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ON CONFORMALLY FLAT p-POWER (α, β)-METRICS

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Abstract. The purpose of this paper is to study the class of conformally flat p-power (α, β)-metrics $F = \alpha(1 + \frac{\beta}{\alpha})^p$, where $p \neq 0$ is a constant. This metric is interesting, because for $p = -1, \frac{1}{2}, 1, 2$ it reduces to the Matsumoto, square-root, Randers and square metric, respectively. We prove that if a p-power (α, β)-metric has relatively isotropic mean Landsberg curvature, then it is either a Riemannian metric or a locally Minkowski metric.
Keywords: Conformally flat metric, p-power (α, β)-metric, mean Landsberg curvature.

1. Introduction

Conformal geometry has many important and interesting applications in physical theories, which have led to increased attention and research. In general relativity, light-like geodesics remain invariant under the conformal relation between pseudo-Riemannian metrics. The Weyl theorem states that the study of conformal and projective properties of a Finsler metric characterizes the metric properties as uniquely [13, 19].

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Two Finsler metrics F and $\tilde{F}$, defined on a differentiable manifold M, are said to be conformally related if $F = e^{\kappa(x)}\tilde{F}$, where $\kappa(x)$ is a scalar function on M and is referred to as the conformal factor. If $\tilde{F}$ is a locally Minkowski metric, we say that F is a conformally flat metric.

S. Kikuchi found a Finsler connection that is conformally invariant and he expressed the conformal flatness of a Finsler metric in terms of this connection [12]. Based on Kikuchi's idea, M. Matsumoto introduced a conformal invariant Finsler connection for Finsler metrics that have a tensor satisfying in a certain condition, weaker than Kikuchi's condition, then the condition that a Finsler metric is conformally flat stated by the terms of this connection [16].

Ichijyō and Hashiguchi proposed a condition that a Randers metric can be conformally flat [10]. Randers metrics are the simplest examples of (α, β)-metrics which made an important class of Finsler metrics. They have wide-ranging applications in physics, biology, etc (see [2]).

A Finsler metric that is represented as $F = \alpha\phi(s),\ s := \beta/\alpha$ is called a (α, β)-metric where $\alpha(x, y) := \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric, $\beta(x, y) := b_i(x)y^i$ is a 1-form and $\phi(s)$ is a C^∞ function that satisfies a certain inequality [15]. The one-form β can be considered as an external force like wind and current. Thus (α, β)-metrics can express the geometry of a Riemannian space impacted by an external force. These metrics have been extensively studied because they are computable and the research on (α, β)-metrics have contributed significantly to the field of Finsler geometry and have suggested several ideas for further studies.

L. Kang considered conformally flat Randers metrics of scalar flag curvature and proved that they are projectively flat and also provided a complete classification of such metrics [11]. Conformally flat (α, β)-metrics with isotropic S-curvature are considered in [3]. It is shown that they are Riemannian or locally Minkowski metric and also classified conformally flat weak Einstein (α, β)-metrics of polynomial type. In [5] it is proved that every non-Riemannian conformally flat weakly Landsberg (α, β)-metric must be a locally Minkowski metric. Chen et al. studied conformally flat (α, β)-metrics with constant flag curvature. Their research showed that these metrics are either locally Minkowski or Riemannian metrics [4]. Tayebi and Razgordani studied conformally flat weak Einstein fourth root (α, β)-metrics and proved that they are also either locally Minkowskian or Riemannian [23]. For more references, refer to [1, 17, 21, 22]

The class of p-power (α, β)-metrics are of the form

$$F = \alpha\Big(1 + \frac{\beta}{\alpha}\Big)^p,$$

where $p \neq 0$ is a constant. For $p = 1$, F is a Randers metric. The Randers metrics were introduced by G. Randers, when he was studying general relativity [18]. Later on, R. S. Ingarden applied this metric to the theory of the electron microscope [2]. If $p = -1$ then $F = \alpha^2/(\beta + \alpha)$ is a Matsumoto metric. This metric was introduced by M. Matsumoto and is also called the slope metric. As a geometrical motivation for the Matsumoto metric, suppose that a person is walking with the constant speed v on a surface S that has the angle θ with respect to sea level and under the gravitational field g. We can embedded S in the Euclidean space $\mathbb{E}^3$ using the parametrization

$$(x, y) \longmapsto (x, y, f(x, y)),$$

where $f(x, y)$ is a smooth function. By use of Okubo's method [2], we obtain the Matsumoto metric

$$F(x, y, \dot{x}, \dot{y}) = \frac{\alpha^2}{v\alpha - \frac{g}{2}\beta},$$

where

$$\alpha := \sqrt{(1 + f_x^2)\dot{x}^2 + 2f_x f_y \dot{x}\dot{y} + (1 + f_y^2)\dot{y}^2},$$
$$\beta := f_x\dot{x} + f_y\dot{y}.$$

In order to obtain the usual form of the Matsumoto metric, we set $v = -\frac{g}{2}$ and in this case, we have $F = \frac{\alpha^2}{\alpha+\beta}$. For more details see [20].

Also, in the case of $p = \frac{1}{2}$, $F = \sqrt{\alpha(\alpha + \beta)}$ that is called square-root (α, β)-metric. In [24] p-power (α, β)-metrics of Einstein-reversible type have been considered and the local structure of a two-dimensional square-root metric is determined. Thus, the class of p-power (α, β)-metrics deserve more attention.

For a Finsler metric F, we have the basic tensors, fundamental tensor $\mathbf{g}_y$ and Cartan torsion $\mathbf{C}$. By taking horizontal covariant derivative of Cartan torsion along the geodesics we obtain the tensor field $\mathbf{L}$ that is called Landsberg curvature. The trace of $\mathbf{C}$ and $\mathbf{L}$ are called the mean Cartan torsion $\mathbf{I}$ and the mean Landsberg curvature $\mathbf{J}$, respectively. A Finsler metric F is called relatively isotropic mean Landsberg curvature if there exists a scalar function $c = c(x)$ on M such that

$$\mathbf{J} + cF\mathbf{I} = 0. \tag{1.1}$$

As to find an explicit example of conformally flat Finsler metric, this paper is devoted to the study of the conformally flat p-power (α, β)-metric, that has relatively isotropic mean Landsberg curvature. The main result is the following.

Theorem 1.1. *Let $F = \alpha\big(1 + \frac{\beta}{\alpha}\big)^p$ be the conformally flat (α, β)-metric on a differentiable manifold M of dimension $n \geqslant 3$, where $p \neq 0$ is a real constant. Suppose that F has relatively isotropic mean Landsberg curvature. Then F reduces to a Riemannian metric or a locally Minkowski metric.*

As the consequences of Theorem 1.1, we have the following conclusions, that retrieve other researcher's results.

Corollary 1.1. *Let $F = \alpha + \beta$ be the conformally flat Randers metric on a differentiable manifold M of the dimension $n \geq 3$. Suppose that F has relatively isotropic mean Landsberg curvature. Then F is a Riemannian metric or a locally Minkowski metric.*

It would be noted that, for $F = \alpha + \beta$, it is proved in [6] that every Randers metric is of isotropic mean Landsberg curvature if and only if it is of isotropic S-curvature. On the other hand every conformally flat Randers metric with isotropic S-curvature is either a locally Minkowski or a Riemannian metric [11]. Thus, using these results, we have additional proof for Corollary 1.1.

Corollary 1.2. *Let $F = \sqrt{\alpha(\alpha+\beta)}$ be the conformally flat square-root (α, β)-metric on a differentiable manifold M of the dimension $n \geq 3$. Suppose that F has relatively isotropic mean Landsberg curvature. Then F is a Riemannian metric or a locally Minkowski metric.*

In [17] the conformally flat square-root (α, β)-metrics of relatively isotropic mean Landsberg curvature is studied and Corollary 1.2 is proved.

Corollary 1.3. *Let $F = \frac{\alpha^2}{\alpha+\beta}$ be the conformally flat Matsumoto metric on a differentiable manifold M of the dimension $n \geq 3$. Suppose that F has relatively isotropic mean Landsberg curvature. Then F is a Riemannian metric or a locally Minkowski metric.*

2. Preliminaries

Let $F = F(x, y)$ be a Finsler metric on an n-dimensional differentiable manifold M and $TM_0 := \bigcup_{x \in M} T_x M - \{0\}$ the slit tangent bundle. The fundamental tensor $(\mathbf{g}_y) = (g_{ij}(x, y))$ of F is a quadratic form on $T_x M$ that is defined

$$g_{ij}(x, y) := \frac{1}{2}[F^2]_{y^i y^j}(x, y).$$

A curve $x = x^i(t)$ on Finsler space (M, F) is called geodesic if satisfies in the following system of ODEs:

$$\frac{d^2 x^i}{dt^2} + G^i(x, \frac{dx}{dt}) = 0,$$

where $G^i = G^i(x, y)$ are called the geodesic coefficients of F and defined by

$$G^i = \frac{1}{4} g^{il} \Big\{ [F^2]_{x^m y^l} y^m - [F^2]_{x^l} \Big\}.$$

In Finsler geometry, there are some geometric quantities that are vanishing for Riemannian metrics and are called non-Riemannian quantities. The Cartan torsion $\mathbf{C}$ is a symmetric trilinear form $\mathbf{C} := C_{ijk} dx^i \otimes dx^j \otimes dx^k$ on TM_0 that is defined as follow

$$C_{ijk} := \frac{1}{4}[F^2]_{y^i y^j y^k} = \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k}.$$

One can see that F is a Riemannian metric if and only if $\mathbf{C} = 0$. Thus it is a non-Riemannian quantity.

The mean Cartan torsion of F is the tensor field $\mathbf{I} := I_i dx^i$, that is defined by

$$I_i := g^{jk} C_{ijk}.$$

Furthermore, one can see that

$$I_i = \frac{\partial}{\partial y^i}\Big[\ln \sqrt{\det(g_{jk})}\Big]$$

The horizontal covariant derivative of Cartan torsion along the geodesics defines the tensor field $\mathbf{L} := L_{ijk} dx^i \otimes dx^j \otimes dx^k$ on the slit tangent bundle TM_0, that is called the Landsberg curvature of F. Thus $L_{ijk} := C_{ijk;m} y^m$, where ";" denoted the horizontal covariant derivative with respect to the Berwald connection of F. Also, the Landsberg curvature can be expressed as the following

$$L_{ijk} = -\frac{1}{2} F F_{y^m} [G^m]_{y^i y^j y^k}. \tag{2.1}$$

A Finsler metric F is called the Landsberg metric if $\mathbf{L} = 0$.

The mean Landsberg curvature $\mathbf{J} := J_i dx^i$ is a non-Riemannian quantity that is obtained by horizontal covariant derivative of the mean Cartan torsion $\mathbf{I}$ along the geodesics of F. Thus

$$J_i := I_{i;m} y^m. \tag{2.2}$$

Also, the mean Landsberg curvature $\mathbf{J}$ can be obtained as following

$$J_i := g^{jk} L_{ijk}.$$

A Finsler metric F is called weak Landsberg metric if $\mathbf{J} = 0$.

A Finsler metric F is called of relatively isotropic mean Landsberg curvature if $\mathbf{J}/\mathbf{I}$, the relative growth rate of the mean Cartan torsion along geodesics of F is isotropic, i.e. there exists a scalar function $c = c(x)$ on M such that

$$\mathbf{J} + cF\mathbf{I} = 0.$$

A Finsler metric F is an (α, β)-metric if $F = \alpha\phi(s)$, $\quad s := \beta/\alpha$, where $\alpha = \sqrt{a_{ij}(x) y^i y^j}$ is a Riemannian metric, $\beta = b_i(x) y^i$ is a 1-form with $||\beta_x|| < b_0$, $x \in M$ and $\phi(s)$ is a positive C^∞ function on $(-b_0, b_0)$ satisfying

$$\phi(s) - s\phi^{'}(s) + (b^2 - s^2)\phi^{''}(s) > 0, \quad |s| \le b < b_0. \tag{2.3}$$

In this case, the metric $F = \alpha\phi(s)$ is a positive definite Finsler metric [9]. The fundamental tensor $F = \alpha\phi(s)$ is given by

$$g_{ij} = \rho a_{ij} + \rho_0 b_i b_j + \rho_1 (b_i \alpha_i + b_j \alpha_i) + \rho_2 \alpha_i \alpha_j,$$

where $\alpha_i := \alpha^{-1} a_{ij} y^j$, and

$$\rho := \phi(\phi - s\phi'), \qquad \rho_0 := \phi\phi'' + \phi'\phi',$$
$$\rho_1 := -s(\phi\phi'' + \phi'\phi') + \phi\phi', \qquad \rho_2 := s\{s(\phi\phi'' + \phi'\phi') - \phi\phi'\}.$$

Let

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i})$$

where $b_{i|j}$ denote the coefficients of the covariant derivative of β with respect to α. Let's denote

$$\begin{array}{lll} r^i{}_j := a^{im} r_{mj}, & r_{00} := r_{ij} y^i y^j, & r_i := b^m r_{mi}, \\ r_0 := r_i y^i, & r_{i0} := r_{im} y^m, & s^i{}_j := a^{im} s_{mj}, \\ s_{i0} := s_{im} y^m, & s_i := b^m s_{mi}, & s_0 := s_i y^i, \end{array}$$

The geodesic coefficients G^i of an (α, β)-metric $F = \alpha\phi(s)$ are given by

$$G^i = G^i_\alpha + \alpha Q s^i{}_0 + \{-2Q\alpha s_0 + r_{00}\}\Big\{\Psi b^i + \Theta \alpha^{-1} y^i\Big\}, \tag{2.4}$$

where G^i_α are the geodesic coefficients of α and

$$\begin{aligned} Q &:= \frac{\phi'}{\phi - s\phi'}, \\ \Theta &:= \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi[\phi - s\phi' + (b^2 - s^2)\phi'']}, \\ \Psi &:= \frac{\phi''}{2[\phi - s\phi' + (b^2 - s^2)\phi'']}. \end{aligned}$$

For more details, see [9].

3. Proof of Theorem 1.1

In this section, we focus on a class of (α, β)-metrics, that contain Randers metric, square metric, square-root metric, Matsumoto metric, etc. This class of metrics is called p-power (α, β)-metrics and is of the form

$$F = \alpha(1 + s)^p, \;\; s := \beta/\alpha,$$

where $p \neq 0$ is a real constant. For more details see [24].

We consider p-power (α, β)-metric with relatively isotropic mean Landsberg curvature and prove Theorem 1.1. In [8], the mean Cartan torsion of an (α, β)-metric is computed.

Lemma 3.1. ([8])*For an (α, β)-metric $F = \alpha\phi(s)$, $s = \beta/\alpha$, the mean Cartan torsion is given by*

$$I_i = -\frac{1}{2F}\frac{\Phi}{\Delta}(\phi - s\phi')h_i, \tag{3.1}$$

where

$$\begin{aligned}
&\Delta := 1 + sQ + (b^2 - s^2)Q',\\
&\Phi := -(n\Delta + 1 + sQ)(Q - sQ') - (b^2 - s^2)(1 + sQ)Q'',\\
&\Psi_1 := \sqrt{b^2 - s^2}\,\Delta^{\frac{1}{2}}\left[\frac{\sqrt{b^2 - s^2}\,\Phi}{\Delta^{\frac{3}{2}}}\right]',\\
&h_j := b_j - \alpha^{-1} s y_j.
\end{aligned}$$

It is well known that, by Deicke's theorem, F is a Riemannian metric if and only if $\mathbf{I} = 0$. Thus from (3.1) we have

Lemma 3.2. *An (α,β)-metric $F = \alpha\phi(s)$, $s := \beta/\alpha$ is a Riemannian metric if and only if $\Phi = 0$.*

From (2.2) and (3.1), one can see that the mean Landsberg curvature of an (α,β)-metric $F = \alpha\phi(s)$, $s = \beta/\alpha$, is given by

$$\begin{aligned}
J_j = \frac{1}{2\alpha^4\Delta}\Bigg\{&\frac{2\alpha^3}{b^2 - s^2}\Big[\frac{\Phi}{\Delta} + (n+1)(Q - sQ')\Big](s_0 + r_0)h_j\\
&+\frac{\alpha^2}{b^2 - s^2}\Big[\Psi_1 + s\frac{\Phi}{\Delta}\Big](r_{00} - 2\alpha Q s_0)h_j\\
&+\alpha\Big[-\alpha^2 Q' s_0 h_j + \alpha Q(\alpha^2 s_j - y_j s_0) + \alpha^2\Delta s_{j0}\\
&+\alpha^2(r_{j0} - 2\alpha Q s_j) - (r_{00} - 2\alpha Q s_0)y_j\Big]\frac{\Phi}{\Delta}\Bigg\},
\end{aligned}\tag{3.2}$$

where $y_j := a_{ij}y^i$. For more details see [8, 14]. From (3.1) and (3.2), we obtain

$$\begin{aligned}
J_j + c(x)FI_j = -\frac{1}{2\alpha^4\Delta}\Bigg\{&\frac{2\alpha^3}{b^2 - s^2}\Big[\frac{\Phi}{\Delta} + (n+1)(Q - sQ')\Big](s_0 + r_0)h_j\\
&+\frac{\alpha^2}{b^2 - s^2}\Big[\Psi_1 + s\frac{\Phi}{\Delta}\Big](r_{00} - 2\alpha Q s_0)h_j + \alpha\Big[-\alpha^2 Q' s_0 h_j\\
&+\alpha Q(\alpha^2 s_j - y_j s_0) + \alpha^2\Delta s_{j0} + \alpha^2(r_{j0} - 2\alpha Q s_j)\\
&-(r_{00} - 2\alpha Q s_0)y_j\Big]\frac{\Phi}{\Delta} + c(x)\alpha^4\Phi(\phi - s\phi')h_j\Bigg\}.
\end{aligned}\tag{3.3}$$

Since we study conformally flat (α,β)-metrics, we need the following Lemma.

Lemma 3.3. ([2]) *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α,β)-metric. Then F is locally Minkowski metric if and only if α is flat and β is parallel with respect to α.*

Now, let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be a conformally flat Finsler metric, it means that, there exists a Minkowski metric $\tilde{F}$ such that $\tilde{F} = e^{\kappa(x)}F$, where $\kappa(x)$ is a scalar function on the manifold. Since $F = \alpha\phi(\beta/\alpha)$, we deduce $\tilde{F} = \tilde{\alpha}\phi(\tilde{\beta}/\tilde{\alpha})$ is an (α, β)-metric, where

$$\tilde{\alpha} = e^{\kappa(x)}\alpha, \quad \tilde{\beta} = e^{\kappa(x)}\beta. \tag{3.4}$$

From (3.4), we have

$$\tilde{a}_{ij} = e^{2\kappa(x)}a_{ij}, \quad \tilde{b}_i = e^{\kappa(x)}b_i.$$

The Christoffel symbols Γ^i_{jk} of α and the Christoffel symbols $\tilde{\Gamma}^i_{jk}$ of $\tilde{\alpha}$ are related by

$$\tilde{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j\kappa_k + \delta^i_k\kappa_j - \kappa^i a_{jk},$$

where $\kappa_i := \frac{\partial\kappa}{\partial x^i}$ and $\kappa^i := a^{ij}\kappa_j$ [7]. Thus, we obtain

$$\tilde{b}_{i\|j} = \frac{\partial\tilde{b}_i}{\partial x^j} - \tilde{b}_s\tilde{\Gamma}^i_{jk} = e^{\kappa}(b_{i|j} - b_j\kappa_i + b_r\kappa^r a_{ij}). \tag{3.5}$$

where $\tilde{b}_{i\|j}$ denote the coefficients of the covariant derivative of $\tilde{\beta}$ with respect to $\tilde{\alpha}$.

Since $\tilde{F}$ is a Minkowski metric, from Lemma 3.3, we have $\tilde{b}_{i\|j} = 0$. Thus

$$b_{i|j} = b_j\kappa_i - b_r\kappa^r a_{ij}. \tag{3.6}$$

From (3.6), we conclude

$$r_{ij} = \frac{1}{2}(\kappa_i b_j + \kappa_j b_i) - b_r\kappa^r a_{ij}, \qquad r_j = -\frac{1}{2}(b^r\kappa_r)b_j + \frac{1}{2}\kappa_j b^2, \tag{3.7}$$

$$r_{i0} = \frac{1}{2}[\kappa_i\beta + (\kappa_r y^r)b_i] - \kappa_r b^r y_i, \qquad s_{ij} = \frac{1}{2}(\kappa_i b_j - \kappa_j b_i), \tag{3.8}$$

$$s_j = \frac{1}{2}(b^r\kappa_r)b_j - \kappa_j b^2, \qquad s_{i0} = \frac{1}{2}[\kappa_i\beta - (\kappa_r y^r)b_i]. \tag{3.9}$$

Further, we have

$$r_{00} = (\kappa_r y^r)\beta - (\kappa_r b^r)\alpha^2, \tag{3.10}$$

$$r_0 = \frac{1}{2}(\kappa_r y^r)b^2 - \frac{1}{2}(\kappa_r b^r)\beta, \tag{3.11}$$

$$s_0 = \frac{1}{2}(\kappa_r b^r)\beta - \frac{1}{2}(\kappa_r y^r)b^2. \tag{3.12}$$

From (3.11) and (3.12), we see that a conformally flat (α, β)-metric satisfyies $r_0 + s_0 = 0$ which means that the 1-form β has constant length with respect to α.

In order to simplify the computations, we take an orthonormal basis at any point x with respect to α such that $\alpha = \sqrt{\sum_{i=1}^n (y^i)^2}$ and $\beta = by^1$, where $b := \|\beta_x\|_\alpha$. Then, we take the following coordinate transformation

$$\psi : (s, u^A) \longrightarrow (y^i),$$

in T_xM, that is

$$y^1 = \frac{s}{\sqrt{b^2 - s^2}}\bar{\alpha}, \quad y^A = u^A, \ \ 2 \leq A \leq n, \tag{3.13}$$

where $\bar{\alpha} = \sqrt{\sum_{i=2}^n (u^A)^2}$. In this case, we have

$$\alpha = \frac{b}{\sqrt{b^2 - s^2}}\bar{\alpha}, \qquad \beta = \frac{bs}{\sqrt{b^2 - s^2}}\bar{\alpha}. \tag{3.14}$$

Then, by (3.6)-(3.14) one can obtain

$$r_{00} = -b\kappa_1\bar{\alpha}^2 + \frac{bs\bar{\kappa}_0\bar{\alpha}}{\sqrt{b^2 - s^2}}, \qquad r_0 = -s_0 = \frac{1}{2}b^2\bar{\kappa}_0, \tag{3.15}$$

$$r_{A0} = \frac{1}{2}\frac{\kappa_A bs\bar{\alpha}}{\sqrt{b^2 - s^2}} - (b\kappa_1)u_A, \quad r_{10} = \frac{1}{2}b\bar{\kappa_0}, \tag{3.16}$$

$$s_A = -\frac{1}{2}\kappa_A b^2, \qquad s_1 = 0, \tag{3.17}$$

$$s_{A0} = \frac{1}{2}\frac{\kappa_A bs\bar{\alpha}}{\sqrt{b^2 - s^2}}, \qquad s_{10} = -\frac{1}{2}b\bar{\kappa}_0, \tag{3.18}$$

$$h_A = -\frac{\sqrt{b^2 - s^2}su_A}{b\bar{\alpha}}, \qquad h_1 = b - \frac{s^2}{b}. \tag{3.19}$$

where $\bar{\kappa}_0 := \kappa_A u^A$.

Now, we can prove Theorem 1.1.

Proof of Theorem 1.1: Since $\tilde{b}_{i\|j} = 0$, we have that $\tilde{b}$ is a real constant. If $\tilde{b} = 0$, then $F = e^{k(x)}\tilde{\alpha}$ is a Riemannian metric. Now, let F is not Riemannian. Suppose that F is a conformally flat (α, β)-metric with relatively isotropic mean Landsberg curvature. By (3.3) and $r_0 + s_0 = 0$, we obtain

$$\begin{aligned} &\frac{\alpha^2}{b^2 - s^2}\big\{\Psi_1 + s\frac{\Phi}{\Delta}\big\}(r_{00} - 2\alpha Q s_0)h_j + \alpha\big\{ - \alpha^2 Q' s_0 h_j \\ &\quad +\alpha Q(\alpha^2 s_j - y_j s_0) + \alpha^2\Delta s_{j0} + \alpha^2(r_{j0} - 2\alpha Q s_j) \\ &\quad -(r_{00} - 2\alpha Q s_0)y_j\big\}\frac{\Phi}{\Delta} - c(x)\alpha^4\Phi(\phi - s\phi')h_j = 0. \end{aligned} \tag{3.20}$$

Putting $j = 1$ in (3.20), we have

$$\begin{aligned} &\frac{\alpha^2}{b^2 - s_2}\big\{\Psi_1 + s\frac{\Phi}{\Delta}\big\}(r_{00} - 2\alpha Q s_0)h_1 + \alpha\big\{ - \alpha^2 Q' s_0 h_1 \\ &\quad +\alpha Q(\alpha^2 s_1 - y_1 s_0) + \alpha^2\Delta s_{10} + \alpha^2(r_{10} - 2\alpha Q s_1) \\ &\quad -(r_{00} - 2\alpha Q s_0)y_1\big\}\frac{\Phi}{\Delta} - c(x)\alpha^4\Phi(\phi - s\phi')h_1 = 0. \end{aligned} \tag{3.21}$$

Substituting (3.14)-(3.19) into (3.21) and then multiplying the resulting equation with $-2\Delta(b^2 - s^2)^{3/2}$ we have

$$b^2\bar{\alpha}^3\Big\{2\sqrt{b^2 - s^2}\Delta\big[bc(x)\Phi(\phi - s\phi') + \Psi_1\sigma_1\big]\bar{\alpha} - \bar{\kappa}_0\big[b^2\Phi Q'(b^2 - s^2)$$

$$+\Phi b^2(sQ+1)-\Delta\Phi b^2-2\Psi_1\Delta(b^2Q+s)\big]\Big\}=0. \tag{3.22}$$

From (3.22), we get

$$\Delta\big[bc(x)\Phi(\phi-s\phi')+\Psi_1\kappa_1\big]=0, \tag{3.23}$$

$$\bar{\kappa}_0\big[b^2\Phi Q'(b^2-s^2)+\Phi b^2(sQ+1)-\Delta\Phi b^2-2\Psi_1\Delta(b^2Q+s)\big]=0. \tag{3.24}$$

Since $\Delta=Q'(b^2-s^2)+sQ+1$, one can see that (3.24) simplify as follow

$$(b^2\Psi_1\Delta Q+\Psi_1\Delta s)\bar{\kappa}_0=0.$$

This means that

$$\Psi_1\Delta(b^2Q+s)\bar{\kappa}_0=0. \tag{3.25}$$

Now let $j=A$ in (3.20), thus we have

$$\begin{aligned}&\frac{\alpha^2}{b^2-s^2}\big[\Psi_1+s\frac{\Phi}{\Delta}\big](r_{00}-2\alpha Qs_0)h_A+\alpha\big[-\alpha^2Q's_0h_A+\\ &\qquad\alpha Q(\alpha^2s_A-y_As_0)+\alpha^2\Delta s_{A0}+\alpha^2(r_{A0}-2\alpha Qs_A)\\ &\qquad-(r_{00}-2\alpha Qs_0)y_A\big]\frac{\Phi}{\Delta}+c(x)\alpha^4\Phi(\phi-s\phi')h_A=0.\end{aligned} \tag{3.26}$$

Putting (3.14)-(3.19) into (3.26) and using the same method as used in the case of $j=1$ and from $\Delta=Q'(b^2-s^2)+sQ+1$, we get

$$\begin{aligned}&(s\Delta+s+b^2Q)b^2\Phi\kappa_A\bar{\alpha}^2-\big[(s\Delta+s+b^2Q)b^2\Phi\\ &\qquad+2s(b^2Q+s)\Psi_1\Delta\big]\bar{\kappa}_0u_A=0,\end{aligned} \tag{3.27}$$

$$s\sqrt{b^2-s^2}[\Phi bc(x)(\phi-s\phi')+\Psi_1\kappa_1]\Delta u_A=0. \tag{3.28}$$

One can easily see that (3.28) is equivalent to (3.23). Also, multiplying (3.27) with u^A implies that

$$s(b^2Q+s)\Psi_1\Delta\bar{\kappa}_0\bar{\alpha}^2=0. \tag{3.29}$$

It is obvious that (3.29) is equivalent to (3.25). Anyway, we showed that a conformally flat (α,β)-metric with relatively isotropic mean Landsberg curvature satisfiy (3.23) and (3.25).

If $b^2Q+s=0$, then we obtain $\phi=k\sqrt{b^2-s^2}$, where k is a constant. This is a contradiction with the assumption that $\phi=(1+s)^p$. Thus $b^2Q+s\neq 0$ and then from (3.25) we conclude that $\Psi_1=0$ or $\kappa_A=0$.

If $\Psi_1=0$, then using (3.23) we obtain that $\Phi=0$, and from Lemma 3.2, we see that F is a Riemannian metric.

If $\Psi_1\neq 0$, then $\kappa_A=0$. In this case , we prove that $\kappa_1=0$.
Simplifying (3.23) and multiplying it by Δ^2, we get

$$\Big\{[(b^2-s^2)\Phi'-s\Phi]\Delta-\frac{3}{2}(b^2-s^2)\Phi\Delta'\Big\}\kappa_1-bc(x)\Delta^2\Phi(\phi-s\phi')=0. \tag{3.30}$$

Let $A_1 := (p-1)s-1$ and $A_2 := (1-p^2)s^2+(2-p)s+p(p-1)b^2+1$. Putting $\phi(s)=(1+s)^p$ into (3.30) and multiplying by $A_1^4A_2^2$ and using Maple program, we obtain

$$A_1A_2^2Ec(x)bp(1+s)^{p-1}+\mathcal{M}_7s^7+\ldots+\mathcal{M}_0=0, \tag{3.31}$$

where

$$\begin{aligned} E := {} & 2n(p+1)(p-1)^2s^3+(p-1)\big[(p-5)n-3\big]s^2 \\ & -\big\{2\big(np^3-(2n+1)p^2+(n+2)p-1\big)b^2+(n+1)(3p-4)\big\}s \\ & +\big[(n+2)p^2-(n+4)p+2\big]b^2+(1+n), \end{aligned} \tag{3.32}$$

and $\mathcal{M}_i,\ (0\le i\le 7)$ are polynomials independent from s. Specially

$$\mathcal{M}_7 := 3(n-1)p(p+1)(p-1)^4\kappa_1. \tag{3.33}$$

Now, we consider the following cases.

Case(I): If $p\ge 2$, is a positive integer constant. Then (3.31) can be rewritten as

$$\mathcal{M}'_{7+p}s^{7+p}+\mathcal{M}'_{6+p}s^{6+p}+\ldots+\mathcal{M}'_0=0. \tag{3.34}$$

where $\mathcal{M}'_{7+p} := 2nbp(p+1)^3(p-1)^5c(x)$. Thus $c(x)=0$ and therefore (3.34) is reduced to

$$\mathcal{M}_7s^7+\ldots+\mathcal{M}_0=0. \tag{3.35}$$

From (3.35) and (3.33) we obtain that $\kappa_1=0$.

Case(II): If $p=1$, then $F=\alpha+\beta$ is a Randers metric. In this case (3.31) is reduced to

$$(\kappa_1-2bc(x))s^2+2(\kappa_1-2bc(x))s+b(\kappa_1b-2c(x))=0. \tag{3.36}$$

From (3.36) it follows that $c(x)=\kappa_1=0$.

Case (III): If p is a positive non-integer constant. Then, from (3.31) and (3.32), we hvave that $c(x)=\mathcal{M}_i=0,\ (0\le i\le 7)$. Thus from (3.33), it follows that $\kappa_1=0$.

Case (IV): If $p=-1$. In this case $F=\frac{\alpha^2}{\alpha+\beta}$, is a Matsumoto metric. Putting $\phi(s)=\frac{1}{1+s}$ in (3.30) and multiplying by $A_1^4A_2^2(1+s)^2$, we get

$$18(2n+1)\kappa_1s^8+\mathcal{D}_7s^6+\ldots+\mathcal{D}_0=0.$$

Thus $\kappa_1=0$.

Case (V): If $p\le -2$ is a negative integer constant. In this case, Eq. (3.31) can be rewritten as follows.

$$\mathcal{N}_{8-p}s^{8-p}+\mathcal{N}_{7-p}s^{7-p}+\ldots+\mathcal{N}_1s+\mathcal{N}_0=0, \tag{3.37}$$

where $\mathcal{N}_{8-p} = 3(n-1)p(p+1)(p-1)^4\kappa_1$. Thus $\kappa_1 = 0$.

Case (VI): If p is a negative non-integer constant. In this case, (3.31) is reduced to

$$A_1 A_2^2 Ec(x)bp + (1+s)^{1-p}[\mathcal{M}_7 s^7 + \ldots + \mathcal{M}_0] = 0. \tag{3.38}$$

From (3.33), and (3.38) we have $\kappa_1 = c(x) = 0$.

Therefore, in any case $\kappa_1 = \kappa_A = 0$, which means that κ is a constant. Thus F is a locally Minkowski metric. This completes the proof. □

It would be noted that, in the Case (I), $\phi(s) = (1+s)^p$ is a polynomials of s. In [5] it is proved that every (α, β)-metric $F = \alpha\phi(s)$ with relatively isotropic mean Landsberg curvature, where $\phi(s)$ is a polynomial of s, is either a Riemannian or a locally Minkowski metric.

4. Conclusion

In this paper, we considered conformally flat p-power (α, β)-metrics. We proved that every conformally flat p-power (α, β)-metrics with relatively isotropic mean Landsberg curvature is a Riemannian or a locally Minkowski metric. Therefore, this class of (α, β)-metrics are classified. The readers are encouraged to consider the class of conformally flat p-power (α, β)-metrics with other Riemannian or non-Riemannian quantities, such as weakly isotropic scalar flag curvature, weak Einstein metric, etc.

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