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APPLICATION OF GENERALIZED WEAK CONTRACTION IN PERIODIC BOUNDARY VALUE PROBLEMS

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Abstract. The main aim of this research article is to formulate some fixed point and coupled fixed point results under generalized weak contraction on Re −metric spaces. As an application, we obtain the solution for periodic boundary value problems and also give an example to demonstrate the applicability of our results. The obtain results generalize and improve several well-known results in the existing literature. **Keywords**: Fixed point, coincidence point, generalized weak contraction, Re −metric space, ordinary differential equations.

1. Introduction

The classical Banach contraction principle has many inferences and huge applicability in mathematical theory and because of this, Banach contraction principle has been improved and generalized in various metric settings. One such interesting and important setting is to establish fixed point results in metric spaces equipped with an arbitrary binary relation. By using the notions of various kinds of binary relations such as partial order, strict order, near order, tolerance, etc. on metric spaces, many researcher have been doing their research attempting to obtain new extensions of the celebrated Banach contraction principle. Among these extensions, we must quote the one due to Alam and Imdad [3], where some relation theoretic analogues of standard metric notions (such as continuity and completeness) were

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used. Further, Ahmadullah et al. [2] extended the above setting for nonlinear contractions and obtained an extension of the Boyd-Wong [5] fixed point theorem to such spaces.

On the other hand, weak contraction was first studied in partially ordered metric spaces by Harjani and Sadarangani [22]. In [7], Choudhury et al. establish some coincidence point results for generalized weak contractions with discontinuous control functions on partially ordered metric spaces. Some of our basic references are ([1], [4], [8]-[12], [14], [16]-[21], [27]).

In this paper, we establish some fixed point and coupled fixed point results under generalized weak contraction on $\mathrm{Re}-$metric spaces. We obtain the solution for periodic boundary value problems and also give an example to show the fruitfulness of our results. Our results generalize and improve the results of Choudhury et al. [7], Ding et al. [13], Harjani and Sadarangani [22] and many other results in the existing literature.

2. Fixed point results

Throughout this article, Re stands for a non-empty binary relation, X for a non-empty set, $\mathbb{N}_0$ for the the set of whole numbers, that is, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{R}$ for the set of all real numbers. For simplicity, if $\tau \in X$, we denote $\xi(\tau)$ by $\xi\tau$.

Definition 2.1. [25]. Suppose $(X,\, d)$ is a metric space and Re is a binary relation on X. Then the triple $(X,\, d,\, \mathrm{Re})$ is called relational metric space or in brief $\mathrm{Re}-$metric space.

Example 2.1. [2]. Let $(X,\, d)$ be the usual metric space and let $(x,\, y) \in \mathrm{Re} \Leftrightarrow x \leq y$, $(x,\, y) \in \mathrm{Re} \Leftrightarrow x \geq y$ or $(x,\, y) \in \mathrm{Re} \Leftrightarrow x \perp y$, ..., then $(X,\, d,\, \mathrm{Re})$ is an $\mathrm{Re}-$metric space.

Definition 2.2. [3]. Let Re be a binary relation on a non-empty set X and τ, $\eta \in X$. We say that τ and η are $\mathrm{Re}-$comparative if either $(\tau,\, \eta) \in \mathrm{Re}$ or $(\eta,\, \tau) \in \mathrm{Re}$. We denote it by $[\tau,\, \eta] \in \mathrm{Re}$.

Definition 2.3. [3]. Let X be a non-empty set and Re be a binary relation on X. A sequence $\{\tau_n\} \subset X$ is called an $\mathrm{Re}-$preserving sequence if $(\tau_n,\, \tau_{n+1}) \in \mathrm{Re}$ for all $n \in \mathbb{N}_0$.

Definition 2.4. [3]. Let X be a non-empty set and ζ be a self-mapping on X. A binary relation Re on X is called $\zeta-$closed if for any $\tau,\, \eta \in X$, $(\tau,\, \eta) \in \mathrm{Re}$ implies $(\zeta\tau,\, \zeta\eta) \in \mathrm{Re}$.

Definition 2.5. [2]. An $\mathrm{Re}-$metric space $(X,\, d,\, \mathrm{Re})$ be said to be $\mathrm{Re}-$complete if every $\mathrm{Re}-$preserving Cauchy sequence in X converges.

Definition 2.6. [2]. Let $(X,\, d,\, \mathrm{Re})$ be an $\mathrm{Re}-$metric space and $\tau \in X$. A self-mapping ζ on X is called $\mathrm{Re}-$continuous at τ if for any $\mathrm{Re}-$preserving sequence $\{\tau_n\}$ such that $\tau_n \xrightarrow{d} \tau$, we have $\zeta(\tau_n) \xrightarrow{d} \zeta(\tau)$. Moreover, ζ is called $\mathrm{Re}-$continuous if it is $\mathrm{Re}-$continuous at each point of X.

Definition 2.7. [23]. Let $(X,\, d)$ be a metric space. Two mappings $\zeta,\, \xi : X \to X$ are said to be compatible if

$$\lim_{n\to\infty} d(\zeta\xi\tau_n,\ \xi\zeta\tau_n) = 0,$$

provided that $\{\tau_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} \zeta\tau_n = \lim_{n\to\infty} \xi\tau_n \in X.$$

Definition 2.8. [24]. Let X be a non-empty set. Two self-mappings ζ and ξ on X are said to be weakly compatible if they commute at their coincidence points, that is, if $\zeta\tau = \xi\tau$ for some $\tau \in X$, then $\zeta\xi\tau = \xi\zeta\tau$.

Definition 2.9. An $\mathrm{Re}-$metric space $(X,\, d,\, \mathrm{Re})$ is said to be $\mathrm{Re}-$regular if for every $\mathrm{Re}-$preserving sequence $\{\tau_n\} \subseteq X$ converges to $\tau \in X$, we have $(\tau_n,\, \tau) \in \mathrm{Re}$ for all $n \in \mathbb{N}_0$.

Definition 2.10. Let X be a non-empty set and ζ be a self-mapping on X. A binary relation Re on X is called $\zeta-$closed with respect to ξ if for any $\tau,\, \eta \in X$, $(\xi\tau,\, \xi\eta) \in \mathrm{Re}$ implies $(\zeta\tau,\, \zeta\eta) \in \mathrm{Re}$.

In [7], Choudhury et al. introduced the following families of functions:

The family Ψ of all functions $\psi : [0, +\infty) \to [0, +\infty)$ which satisfied the following conditions:

(i_ψ) ψ is continuous and non-decreasing,

(ii_ψ) $\psi(\tau) = 0$ if and only if $\tau = 0$.

The family Θ of all functions $\theta : [0,\, +\infty) \to [0,\, +\infty)$ satisfying

(i_θ) θ is bounded on any bounded interval in $[0,\, +\infty)$,

(ii_θ) θ is continuous and $\theta(0) = 0$.

Theorem 2.1. *Let* $(X,\, d,\, \mathrm{Re})$ *be an* $\mathrm{Re}-$*metric space,* ζ *and* ξ *be two self mappings on* X *satisfying*

(i) Re *is* $\zeta-$*closed with respect to* ξ *and* $\zeta(X) \subseteq \xi(X)$,

(ii) *there exists* $\tau_0 \in X$ *such that* $(\xi\tau_0,\, \zeta\tau_0) \in \mathrm{Re}$,

(iii) *there exist* $\psi \in \Psi$ *and* $\varphi,\, \theta \in \Theta$ *such that*

$$\psi(\tau) \le \varphi(\eta) \Rightarrow \tau \le \eta, \tag{2.1}$$

for any sequence $\{\tau_n\}$ *in* $[0,\, +\infty)$ *with* $\tau_n \to \tau > 0$,

$$\psi(\tau) - \overline{\lim}\varphi(\tau_n) + \underline{\lim}\theta(\tau_n) > 0, \tag{2.2}$$

and

$$\psi(d(\zeta\tau,\ \zeta\eta)) \le \varphi(d(\xi\tau,\ \xi\eta)) - \theta(d(\xi\tau,\ \xi\eta)), \tag{2.3}$$

for all $\tau,\ \eta \in X$ *such that* $(\xi\tau,\ \xi\eta) \in \mathrm{Re}$. *Also assume that one of the following conditions holds.*

(a) $(X,\ d,\ \mathrm{Re})$ *is* $\mathrm{Re}-$*complete,* ζ *and* ξ *are* $\mathrm{Re}-$*continuous and the pair* $(\zeta,\ \xi)$ *is compatible,*

(b) $(\xi(X),\ d,\ \mathrm{Re})$ *is* $\mathrm{Re}-$*complete and* $(X,\ d,\ \mathrm{Re})$ *is* $\mathrm{Re}-$*regular,*

(c) $(X,\ d, \mathrm{Re})$ *is* $\mathrm{Re}-$*complete,* ξ *is* $\mathrm{Re}-$*continuous and* Re *is* $\xi-$*closed, the pair* $(\zeta,\ \xi)$ *is compatible and* $(X,\ d,\ \mathrm{Re})$ *is* $\mathrm{Re}-$*regular.*

Then ζ *and* ξ *have a coincidence point. Moreover, if*

(iv) *for each* $\tau,\ \eta \in X$ *there exists* $\gamma \in X$ *such that* $\zeta\gamma$ *is* $\mathrm{Re}-$*comparative to* $\zeta\tau$ *and* $\zeta\eta$, *and also the pair* $(\zeta,\ \xi)$ *is weakly compatible.*

Then ζ *and* ξ *have a unique common fixed point.*

Proof. Let $\tau_0 \in X$ be arbitrary and $\zeta(X) \subseteq \xi(X)$, there exists $\tau_1 \in X$ such that $\zeta\tau_0 = \xi\tau_1$. By (ii), we have $(\xi\tau_0,\ \zeta\tau_0) \in \mathrm{Re}$, that is, $(\xi\tau_0,\ \xi\tau_1) \in \mathrm{Re}$. Since Re is $\zeta-$closed with respect to ξ, $(\zeta\tau_0,\ \zeta\tau_1) \in \mathrm{Re}$. Now $\zeta\tau_1 \in \zeta(X) \subseteq \xi(X)$, there exists $\tau_2 \in X$ such that $\zeta\tau_1 = \xi\tau_2$. Then $(\xi\tau_1,\ \xi\tau_2) \in \mathrm{Re}$. Since Re is $\zeta-$closed with respect to ξ, $(\zeta\tau_1,\ \zeta\tau_2) \in \mathrm{Re}$. Proceeding in the similar manner, we get a sequence $\{\tau_n\}_{n\in\mathbb{N}_0}$ such that $(\xi\tau_n,\ \xi\tau_{n+1}) \in \mathrm{Re}$ and

$$\xi\tau_{n+1} = \zeta\tau_n \text{ for all } n \in \mathbb{N}_0. \tag{2.4}$$

Thus the sequence $\{\xi\tau_n\}_{n\in\mathbb{N}_0}$ is $\mathrm{Re}-$preserving. Let $\varpi_n = d(\xi\tau_n,\ \xi\tau_{n+1})$ for all $n \in \mathbb{N}_0$. Since $(\xi\tau_{n+1},\ \xi\tau_{n+2}) \in \mathrm{Re}$, by using contractive condition (2.3) and (2.4), we have

$$\begin{aligned} \psi(d(\xi\tau_{n+1},\ \xi\tau_{n+2})) &= \psi(d(\zeta\tau_n,\ \zeta\tau_{n+1})) \\ &\leq \varphi(d(\xi\tau_n,\ \xi\tau_{n+1})) - \theta(d(\xi\tau_n,\ \xi\tau_{n+1})), \end{aligned}$$

which implies that

$$\psi(\varpi_{n+1}) \leq \varphi(\varpi_n) - \theta(\varpi_n), \tag{2.5}$$

which follows, by the fact $\theta \geq 0$, that $\psi(\varpi_{n+1}) \leq \varphi(\varpi_n)$ and so by (2.1) we find that $\varpi_{n+1} \leq \varpi_n$ for all $n \in \mathbb{N}_0$. Consequently $\{\varpi_n\}$ is a non-increasing sequence and so there exists an $\varpi \geq 0$ such that

$$\lim_{n\to\infty} \varpi_n = \lim_{n\to\infty} d(\xi\tau_n,\ \xi\tau_{n+1}) = \varpi. \tag{2.6}$$

Taking limit supremum on both sides of (2.5), using (2.6) and the continuity of ψ, we obtain

$$\psi(\varpi) \leq \overline{\lim}\varphi(\varpi_n) - \underline{\lim}\theta(\varpi_n) \Rightarrow \psi(\varpi) - \overline{\lim}\varphi(\varpi_n) + \underline{\lim}\theta(\varpi_n) \leq 0,$$

which contradicts (2.2). Thus

$$\varpi = \lim_{n\to\infty} \varpi_n = \lim_{n\to\infty} d(\xi\tau_n,\ \xi\tau_{n+1}) = 0. \tag{2.7}$$

One can easily obtain that $\{\xi\tau_n\}_{n\in\mathbb{N}_0}$ is a Cauchy sequence in X. Consequently $\{\xi\tau_n\}_{n\in\mathbb{N}_0}$ is an $\mathrm{Re}-$preserving Cauchy sequence in X. Now, we shall demonstrate that ζ and ξ have a coincidence point between cases $(a)-(c)$.

Suppose that (a) holds, that is, $(X,\ d,\ \mathrm{Re})$ is $\mathrm{Re}-$complete, ζ and ξ are $\mathrm{Re}-$continuous and $(\zeta,\ \xi)$ is compatible. Since $(X,\ d,\ \mathrm{Re})$ is $\mathrm{Re}-$complete, there exists $\delta\in X$ such that $\{\xi\tau_n\}\xrightarrow{d}\delta$ and (2.4) implies that $\{\zeta\tau_n\}\xrightarrow{d}\delta$. Since ζ and ξ are $\mathrm{Re}-$continuous, $\{\zeta\xi\tau_n\}\xrightarrow{d}\zeta\delta$ and $\{\xi\xi\tau_n\}\xrightarrow{d}\xi\delta$. Also the fact that the pair $(\zeta,$ $\xi)$ is compatible implies

$$d(\xi\delta,\ \zeta\delta)=\lim_{n\to\infty}d(\xi\xi\tau_{n+1},\ \zeta\xi\tau_n)=\lim_{n\to\infty}d(\xi\zeta\tau_n,\ \zeta\xi\tau_n)=0,$$

that is, δ is a coincidence point of ζ and ξ.

Suppose now (b) holds, that is, $(\xi(X),\ d,\ \mathrm{Re})$ is $\mathrm{Re}-$complete and $(X,\ d,\ \mathrm{Re})$ is regular. As $\{\xi\tau_n\}$ is an $\mathrm{Re}-$preserving Cauchy sequence in $\mathrm{Re}-$complete space $(\xi(X),\ d)$, so there exists $\eta\in\xi(X)$ such that $\{\xi\tau_n\}\xrightarrow{d}\eta$. Let $\delta\in X$ be any point such that $\eta=\xi\delta$, then $\{\xi\tau_n\}\xrightarrow{d}\xi\delta$. Since $(X,\ d,\ \mathrm{Re})$ is $\mathrm{Re}-$regular and $\{\xi\tau_n\}$ is $\mathrm{Re}-$preserving which converges to $\xi\delta$, we obtain that $(\xi\tau_n,\ \xi\delta)\in\mathrm{Re}$ for all $n\in\mathbb{N}_0$. Using contractive condition (2.3), we have

$$\psi(d(\xi\tau_{n+1},\ \zeta\delta))=\psi(d(\zeta\tau_n,\ \zeta\delta))\leq\varphi(d(\xi\tau_n,\ \xi\delta))-\theta(d(\xi\tau_n,\ \xi\delta)).$$

Letting $n\to\infty$ in the above inequality and by using (ii_θ) of θ, φ and $\{\xi\tau_n\}\xrightarrow{d}\xi\delta$, we get $\psi(d(\xi\delta,\ \zeta\delta))=0$. It follows, from (ii_ψ), that $d(\xi\delta,\ \zeta\delta)=0$, that is, δ is a coincidence point of ζ and ξ.

Suppose now that (c) holds, that is, $(X,\ d,\ \mathrm{Re})$ is $\mathrm{Re}-$complete, ξ is $\mathrm{Re}-$continuous and Re is $\xi-$closed, the pair $(\zeta,\ \xi)$ is compatible and $(X,\ d,\ \mathrm{Re})$ is $\mathrm{Re}-$regular. Since $(X,\ d,\ \mathrm{Re})$ is $\mathrm{Re}-$complete, there exists $\delta\in X$ such that $\{\xi\tau_n\}\xrightarrow{d}\delta$, then (2.4) follows that $\{\zeta\tau_n\}\xrightarrow{d}\delta$. Since ξ is $\mathrm{Re}-$continuous, $\{\xi\xi\tau_n\}\xrightarrow{d}\xi\delta$. Furthermore $\{\xi\xi\tau_n\}\xrightarrow{d}\xi\delta$ and the pair $(\zeta,\ \xi)$ is $\mathrm{Re}-$compatible, so we have

$$\lim_{n\to\infty}d(\xi\xi\tau_{n+1},\ \zeta\xi\tau_n)=\lim_{n\to\infty}d(\xi\zeta\tau_n,\ \zeta\xi\tau_n)=0.$$

These facts together implies that $\{\zeta\xi\tau_n\}\xrightarrow{d}\xi\delta$.

Also, since $(X,\ d,\ \mathrm{Re})$ is $\mathrm{Re}-$regular and $\{\xi\tau_n\}$ is $\mathrm{Re}-$preserving which converges to δ, $(\xi\tau_n,\ \delta)\in\mathrm{Re}$, which, by the fact that Re is $\xi-$closed, implies $(\xi\xi\tau_n,$ $\xi\delta)\in\mathrm{Re}$. Using the contractive condition (2.3), we get

$$\psi(d(\zeta\xi\tau_n,\ \zeta\delta))\leq\varphi(d(\xi\xi\tau_n,\ \xi\delta))-\theta(d(\xi\xi\tau_n,\ \xi\delta)).$$

Taking limit $n\to\infty$ and by using (ii_θ) of θ, φ and $\{\xi\xi\tau_n\}\xrightarrow{d}\xi\delta$, $\{\zeta\xi\tau_n\}\xrightarrow{d}\xi\delta$, we get $\psi(d(\xi\delta,\ \zeta\delta))=0$. It follows, from (ii_ψ), that $d(\xi\delta,\ \zeta\delta)=0$, that is, δ is a coincidence point of ζ and ξ.

Consequently the set of coincidence points of ζ and ξ is non-empty. Let τ and η be two coincidence points of ζ and ξ, that is, $\zeta\tau = \xi\tau$ and $\zeta\eta = \xi\eta$. Now, we claim that $\xi\tau = \xi\eta$. By the assumption, there exists $\alpha \in X$ such that $\zeta\alpha$ is $\mathrm{Re}-$comparative with $\zeta\tau$ and $\zeta\eta$. Put $\alpha_0 = \alpha$ and choose $\alpha_1 \in X$ so that $\xi\alpha_0 = \zeta\alpha_1$. One can inductively define the sequence $\{\xi\alpha_n\}$ where $\xi\alpha_{n+1} = \zeta\alpha_n$ for all $n \in \mathbb{N}_0$. Hence $\zeta\tau = \xi\tau$ and $\zeta\alpha = \zeta\alpha_0 = \xi\alpha_1$ are $\mathrm{Re}-$comparative. Suppose that $(\xi\alpha_1, \xi\tau) \in \mathrm{Re}$. We claim that $(\xi\alpha_n, \xi\tau) \in \mathrm{Re}$ for each $n \in \mathbb{N}$. For this, we shall use mathematical induction. As $(\xi\alpha_1, \xi\tau) \in \mathrm{Re}$ and so our claim is true for $n = 1$.

Now, suppose that $(\xi\alpha_n, \xi\tau) \in \mathrm{Re}$ holds for some $n > 1$. Since Re is $\zeta-$closed with respect to ξ, we get $(\zeta\alpha_n, \zeta\tau) \in \mathrm{Re}$, that is, $(\xi\alpha_{n+1}, \xi\tau) \in \mathrm{Re}$. Similarly one can show that $(\xi\tau, \xi\alpha_{n+1}) \in \mathrm{Re}$. Thus our claim is proved. Hence $\xi\alpha_n$ and $\xi\tau$ are $\mathrm{Re}-$comparative, for each $n \in \mathbb{N}_0$.

Let $\pi_n = d(\xi\tau, \xi\alpha_n)$ for all $n \in \mathbb{N}_0$. Since $(\xi\alpha_n, \xi\tau) \in \mathrm{Re}$, by using the contractive condition (2.3) and (2.4), we have

$$\psi(d(\xi\tau,\ \xi\alpha_{n+1})) = \psi(d(\zeta\tau,\ \zeta\alpha_n)) \leq \varphi(d(\xi\tau,\ \xi\alpha_n)) - \theta(d(\xi\tau,\ \xi\alpha_n)),$$

which implies that

$$\psi(\pi_{n+1}) \leq \varphi(\pi_n) - \theta(\pi_n), \tag{2.8}$$

which, by the fact that $\theta \geq 0$, implies $\psi(\pi_{n+1}) \leq \varphi(\pi_n)$ and so by (2.1), it follows that $\pi_{n+1} \leq \pi_n$ for all $n \in \mathbb{N}_0$. Thus $\{\pi_n\}$ is a monotone non-increasing sequence. Hence there exists an $\pi \geq 0$ such that

$$\lim_{n\to\infty} \pi_n = d(\xi\tau,\ \xi\alpha_n) = \pi. \tag{2.9}$$

Taking limit supremum on both sides of (2.8), using (2.9) and the continuity of ψ, we obtain

$$\psi(\pi) \leq \overline{\lim}\varphi(\pi_n) - \underline{\lim}\theta(\pi_n), \text{ that is, } \psi(\pi) - \overline{\lim}\varphi(\pi_n) + \underline{\lim}\theta(\pi_n) \leq 0.$$

This contradicts (2.2). Thus we must have

$$\pi = \lim_{n\to\infty} \pi_n = \lim_{n\to\infty} d(\xi\tau,\ \xi\alpha_n) = 0. \tag{2.10}$$

Similarly, one can obtain that

$$\lim_{n\to\infty} d(\xi\eta,\ \xi\alpha_n) = 0. \tag{2.11}$$

Hence, by (2.10) and (2.11), we get

$$\xi\tau = \xi\eta. \tag{2.12}$$

Since $\xi\tau = \zeta\tau$, therefore by weak compatibility of ξ and ζ, we have $\xi\xi\tau = \xi\zeta\tau = \zeta\xi\tau$. Let $\delta = \xi\tau$, then $\xi\delta = \zeta\delta$, that is, δ is a coincidence point of ζ and ξ. Then from (2.12) with $\eta = \delta$, it follows that $\xi\tau = \xi\delta$, that is, $\delta = \xi\delta = \zeta\delta$. Hence δ is a common fixed point of ζ and ξ. To prove the uniqueness, assume that γ is another common fixed point of ζ and ξ. Then by (2.12) we have $\gamma = \xi\gamma = \xi\delta = \delta$, that is, the common fixed point of ζ and ξ is unique. $\square$

If we take $\psi = I$ (the identity mapping) and $\theta(t) = 0$ for all $t \geq 0$ in Theorem 2.1, we have the following corollary.

Corollary 2.1. *Let* $(X,\, d,\, \mathrm{Re})$ *be an* $\mathrm{Re}-$*metric space,* ζ *and* ξ *be two self mappings on* X *satisfying conditions* (i) *and* (ii) *of Theorem 2.1 and*

(i) *there exists some* $\varphi \in \Theta$ *such that for any sequence* $\{\tau_n\}$ *in* $[0,\, +\infty)$ *with* $\tau_n \to \tau > 0$,

$$\overline{\lim}\varphi(\tau_n) < \tau,$$

and

$$d(\zeta\tau,\ \zeta\eta) \leq \varphi(d(\xi\tau,\ \xi\eta)),$$

for all $\tau,\, \eta \in X$ *such that* $(\xi\tau,\, \xi\eta) \in \mathrm{Re}$. *Also assume that one of the conditions* $(a)-(c)$ *of Theorem 2.1 holds. Then* ζ *and* ξ *have a coincidence point. Moreover, if condition* (iv) *of Theorem 2.1 holds, then* ζ *and* ξ *have a unique common fixed point.*

If we take $\theta(t) = 0$ and $\varphi(t) = k\psi(t)$ with $0 \leq k < 1$ and for all $t \geq 0$ in Theorem 2.1, we have the following corollary.

Corollary 2.2. *Let* $(X,\, d,\, \mathrm{Re})$ *be an* $\mathrm{Re}-$*metric space,* ζ *and* ξ *be two self mappings on* X *satisfying conditions* (i) *and* (ii) *of Theorem 2.1 and*

(i) *there exist some* $\psi \in \Psi$ *and* $k \in [0,\, 1)$ *such that*

$$\psi(d(\zeta\tau,\ \zeta\eta)) \leq k\psi(d(\xi\tau,\ \xi\eta)),$$

for all $\tau,\, \eta \in X$ *such that* $(\xi\tau,\, \xi\eta) \in \mathrm{Re}$. *Also assume that one of the conditions* $(a)-(c)$ *of Theorem 2.1 holds. Then* ζ *and* ξ *have a coincidence point. Moreover, if condition* (iv) *of Theorem 2.1 holds, then* ζ *and* ξ *have a unique common fixed point.*

Taking $\varphi = \psi$ in Theorem 2.1, we have the following corollary.

Corollary 2.3. *Let* $(X,\, d,\, \mathrm{Re})$ *be an* $\mathrm{Re}-$*metric space,* ζ *and* ξ *be two self mappings on* X *satisfying conditions* (i) *and* (ii) *of Theorem 2.1 and*

(i) *there exist some* $\psi \in \Psi$ *and* $\theta \in \Theta$ *such that for any sequence* $\{\tau_n\}$ *in* $[0,\, +\infty)$ *with* $\tau_n \to \tau > 0$,

$$\overline{\lim}\theta(\tau_n) > 0,$$

and

$$\psi(d(\zeta\tau,\ \zeta\eta)) \leq \psi(d(\xi\tau,\ \xi\eta)) - \theta(d(\xi\tau,\ \xi\eta)),$$

for all $\tau,\, \eta \in X$ *such that* $(\xi\tau,\, \xi\eta) \in \mathrm{Re}$. *Also assume that one of the conditions* $(a)-(c)$ *of Theorem 2.1 holds. Then* ζ *and* ξ *have a coincidence point. Moreover, if condition* (iv) *of Theorem 2.1 holds, then* ζ *and* ξ *have a unique common fixed point.*

If we take $\psi = \varphi = I$ (the identity mappings) in Theorem 2.1, we have the following corollary.

Corollary 2.4. *Let $(X,\ d,\ \mathrm{Re})$ be an $\mathrm{Re}-$metric space, ζ and ξ be two self mappings on X satisfying conditions (i) and (ii) of Theorem 2.1 and*

(i) there exists some $\theta \in \Theta$ such that for any sequence $\{\tau_n\}$ in $[0,\ +\infty)$ with $\tau_n \to \tau > 0$,

$$\overline{\lim}\theta(\tau_n) > 0,$$

and

$$d(\zeta\tau,\ \zeta\eta) \leq d(\xi\tau,\ \xi\eta) - \theta(d(\xi\tau,\ \xi\eta)),$$

for all $\tau,\ \eta \in X$ such that $(\xi\tau,\ \xi\eta) \in \mathrm{Re}$. Also assume that one of the conditions $(a)-(c)$ of Theorem 2.1 holds. Then ζ and ξ have a coincidence point. Moreover, if condition (iv) of Theorem 2.1 holds, then ζ and ξ have a unique common fixed point.

If we take $\psi = \varphi = I$ (the identity mapping) and $\theta(t) = (1-k)t$, for all $t \geq 0$, where $0 \leq k < 1$ in Theorem 2.1, we have the following corollary.

Corollary 2.5. *Let $(X,\ d,\ \mathrm{Re})$ be an $\mathrm{Re}-$metric space, ζ and ξ be two self mappings on X satisfying conditions (i) and (ii) of Theorem 2.1 and*

(i) there exists $k \in [0,\ 1)$ such that

$$d(\zeta\tau,\ \zeta\eta) \leq kd(\xi\tau,\ \xi\eta),$$

for all $\tau,\ \eta \in X$ such that $(\xi\tau,\ \xi\eta) \in \mathrm{Re}$. Also assume that one of the conditions $(a)-(c)$ of Theorem 2.1 holds. Then ζ and ξ have a coincidence point. Moreover, if condition (iv) of Theorem 2.1 holds, then ζ and ξ have a unique common fixed point.

If we take $\xi = I$ (the identity mapping) in Corollary 2.5, we have the following corollary.

Corollary 2.6. *Let $(X,\ d,\ \mathrm{Re})$ be an $\mathrm{Re}-$metric space and ζ be a self mapping on X such that*

$$d(\zeta\tau,\ \zeta\eta) \leq kd(\tau,\ \eta),$$

for all $\tau,\ \eta \in X$ such that $(\tau,\ \eta) \in \mathrm{Re}$, where $k \in [0,\ 1)$. Also suppose that either

(a) ζ is $\mathrm{Re}-$continuous or,

(b) $(X,\ d,\ \mathrm{Re})$ is $\mathrm{Re}-$regular.

There exists $\tau_0 \in X$ such that $(\tau_0,\ \zeta\tau_0) \in \mathrm{Re}$ and Re is $\zeta-$closed. Then ζ has a fixed point.

Example 2.2. Let $X = [0, 1]$, equipped with the usual metric $d : X \times X \to [0, +\infty)$ with the natural ordering of real numbers $\leq$. Let $\zeta, \xi : X \to X$ be defined as

$$\zeta\tau = \frac{\tau^2}{3} \text{ and } \xi\tau = \tau^2, \text{ for all } \tau \in X.$$

Define $\varphi, \psi, \theta : [0, +\infty) \to [0, +\infty)$ as follows

$$\psi(t) = t^2, \ \varphi(t) = \begin{cases} \frac{1}{3}[t]^2, \text{ if } 3 < t < 4, \\ \frac{1}{9}t^2, \text{ otherwise,} \end{cases} \quad \text{and } \theta(t) = \begin{cases} \frac{1}{9}[t]^2, \text{ if } 3 < t < 4, \\ 0, \text{ otherwise.} \end{cases}$$

Then ψ, φ and θ have all the properties mentioned in Theorem 2.1. One can easily see that the contractive condition of Theorem 2.1 is satisfied for all $\tau, \eta \in X$. Furthermore, all the other conditions of Theorem 2.1 are also satisfied and $\delta = 0$ is a unique fixed point of ζ and ξ.

3. Coupled coincidence point results

Given $n \in \mathbb{N}$ with $n \geq 2$, let X^n be the nth Cartesian product $X \times X \times ... \times X$ (n times).

Definition 3.1. Let Re be a binary relation on X and the product space X^2 associated with the following binary relation: for all $\varepsilon_1 = (\tau_1, \eta_1)$ and $\varepsilon_2 = (\tau_2, \eta_2) \in X^2$, we have

$$(\varepsilon_1, \ \varepsilon_2) \in \mathcal{S} \Leftrightarrow (\tau_1, \ \tau_2) \in \text{Re} \text{ and } (\eta_1, \ \eta_2) \in \text{Re}\,.$$

Definition 3.2. [1]**.** Let (X, d) be a metric space. Define $\Delta_n : X^n \times X^n \to [0, +\infty)$, for $A = (a_1, a_2, ..., a_n)$, $B = (b_1, b_2, ..., b_n) \in X^n$, by

$$\Delta_n(A, \ B) = \frac{1}{n}\sum_{i=1}^{n} d(a_i, \ b_i).$$

Then Δ_n is metric on X^n.

Definition 3.3. [15]. Let $F : X^2 \to X$ be a given mapping. An element $(\tau, \eta) \in X^2$ is called a coupled fixed point of F if

$$F(\tau, \ \eta) = \tau \text{ and } F(\eta, \ \tau) = \eta.$$

Definition 3.4. [26]**.** Let $F : X^2 \to X$ and $G : X \to X$ be given mappings. An element $(\tau, \eta) \in X^2$ is called a coupled coincidence point of the mappings F and G if $F(\tau, \eta) = G\tau$ and $F(\eta, \tau) = G\eta$.

Definition 3.5. [26]**.** Let $F : X^2 \to X$ and $G : X \to X$ be given mappings. An element $(\tau, \eta) \in X^2$ is called a common coupled fixed point of the mappings F and G if $\tau = F(\tau, \eta) = G\tau$ and $\eta = F(\eta, \tau) = G\eta$.

Definition 3.6. [6]**.** Let $(X,\, d)$ be a metric space. Mappings $F : X^2 \to X$ and $G : X \to X$ are said to be compatible if

$$\begin{aligned}\lim_{n\to\infty} d(GF(\alpha_n,\ \beta_n),\ F(G\alpha_n,\ G\beta_n)) &= 0,\\ \lim_{n\to\infty} d(GF(\beta_n,\ \alpha_n),\ F(G\beta_n,\ G\alpha_n)) &= 0,\end{aligned}$$

whenever $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in X such that

$$\begin{aligned}\lim_{n\to\infty} F(\alpha_n,\ \beta_n) &= \lim_{n\to\infty} G\alpha_n = \alpha \in X,\\ \lim_{n\to\infty} F(\beta_n,\ \alpha_n) &= \lim_{n\to\infty} G\beta_n = \beta \in X.\end{aligned}$$

Definition 3.7. [6]. Let X be a non-empty set. Two mappings $G : X \to X$ and $F : X^2 \to X$ are said to be weakly compatible if they commute at their coupled coincidence points, that is, if $F(\tau,\, \eta) = G\tau$ and $F(\eta,\, \tau) = G\eta$ for some $(\tau,\, \eta) \in X^2$, then $GF(\tau,\, \eta) = F(G\tau,\, G\eta)$ and $GF(\eta,\, \tau) = F(G\eta,\, G\tau)$.

Definition 3.8. Let X be a non-empty set and $F : X^2 \to X$ be a mapping. A binary relation Re on X is called $F-$closed if for any $(\tau_1,\, \eta_1),\, (\tau_2,\, \eta_2) \in X^2$, $(\tau_1,\, \tau_2) \in \mathrm{Re}$ and $(\eta_1,\, \eta_2) \in \mathrm{Re}$ implies $(F(\tau_1,\, \eta_1),\, F(\tau_2,\, \eta_2)) \in \mathrm{Re}$ and $(F(\eta_1,\, \tau_1),\, F(\eta_2,\, \tau_2)) \in \mathrm{Re}$.

Definition 3.9. Let X be a non-empty set, $F : X^2 \to X$ and $G : X \to X$ be given mappings. A binary relation Re on X is called $F-$closed with respect to G if for any $(\tau_1,\, \eta_1),\, (\tau_2,\, \eta_2) \in X^2$, $(G\tau_1,\, G\tau_2) \in \mathrm{Re}$ and $(G\eta_1,\, G\eta_2) \in \mathrm{Re}$ implies $(F(\tau_1,\, \eta_1),\, F(\tau_2,\, \eta_2)) \in \mathrm{Re}$ and $(F(\eta_1,\, \tau_1),\, F(\eta_2,\, \tau_2)) \in \mathrm{Re}$.

Lemma 3.1. *Let $(X,\, d,\, \mathrm{Re})$ be an $\mathrm{Re}-$metric space, $F : X^2 \to X$ an $G : X \to X$ be given mappings. Define the mappings $\Phi_F,\, \Phi_G : X^2 \to X^2$, for all $\varepsilon = (\tau,\, \eta) \in X^2$, by*

$$\Phi_F(\varepsilon) = (F(\tau,\ \eta),\ F(\eta,\ \tau)) \text{ and } \Phi_G(\varepsilon) = (G\tau,\ G\eta).$$

Then

(1) $(X,\, d,\, \mathrm{Re})$ is $\mathrm{Re}-$complete if and only if $(X^2,\, \Delta_2,\, S)$ is $\mathcal{S}-$complete.

(2) If $(X,\, d,\, \mathrm{Re})$ is $\mathrm{Re}-$regular, then $(X^2,\, \Delta_2,\, \mathcal{S})$ is also $\mathcal{S}-$regular.

(3) If F is $\mathrm{Re}-$continuous, then Φ_F is $\mathcal{S}-$continuous.

(4) Re is $F-$closed if and only if $\mathcal{S}$ is Φ_F-closed.

(5) Re is $F-$closed with respect to G if and only if $\mathcal{S}$ is Φ_F-closed with respect to Φ_G.

(6) If there exist two elements $\tau_0,\, \eta_0 \in X$ with $(G\tau_0,\, F(\tau_0,\, \eta_0)) \in \mathrm{Re}$ and $(G\eta_0,\, F(\eta_0,\, \tau_0)) \in \mathrm{Re}$, then there exists a point $\varepsilon_0 = (\tau_0,\, \eta_0) \in X^2$ such that $(\Phi_G(\varepsilon_0),\, \Phi_F(\varepsilon_0)) \in \mathcal{S}$.

(7) If $F(X^2) \subseteq G(X)$, then $\Phi_F(X^2) \subseteq \Phi_G(X^2)$.

(8) If F and G are compatible in $(X,\, d)$, then Φ_F and Φ_G are also compatible in $(X^2,\, \Delta_2)$.

(9) If F and G are weak compatible in X, then Φ_F and Φ_G are also weak compatible in X^2.

(10) A point $\varepsilon = (\tau,\, \eta) \in X^2$ is a coupled coincidence point of F and G if and only if it is a coincidence point of Φ_F and Φ_G.

(11) A point $\varepsilon = (\tau,\, \eta) \in X^2$ is a coupled fixed point of F if and only if it is a fixed point of Φ_F.

Proof. Statements (1), (2), (3), (5), (6), (7), (8), (9), (10) and (11) are obvious.

(4) Assume that Re is $F-$closed and let $\varepsilon_1 = (\tau_1,\, \eta_1),\, \varepsilon_2 = (\tau_2,\, \eta_2) \in X^2$ be such that $(\varepsilon_1,\, \varepsilon_2) \in \mathcal{S}$. Then $(\tau_1,\, \tau_2) \in \mathrm{Re}$ and $(\eta_1,\, \eta_2) \in \mathrm{Re}$. Since Re is $F-$closed, we have $(F(\tau_1,\, \eta_1),\, F(\tau_2,\, \eta_2)) \in \mathrm{Re}$ and $(F(\eta_1,\, \tau_1),\, F(\eta_2,\, \tau_2)) \in \mathrm{Re}$, which implies that $(\Phi_F(\varepsilon_1),\, \Phi_F(\varepsilon_2)) \in \mathcal{S}$. Hence $\mathcal{S}$ is Φ_F-closed.

(5) Assume that Re is $F-$closed with respect to G and let $\varepsilon_1 = (\tau_1,\, \eta_1)$, $\varepsilon_2 = (\tau_2,\, \eta_2) \in X^2$ be such that $(\Phi_G(\varepsilon_1),\, \Phi_G(\varepsilon_2)) \in \mathcal{S}$. Then $(G\tau_1,\, G\tau_2) \in \mathrm{Re}$ and $(G\eta_1,\, G\eta_2) \in \mathrm{Re}$. Since Re is $F-$closed with respect to G, we have $(F(\tau_1,\, \eta_1),\, F(\tau_2,\, \eta_2)) \in \mathrm{Re}$ and $(F(\eta_1,\, \tau_1),\, F(\eta_2,\, \tau_2)) \in \mathrm{Re}$, which implies that $(\Phi_F(\varepsilon_1),\, \Phi_F(\varepsilon_2)) \in \mathcal{S}$. Hence $\mathcal{S}$ is Φ_F-closed with respect to Φ_G. $\square$

Theorem 3.1. *Let $(X,\, d,\, \mathrm{Re})$ be an $\mathrm{Re}-$metric space, $F : X^2 \to X$ and $G : X \to X$ be two mappings satisfying*

(i) Re is $F-$closed with respect to G and $F(X^2) \subseteq G(X)$,

(ii) there exist two elements $\tau_0,\, \eta_0 \in X$ with

$$(G\tau_0,\ F(\tau_0,\ \eta_0)) \in \mathrm{Re} \ \textit{ and } (G\eta_0,\ F(\eta_0,\ \tau_0)) \in \mathrm{Re},$$

(iii) there exist $\psi \in \Psi$ and $\varphi,\, \theta \in \Theta$ satisfying (2.1), (2.2) and

$$\begin{aligned} (3.1) \quad & \psi\left(\frac{d(F(\tau_1,\ \eta_1),\ F(\tau_2,\ \eta_2)) + d(F(\eta_1,\ \tau_1),\ F(\eta_2,\ \tau_2))}{2}\right) \\ \leq \quad & \varphi\left(\frac{d(G\tau_1,\ G\tau_2) + d(G\eta_1,\ G\eta_2))}{2}\right) - \theta\left(\frac{d(G\tau_1,\ G\tau_2) + d(G\eta_1,\ G\eta_2))}{2}\right), \end{aligned}$$

for all $\tau_1,\, \eta_1,\, \tau_2,\, \eta_2 \in X$, where $(G\tau_1,\, G\tau_2) \in \mathrm{Re}$ and $(G\eta_1,\, G\eta_2) \in \mathrm{Re}$.

(iv) G is $\mathrm{Re}-$continuous and Re is $G-$closed,

(v) the pair $\{F,\, G\}$ is $\mathrm{Re}-$compatible.

(vi) F is $\mathrm{Re}-$continuous or $(X,\, d,\, \mathrm{Re})$ is $\mathrm{Re}-$regular.

Then F and G have a coupled coincidence point. Furthermore, suppose that

(vii) for every $(\tau,\, \eta),\, (\tau^*, \eta^*) \in X^2$, there exists a point $(\tau',\, \eta') \in X^2$ such that $(F(\tau',\, \eta'),\, F(\eta',\, \tau'))$ is $\mathrm{Re}-$comparative to $(F(\tau,\, \eta),\, F(\eta,\, \tau))$ and $(F(\tau^*,\, \eta^*),\, F(\eta^*,\, \tau^*))$, and also the pair $(F,\, G)$ is weakly compatible.

Then F and G have a unique coupled common fixed point.

Proof. One can easily obtain that the contractive condition (3.1) means that

$$\begin{aligned} & \psi(\Delta_2(\Phi_F(\varepsilon_1),\ \Phi_F(\varepsilon_2))) \\ \leq\ & \varphi(\Delta_2(\Phi_G(\varepsilon_1),\ \Phi_G(\varepsilon_2))) - \theta(\Delta_2(\Phi_G(\varepsilon_1),\ \Phi_G(\varepsilon_2))), \end{aligned}$$

where $\varepsilon_1 = (\tau_1,\ \eta_1)$ and $\varepsilon_2 = (\tau_2,\ \eta_2) \in X^2$. It is only require to apply Theorem 2.1 to the mappings $\zeta = \Phi_F$ and $\xi = \Phi_G$ in the relational metric space $(X^2,\ \Delta_2,\ \mathcal{S})$ using Lemma 3.1. □

Corollary 3.1. *Let* $(X,\ d,\ \mathrm{Re})$ *be an* $\mathrm{Re}-$*metric space and* $F : X^2 \to X$ *be a mapping satisfying*

(*i*) Re *is* $F-$*closed,*

(*ii*) *there exist two elements* $\tau_0,\ \eta_0 \in X$ *with*

$$(\tau_0,\ F(\tau_0,\ \eta_0)) \in \mathrm{Re} \ \text{ and } (\eta_0,\ F(\eta_0,\ \tau_0)) \in \mathrm{Re},$$

(*iii*) *there exist* $\psi \in \Psi$ *and* $\varphi,\ \theta \in \Theta$ *satisfying (2.1), (2.2) and*

$$\begin{aligned} & \psi\left(\frac{d(F(\tau_1,\ \eta_1),\ F(\tau_2,\ \eta_2)) + d(F(\eta_1,\ \tau_1),\ F(\eta_2,\ \tau_2))}{2}\right) \\ \leq\ & \varphi\left(\frac{d(\tau_1,\ \tau_2) + d(\eta_1,\ \eta_2))}{2}\right) - \theta\left(\frac{d(\tau_1,\ \tau_2) + d(\eta_1,\ \eta_2))}{2}\right), \end{aligned}$$

for all $\tau_1,\ \eta_1,\ \tau_2,\ \eta_2 \in X$, *with* $(\tau_1,\ \tau_2) \in \mathrm{Re}$ *and* $(\eta_1,\ \eta_2) \in \mathrm{Re}$.

(*iv*) F *is* $\mathrm{Re}-$*continuous or* $(X,\ d,\ \mathrm{Re})$ *is* $\mathrm{Re}-$*regular.*

Then F *has a coupled fixed point.*

In a similar way, we may state the results analog of Corollary 2.1, Corollary 2.2, Corollary 2.3, Corollary 2.4 and Corollary 2.5 for Theorem 3.1 and Corollary 3.1.

4. Application to ordinary differential equations

In this segment, first we obtain the solution for the following first-order periodic problem:

$$\begin{cases} u'(t) = f(t,\ u(t)),\ t \in [0,\ T], \\ u(0) = u(T) \end{cases} \tag{4.1}$$

where $T > 0$ and $f : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Consider the space $X = C(I,\ \mathbb{R})$ $(I = [0,\ T])$ of all continuous functions from I to $\mathbb{R}$, which is a complete metric space with respect to the sup metric

$$d(\tau,\ \eta) = \sup_{t \in I} |\tau(t) - \eta(t)|\,, \text{ for all } \tau,\ \eta \in X.$$

Definition 4.1. A lower solution for (4.1) is a function $a \in C^1(I,\ \mathbb{R})$ such that

$$a'(t) \leq f(t,\ a(t)) \text{ for } t \in I \text{ and } a(0) = a(T) = 0.$$

Theorem 4.1. *Consider problem (4.1) with* $f : I \times \mathbb{R} \to \mathbb{R}$ *is continuous and for* $\tau,\ \eta \in X$ *with* $\tau \leq \eta$,

$$0 \leq f(t,\ \eta) + \lambda\eta - f(t,\ \tau) - \lambda\tau \leq \frac{\lambda}{3}(\eta - \tau),$$

Then the existence of a lower solution of (4.1) provides the existence of a solution of (4.1).

Proof. Equation (4.1) is equivalent to the integral equation

$$u(t) = \int_0^T G(t,\ s)[f(s,\ u(s)) + \lambda u(s)]ds,$$

where $G(t,\ s)$ is the Green function given by

$$G(t,\ s) = \begin{cases} \dfrac{e^{\lambda(T+s-t)}}{e^{\lambda T} - 1}, & 0 \leq s < t \leq T, \\ \dfrac{e^{\lambda(s-t)}}{e^{\lambda T} - 1}, & 0 \leq t < s \leq T. \end{cases}$$

Define now the mapping $\zeta : X \to X$ by

$$\zeta(\tau)(t) = \int_0^T G(t,\ s)[f(s,\ \tau(s)) + \lambda\tau(s)]ds,$$

and a binary relation

$$\mathrm{Re} = \left\{(\tau,\ \eta) \in C^1(I,\ \mathbb{R}) \times C^1(I,\ \mathbb{R}) : \tau(t) \leq \eta(t),\ \text{for all } t \in I\right\}.$$

For any $(\tau,\ \eta) \in \mathrm{Re}$, that is, $\tau \leq \eta$, then by using our assumption, we have

$$f(t,\ \tau) + \lambda\tau \leq f(t,\ \eta) + \lambda\eta.$$

Since $G(t,\ s) > 0$, that for $t \in I$, it implies

$$\begin{aligned} \zeta(\tau)(t) &= \int_0^T G(t,\ s)[f(s,\ \tau(s)) + \lambda\tau(s)]ds \\ &\leq \int_0^T G(t,\ s)[f(s,\ \eta(s)) + \lambda\eta(s)]ds \\ &= \zeta(\eta)(t). \end{aligned}$$

Thus, $(\zeta\tau,\ \zeta\eta) \in \mathrm{Re}$, that is, Re is $\zeta-$closed. Now, for all $(\tau,\ \eta) \in \mathrm{Re}$, we have

$$\begin{aligned}
& d(\zeta(\tau),\ \zeta(\eta)) \\
= \ & \sup_{t\in I} |\zeta(\tau)(t) - \zeta(\eta)(t)| \\
= \ & \sup_{t\in I} \left| \int_0^T G(t,\ s)[f(s,\ \tau(s)) + \lambda\tau(s) - f(s,\ \eta(s)) - \lambda\eta(s)]ds \right| \\
\leq \ & \sup_{t\in I} \left| \int_0^T G(t,\ s) \cdot \frac{\lambda}{3}(\tau(s) - \eta(s))ds \right| \\
\leq \ & \frac{\lambda}{3} d(\tau,\ \eta) \sup_{t\in I} \left| \int_0^T G(t,\ s)ds \right| \\
\leq \ & \frac{\lambda}{3} d(\tau,\ \eta) \sup_{t\in I} \left| \int_0^t \frac{e^{\lambda(T+s-t)}}{e^{\lambda T} - 1} ds + \int_t^T \frac{e^{\lambda(s-t)}}{e^{\lambda T} - 1} ds \right| \\
\leq \ & \frac{1}{3} d(\tau,\ \eta).
\end{aligned}$$

Thus

$$d(\zeta(\tau),\ \zeta(\eta)) \leq \frac{1}{3} d(\tau,\ \eta).$$

Thus the contractive condition of Corollary 2.6 is satisfied with $k = 1/3 < 1$. Finally, let $a \in X$ be a lower solution of (4.1). Then

$$a'(s) + \lambda a(s) \leq f(s,\ a(s)) + \lambda a(s), \text{ for } t \in I.$$

Multiplying both sides by $G(t,\ s)$, we get

$$\int_0^T a'(s)G(t,\ s)ds + \lambda \int_0^T a(t)G(t,\ s)ds \leq \zeta(a)(t), \text{ for } t \in I.$$

Then, for all for $t \in I$, we have

$$\int_0^t a'(s) \frac{e^{\lambda(T+s-t)}}{e^{\lambda T} - 1} ds + \int_t^T a'(s) \frac{e^{\lambda(s-t)}}{e^{\lambda T} - 1} ds + \lambda \int_0^T a(s)G(t,\ s)ds \leq \zeta(a)(t).$$

Using integration by parts and using $a(0) = a(T) = 0$, for all $t \in I$, we get $a(t) \leq \zeta(a)(t)$, that is, $(a,\ \zeta a) \in \mathrm{Re}$. Thus all the hypothesis of Corollary 2.6 are satisfied. Consequently, ζ has a fixed point $\tau \in X$ which is the solution to (4.1) in $X = C(I,\ \mathbb{R})$. $\square$

Now, we study the existence and uniqueness of solution for the following two-point boundary value problem:

$$
(4.2) \qquad \begin{cases} -\tau''(t) = h(t,\ \tau(t),\ \tau(t)),\ \tau \in (0,\ +\infty),\ t \in [0,\ 1], \\ \tau(0) = \tau(1) = 0. \end{cases}
$$

where $h : [0,\ 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function. The space $X = C^2(I,\ \mathbb{R})$ $(I = [0,\ 1])$ denote the set of all continuous functions from I to $\mathbb{R}$, which is a complete metric space with respect to the sup metric

$$
d(\tau,\ \eta) = \sup_{t \in I} |\tau(t) - \eta(t)|\,,\ \text{for all } \tau,\ \eta \in X.
$$

Theorem 4.2. *Under the assumptions*

(i) $h : [0,\ 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is continuous.*

(ii) Suppose that for all $t \in I$, $\tau_1 \leq \tau_2$ *and* $\eta_1 \leq \eta_2$,

$$
0 \leq h(t,\ \tau_2,\ \eta_2) - h(t,\ \tau_1,\ \eta_1) \leq |(\tau_2 - \tau_1) + (\eta_2 - \eta_1)|\,.
$$

(iii) There exists a point $(a,\ b) \in C^2(I,\ \mathbb{R}) \times C^2(I,\ \mathbb{R})$ *such that*

$$
(4.3) \qquad \begin{cases} -a''(t) \leq h(t,\ a(t),\ b(t)),\ t \in [0,\ 1], \\ -b''(t) \leq h(t,\ b(t),\ a(t)),\ t \in [0,\ 1], \\ a(0) = a(1) = b(0) = b(1) = 0. \end{cases}
$$

Then (4.2) has one and only one solution in $C^2(I,\ \mathbb{R})$.

Proof. It is obvious that the solution (in $C^2(I,\ \mathbb{R})$) of (4.2) is equivalent to the solution (in $C(I,\ \mathbb{R})$) of the following Hammerstein integral equation:

$$
\tau(t) = \int_0^1 G(t,\ s)h(s,\ \tau(s),\ \tau(s))ds \text{ for } t \in [0,\ 1],
$$

where $G(t,\ s)$ is the Green function of differential operator $-\dfrac{d^2}{dt^2}$ with Dirichlet boundary condition $\tau(0) = \tau(1) = 0$, that is,

$$
(4.4) \qquad G(t,\ s) = \begin{cases} t(1-s),\ 0 \leq t \leq s \leq 1, \\ s(1-t),\ 0 \leq s \leq t \leq 1. \end{cases}
$$

Define $\varphi,\ \psi,\ \theta : [0,\ +\infty) \to [0,\ +\infty)$ as follows

$$
\psi(t) = t^2,\ \varphi(t) = \begin{cases} \dfrac{1}{32}[t]^2,\ \text{if } 3 < t < 4, \\ \dfrac{1}{64}t^2,\ \text{otherwise,} \end{cases} \quad \text{and } \theta(t) = \begin{cases} \dfrac{1}{64}[t]^2,\ \text{if } 3 < t < 4, \\ 0,\ \text{otherwise.} \end{cases}
$$

Define now the mapping $F : X^2 \to X$ by

$$F(\tau,\ \eta)(t) = \int_0^1 G(t,\ s)h(s,\ \tau(s),\ \eta(s))ds,\ t \in [0,\ 1] \text{ and } \tau,\ \eta \in X,$$

and a binary relation

$$\mathrm{Re} = \left\{(\tau,\ \eta) \in C^2(I,\ \mathbb{R}) \times C^2(I,\ \mathbb{R}) : \tau(t) \leq \eta(t),\ \text{for all } t \in I\right\}.$$

For $(\tau_1,\ \tau_2) \in \mathrm{Re}$ and $(\eta_1,\ \eta_2) \in \mathrm{Re}$, that is, $\tau_1 \leq \tau_2$ and $\eta_1 \leq \eta_2$, then by using our assumption, we have

$$h(t,\ \tau_1,\ \eta_1) \leq h(t,\ \tau_2,\ \eta_2).$$

Since $G(t,\ s) > 0$, that for $t \in I$, it implies

$$\begin{aligned} F(\tau_1,\ \eta_1)(t) &= \int_0^1 G(t,\ s)h(s,\ \tau_1(s),\ \eta_1(s))ds \\ &\leq \int_0^1 G(t,\ s)h(s,\ \tau_2(s),\ \eta_2(s))ds \\ &= F(\tau_1,\ \eta_1)(t). \end{aligned}$$

Thus, $(F(\tau_1, \eta_1), F(\tau_2, \eta_2)) \in \mathrm{Re}$. Similarly, one can obtain that $(F(\eta_1, \tau_1), F(\eta_2, \tau_2)) \in \mathrm{Re}$. Consequently Re is $F-$closed.

Let $\tau_1, \eta_1, \tau_2, \eta_2 \in X$ such that $(\tau_1, \tau_2) \in \mathrm{Re}$ and $(\eta_1, \eta_2) \in \mathrm{Re}$. From (ii), we have

$$\begin{aligned} & d(F(\tau_1,\ \eta_1),\ F(\tau_2,\ \eta_2)) \\ = & \sup_{t \in I} |F(\tau_1,\ \eta_1)(t) - F(\tau_2,\ \eta_2)(t)| \\ = & \sup_{t \in I} \int_0^1 G(t,\ s)[h(s,\ \tau_1(s),\ \eta_1(s)) - h(s,\ \tau_2(s),\ \eta_2(s))]ds \\ \leq & \sup_{t \in I} \int_0^1 G(t,\ s) \cdot |(\tau_1(s) - \tau_2(s)) + (\eta_1(s) - \eta_2(s))|\, ds \\ \leq & (d(\tau_1,\ \tau_2) + d(\eta_1,\ \eta_2)) \sup_{t \in I} \int_0^1 G(t,\ s)ds. \end{aligned}$$

It is easy to verify that

$$\int_0^1 G(t,\ s)ds = -\frac{t^2}{2} + \frac{t}{2} \text{ and } \sup_{t \in [0,\ 1]} \int_0^1 G(t,\ s)ds = \frac{1}{8}.$$

It follows that

$$d(F(\tau_1,\ \eta_1),\ F(\tau_2,\ \eta_2)) \leq \frac{1}{8}(d(\tau_1,\ \tau_2) + d(\eta_1,\ \eta_2)).$$

Similarly

$$d(F(\eta_1,\ \tau_1),\ F(\eta_2,\ \tau_2)) \leq \frac{1}{8}(d(\tau_1,\ \tau_2) + d(\eta_1,\ \eta_2)).$$

Combining them, we get

$$\begin{aligned} & \frac{d(F(\tau_1,\ \eta_1),\ F(\tau_2,\ \eta_2)) + d(F(\eta_1,\ \tau_1),\ F(\eta_2,\ \tau_2))}{2} \\ \leq\ & \frac{1}{8}\left(\frac{d(\tau_1,\ \tau_2) + d(\eta_1,\ \eta_2)}{2}\right). \end{aligned}$$

Thus

$$\begin{aligned} & \psi\left(\frac{d(F(\tau_1,\ \eta_1),\ F(\tau_2,\ \eta_2)) + d(F(\eta_1,\ \tau_1),\ F(\eta_2,\ \tau_2))}{2}\right) \\ =\ & \left(\frac{d(F(\tau_1,\ \eta_1),\ F(\tau_2,\ \eta_2)) + d(F(\eta_1,\ \tau_1),\ F(\eta_2,\ \tau_2))}{2}\right)^2 \\ \leq\ & \frac{1}{64}\left(\frac{d(\tau_1,\ \tau_2) + d(\eta_1,\ \eta_2)}{2}\right)^2 \\ \leq\ & \varphi\left(\frac{d(\tau_1,\ \tau_2) + d(\eta_1,\ \eta_2)}{2}\right) - \theta\left(\frac{d(\tau_1,\ \tau_2) + d(\eta_1,\ \eta_2)}{2}\right), \end{aligned}$$

which is the contractive condition of Corollary 3.1. By (iii), there exists a point $(a, b) \in C^2(I, \mathbb{R}) \times C^2(I, \mathbb{R})$ such that

$$-a''(s) \leq h(s,\ a(s),\ b(s)),\ s \in [0,\ 1].$$

Multiplying by $G(t, s)$, we get

$$\int_0^1 -a''(s)G(t,\ s)ds \leq F(a,\ b)(t),\ t \in [0,\ 1].$$

Then, for all $t \in [0, 1]$, we have

$$-(1-t)\int_0^t sa''(s)ds - t\int_t^1 (1-s)a''(s)ds \leq F(a,\ b)(t).$$

Using an integration by parts and $a(0) = a(1) = 0$, for all $t \in [0, 1]$, we get

$$-(1-t)(ta'(t) - a(t)) - t(-(1-t)a'(t) - a(t)) \leq F(a,\ b)(t).$$

Thus, we have

$$a(t) \leq F(a,\ b)(t),\ \text{for } t \in [0,\ 1].$$

This implies that $(a,\, F(a,\, b)) \in \mathrm{Re}$. Similarly, one can show that $(b,\, F(b,\, a)) \in \mathrm{Re}$. Thus all the hypothesis of Corollary 3.1 are satisfied. Consequently, F has a coupled fixed point $(\tau,\, \eta) \in X^2$ which is the solution to (4.3) in $X = C^2(I,\, \mathbb{R})$. $\square$

Conclusion 4.1. *Using the same technique, one can easily obtain the multidimensional version of Theorem 2.1.*

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