^2\eta(X), \tag{2.7}$$

$$R(X,Y)\xi = -\alpha^2\{\eta(Y)X - \eta(X)Y\}, \tag{2.8}$$

$$(\nabla_X\eta)Y = \alpha\{g(X,Y) - \eta(X)\eta(Y)\}, \tag{2.9}$$

$$(\nabla_X\phi)Y = \alpha\{g(\phi X,Y)\xi - \eta(Y)\phi X\}, \tag{2.10}$$

$$\nabla_X\xi = \alpha\{X - \eta(X)\xi\}, \tag{2.11}$$

for all vector fields X, Y, Z and $W \in \chi(M)$.
Now, we state two Theorems which will be used in the next sections.

Theorem 2.1. *[15] A 3-dimension Riemannian manifold is Einstein if and only if it is manifold of constant curvature.*

Theorem 2.2. *[15] A Riemannian manifold is projectively flat if and only if the manifold is of constant curvature.*

3. ϕ-projectively semisymmetric 3-dimensional α-para Kenmotsu manifolds

Let M be a 3-dimensional α-para Kenmotsu manifold. Therefore $P(X,Y).\phi = 0$ turns into

$$(P(X,Y).\phi)Z = P(X,Y)\phi Z - \phi P(X,Y)Z = 0, \tag{3.1}$$

for any vector fields X, Y and Z.
Now, in view of (1.1), (2.5) we have

$$\begin{aligned} P(X,Y)\phi Z &= (\frac{r}{2} + 2\alpha^2)\{g(Y,\phi Z)X - g(X,\phi Z)Y\} \\ &\quad -(\frac{r}{2} + 3\alpha^2)\{g(Y,\phi Z)\eta(X) - g(X,\phi Z)\eta(Y)\}\xi \\ &\quad -\frac{1}{2}\{(\frac{r}{2} + \alpha^2)g(Y,\phi Z)X - (\frac{r}{2} + \alpha^2)g(X,\phi Z)Y\}. \end{aligned} \tag{3.2}$$

Similarly, we obtain

$$\phi(P(X,Y)Z) = \phi[R(X,Y)\phi Z \quad - \quad \frac{1}{2}\{S(Y,Z)X - S(X,Z)Y\}]. \tag{3.3}$$

By virtue of (3.2) and (3.3), we get from (3.1)

$$\begin{aligned} & \phi[R(X,Y)\phi Z - \frac{1}{2}\{S(Y,Z)X - S(X,Z)Y\}] \\ = \quad & (\frac{r}{2} + 2\alpha^2)\{g(Y,\phi Z)X - g(X,\phi Z)Y\} \\ & -(\frac{r}{2} + 3\alpha^2)\{g(Y,\phi Z)\eta(X) - g(X,\phi Z)\eta(Y)\}\xi \\ & -\frac{1}{2}\{(\frac{r}{2} + \alpha^2)g(Y,\phi Z)X - (\frac{r}{2} + \alpha^2)g(X,\phi Z)Y\}. \end{aligned} \tag{3.4}$$

Putting $X = \xi$ in (3.4) we have

$$\begin{aligned} & \phi[R(\xi,Y)\phi Z - \frac{1}{2}\{S(Y,Z)\xi - S(\xi,Z)Y\}] \\ = \quad & (\frac{r}{2} + 2\alpha^2)\{g(Y,\phi Z)\xi\} - (\frac{r}{2} + 3\alpha^2)\{g(Y,\phi Z)\}\xi \\ & -\frac{1}{2}\{(\frac{r}{2} + \alpha^2)g(Y,\phi Z)\xi\}. \end{aligned} \tag{3.5}$$

Using (2.7), (2.8) in (3.5) yields

$$\begin{aligned} & (\frac{r}{2} + 2\alpha^2)\{g(Y,\phi Z)\xi\} - (\frac{r}{2} + 3\alpha^2)\{g(Y,\phi Z)\}\xi \\ & -\frac{1}{2}\{(\frac{r}{2} + \alpha^2)g(Y,\phi Z)\xi\} = 0, \end{aligned} \tag{3.6}$$

which implies

$$\frac{1}{2}(\frac{r}{2} + 3\alpha^2)g(Y,\phi Z)\xi = 0, \tag{3.7}$$

Since $g(Y,\phi Z) \neq 0$, in (3.7) taking inner product with ξ we have

$$r = -6\alpha^2. \tag{3.8}$$

Substituting (3.8) in (2.6) we obtain

$$S(X,Y) = -2\alpha^2 g(X,Y). \tag{3.9}$$

Therefore the manifold is an Einstein manifold.
It is known [15] that a 3-dimension Riemannian manifold is Einstein if and only if it is manifold of constant curvature. Again a Riemannian manifold is projectively flat if and only if the manifold is of constant curvature.
By the above discussion we have the following:

Theorem 3.1. *In a 3-dimensional α-para Kenmotsu manifold M, the following conditions are equivalent:*

(a) ϕ-projectively semi-symmetric,

(b) the scalar curvature $r = -6\alpha^2$,

(c) the manifold M is of constant curvature,

(d) M is an Einstein manifold.

4. projectively semi-symmetric 3-dimensional α-para Kenmotsu manifolds

In view of (1.1), the projective curvature tensor is given by

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y]. \tag{4.1}$$

Now from the above equation with the help of (2.5), (2.6) we have

$$P(U,V)\xi = 0, \tag{4.2}$$

for any vector fields U, V. We suppose that a 3-dimensional α-para Kenmotsu manifold is projectively semi-symmetric, that is,

$$(R(X,Y).P)(U,V) = 0. \tag{4.3}$$

This implies

$$\begin{aligned} R(X,Y)P(U,V)W - P(R(X,Y)U,V)W &- P(U,R(X,Y)V)W \\ &- P(U,V)R(X,Y)W = 0. \end{aligned} \tag{4.4}$$

Using $Y = U = W = \xi$ in (4.4) we have

$$\begin{aligned} R(X,\xi)P(\xi,V)\xi - P(R(X,\xi)\xi,V)\xi &- P(\xi,R(X,\xi)V)\xi \\ &- P(\xi,V)R(X,\xi)\xi = 0. \end{aligned} \tag{4.5}$$

Therefore from (4.2) and (4.5) we get

$$-P(\xi,V)R(X,\xi)\xi = 0. \tag{4.6}$$

In view of (2.5), (2.6), (2.8) we obtain from (4.6)

$$-\alpha^2 P(\xi,V)X = 0. \tag{4.7}$$

Since $\alpha \neq 0$, the above equation implies

$$P(\xi,V)X = 0. \tag{4.8}$$

Therefore

$$R(\xi, V)X - \frac{1}{2}[S(V, X)\xi - S(\xi, X)V] = 0. \tag{4.9}$$

Applying (2.5), (2.6), (2.8) in (4.9) we get

$$\frac{1}{2}(\frac{r}{2} + 3\alpha^2)g(\phi X, \phi V) = 0. \tag{4.10}$$

Since $g(\phi X, \phi V) \neq 0$, we have

$$r = -6\alpha^2. \tag{4.11}$$

Substituting (4.11) in (2.6) we obtain

$$S(X, Y) = -2\alpha^2 g(X, Y). \tag{4.12}$$

Thus the manifold is an Einstein manifold.
In view of Theorems 2.1 and 2.2 we have the following:

Theorem 4.1. *In a 3-dimensional α-para Kenmotsu manifold M, the following conditions are equivalent:*

(a) projectively semi-symmetric,

(b) the scalar curvature $r = -6\alpha^2$,

(c) the manifold M is of constant curvature,

(d) M is an Einstein manifold.

5. Projectively pseudosymmetric 3-dimensional α-para Kenmotsu manifolds

A Riemannian manifold is said to be projectively pseudosymmetric [12] if at every point of the manifold the following relation holds

$$(R(X, Y).R)(U, V)W = L_R((X \wedge Y).R)(U, V)W), \tag{5.1}$$

for any vector fields X, Y, U, V, W; where L_R is some function of M. The endomorphism $X \wedge Y$ is defined by

$$(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y. \tag{5.2}$$

A Riemannian manifold is said to be projectively pseudosymmetric if it satisfies the condition

$$(R(X, Y).P)(U, V)W = L_P((X \wedge Y).P)(U, V)W), \tag{5.3}$$

where $L_P(\neq \alpha^2)$ is some function on M.
Let us suppose that 3-dimensional α-para Kenmotsu manifold M satisfies the condition

$$(R(X,Y).P)(U,V)W = L_P((X \wedge Y).P)(U,V)W). \tag{5.4}$$

Putting $Y = W = U = \xi$ (5.4) we have

$$(R(X,\xi).P)(\xi,V)\xi = L_P((X \wedge \xi).P)(\xi,V)\xi), \tag{5.5}$$

Now

$$\begin{aligned} L_P((X \wedge \xi).P)(\xi,V)\xi) &= L_P[(X \wedge \xi)P(\xi,V)\xi - P((X \wedge \xi)\xi,V)\xi \\ &\quad -P(\xi,(X \wedge \xi)V)\xi - P(\xi,V)(X \wedge \xi)\xi]. \end{aligned} \tag{5.6}$$

Using (4.2) in (5.6), we get

$$\begin{aligned} L_P((X \wedge \xi).P)(\xi,V)\xi) &= -L_P\{P(\xi,V)(X \wedge \xi)\xi\} \\ &= -L_P\{P(\xi,V)(X - \eta(X)\xi)\} \\ &= -L_P P(\xi,V)X. \end{aligned} \tag{5.7}$$

In view of (4.7), (5.7) we have from (5.5)

$$-\alpha^2 P(\xi,V)X = -L_P P(\xi,V)X. \tag{5.8}$$

Therefore

$$(L_P - \alpha^2)P(\xi,V)X = 0. \tag{5.9}$$

By assumption $L_P \neq \alpha^2$ and hence

$$P(\xi,V)X = 0, \tag{5.10}$$

The above equation same as (4.8), hence it follows that

$$r = -6\alpha^2. \tag{5.11}$$

Substituting (5.11) in (2.6) we obtain

$$S(X,Y) = -2\alpha^2 g(X,Y). \tag{5.12}$$

Thus the manifold is an Einstein manifold.
In view of Theorems 2.1 and 2.2 we can state the following:

Theorem 5.1. *In a 3-dimensional α-para Kenmotsu manifold M, the following conditions are equivalent:*

(a) Projectively pseudosymmetric,

(b) the scalar curvature $r = -6\alpha^2$,

(c) the manifold M is of constant curvature,

(d) M is an Einstein manifold.

6. 3-dimensional α-Para Kenmotsu manifolds satisfying $P.S = 0$

In this section we study 3-dimensional α-Para Kenmotsu manifolds satisfying $P.S = 0$. Therefore we have

$$(P(X,Y)\cdot S)(U,V) = 0. \tag{6.1}$$

This implies

$$S(P(X,Y)U,V) + S(U,P(X,Y)V) = 0. \tag{6.2}$$

Putting $U = \xi$ in (6.2) we have

$$S(P(X,Y)\xi,V) + S(\xi,P(X,Y)V) = 0. \tag{6.3}$$

Using (4.2) in (6.3), we get

$$S(\xi,P(X,Y)V) = 0. \tag{6.4}$$

Therefore from (2.7) and (6.4) we obtain

$$-2\alpha^2 g(P(X,Y)V,\xi) = 0. \tag{6.5}$$

This implies

$$g(R(X,Y)V,\xi) - \frac{1}{2}[S(Y,Z)\eta(X) - S(X,Z)\eta(Y)] = 0. \tag{6.6}$$

Using (2.5) in (6.6) we have

$$\eta(Y)\{\alpha^2 g(X,V) - \frac{1}{2}S(X,V)\} = \eta(X)\{\alpha^2 g(Y,V) + \frac{1}{2}S(Y,V)\}. \tag{6.7}$$

Putting $Y = \xi$ in (6.7), we get

$$S(X,V) = 2\alpha^2 g(X,V). \tag{6.8}$$

Thus the manifold is an Einstein manifold.
In view of Theorems 2.1 and 2.2 we have the following:

Theorem 6.1. *In a 3-dimensional α-para Kenmotsu manifold M, the following conditions are equivalent:*

(a) P.S=0,

(b) the scalar curvature $r = -6\alpha^2$,

(c) the manifold M is of constant curvature,

(d) M is an Einstein manifold.

REFERENCES

1. A. M. Blaga: *η-Ricci solitons on para-Kenmotsu manifolds*, (2014, reprint), arXiv: 1402.0223v3.
2. E. Boeckx, O. Kowalski and L. Vanhecke: *Riemannian manifolds of conullity two*, Singapore World Sci. Publishing, 1996.
3. E. Cartan: *Sur une classe remarqable d' espaces de Riemannian*, Bull. Soc. Math. France., **54**(1962), 214-264.
4. P. Dacko: *On almost para-cosymplectic manifolds*, Tsukuba J. Math., **28**(2004), 193–213.
5. S. Erdem: *On almost (para)contact (hyperbolic) metric manifolds and harmonicity of (ϕ, ϕ')-holomorphic maps between them*, Houst. J. Math., **28**(2002), 21–45.
6. O. Kowalski: *An explicit classification of* 3- *dimensional Riemannian spaces satisfying* $R(X, Y).R = 0$, Czechoslovak Math. J. **46**(121)(1996), 427-474.
7. M. Manev and M. Staikova: *On almost paracontact Riemannian manifolds of type* (n, n), J. Geom., **72**(2001), 108–114.
8. G. Nakova and S. Zamkovoy: *Almost paracontact manifolds*, (2009, reprint) arXiv:0806.3859v2.
9. G. Sooś: *Über die geodätischen Abbildungen von Riemannaschen Räumen auf projektiv symmetrische Riemannsche Räume*, Acta. Math. Acad.Sci. Hungar. Tom **9**(1958), 359-361.
10. Z. I. Szabó: *Structure theorems on Riemannian spaces satisfying* $R(X, Y).R = 0$*, the local version*, J. Diff. Geom., **17**(1982), 531-582.
11. K. Srivastava and S. K. Srivastava: *On a class of α-Para Kenmotsu Manifolds,* Mediterr. J. Math., (2014), DOI 10.1007/s00009-014-0496-9.
12. L. Verstraelen: *Comments on pseudosymmetry in the sense of Ryszard Deszcz, In: Geometry and Topology of submanifolds*, VI. River Edge, NJ: World Sci. Publishing, 1994, 199-209.
13. J. Welyczko: *Slant curves in* 3*-dimensional normal almost paracontact metric manifolds,* Mediterr. J. Math., **11**(2014), 965–978.
14. J. Welyczko: *On Legendre curves in* 3*-dimensional normal almost paracontact metric manifolds,* Result Math., **54**(2009), 377–387.
15. K. Yano and M. Kon: *Structures on manifolds*, Series in Pure Math Vol-3, World Sci. Publ. Co. Pte. Ltd. 1984.
16. K. Yano and S. Bochner: *Curvature and Betti numbers*, Annals of mathematics studies, **32**,Princeton university press,1953.
17. S. Zamkovoy: *Canonical connections on paracontact manifolds*, Ann. Glob. Anal. Geom., **36**(2009), 37–60.

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