FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 40, No 1 (2025), 185-196
https://doi.org/10.22190/FUMI240301014A
Original Scientific Paper

ON GENERALIZED FRAMED AND FRENET-TYPE FRAMED BERTRAND CURVES IN EUCLIDEAN 3-SPACE

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Abstract. In this paper, we consider Framed and Frenet-type Framed Bertrand curves. We generalized the notion of Framed and Frenet-type Framed Bertrand curves in Euclidean 3-space. According to this generalization, the Bertrand curve conditions of a given Framed and Frenet-type Framed Bertrand curves are obtained and the relations between the moving frames and curvature functions are given.
Keywords: Euclidean space, Bertrand curve, framed curve.

1. Introduction

The relations between Frenet vectors of one space curve and Frenet vectors of another space curve in Euclidean 3-space result in many known pairs of curves. One of them is Bertrand curves. In 1845, Saint Venant [21] proposed the question of whether the principal normal of a curve is the principal normal of another on the surface generated by the principal normal of the given one. Bertrand answered this question in [2], published in 1850. He proved that a necessary and sufficient condition for the existence of the second curve is required; in fact, a linear relationship calculated with constant coefficients should exist between the first and second curvatures of the given original curve. In other words, if we denote the first and second curvatures of a given curve by k_1 and k_2 respectively, we have $\lambda k_1 + \mu k_2 = 1$,

Received: March 01, 2024, revised: April 16, 2024, accepted: April 16, 2024
Communicated by Mića Stanković
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2020 *Mathematics Subject Classification.* Primary 53A04; Secondary 53D35

$\lambda \in \mathbb{R}_0$, $\mu \in \mathbb{R}$. Since 1850, after the paper of Bertrand, the pairs of curves like this have been called conjugate Bertrand curves, or more commonly Bertrand curves.

Research related to Bertrand curves and their topological and geometrical properties and characterizations was considered by many researchers in Minkowski space–time, especially in Minkowski 3-space as well as in Euclidean space (see [1],[4] and [15] -[19]).

If a space curve has a singular point or points, it is obvious that Frenet frame cannot be constructed at these points. In order to study the geometric properties of the curve at these points, the question of how to calculate similar concepts such as Frenet frame and therefore curvature was answered for the first time with the concept of framed curve defined by S. Honda and M. Takahashi [6]. A framed curve is a smooth curve with a moving frame which may have singular point. Morever, T. Fukunaga and M. Takahashi gave existence conditions of framed curves for smooth curves in Euclidean 3-space [5]. Many curves in Euclidean 3-space whose geometric properties are well known and their properties have been revisited for framed curves and many works have been published on this subject. For example, the conditions for framed curves to be Bertrand and Mannheim curves were first studied by S. Honda and M. Takahashi [7], [8]. In addition, Y. Wang, D. Pei and R. Gao defined an adapted frame for framed curves and studied framed rectifying curves in Euclidean 3- space [20] (see also [9]-[13]).

In the known classical definition of Bertrand curves, the principal normal vector fields are linearly dependent at opposite points of pairs of space curves. Instead of this classical definition, a new approach was given by Zhang and Pei [22] by taking the principal normal vector field of a given space curve in the normal plane of the other space curve and the angle between the principal normals as constant, and Bertrand curves in Minkowski 3-space were studied. This generalization was studied in Euclidean 3-space by Demir and İlarslan [3].

In this study, the characterizations of framed and Frenet-type framed curves as Bertrand curves according to this new approach are obtained and the relations between the moving frames and curvature functions on the curves are given.

2. Preliminaries

In this section, we give some well known results from Euclidean geometry([7, 14]). Let $\mathbb{E}^3$ be the 3-dimensional Euclidean space equipped with the inner product $\langle \overrightarrow{X}, \overrightarrow{Y} \rangle = x_1y_1 + x_2y_2 + x_3y_3$, where $\overrightarrow{X} = (x_1, x_2, x_3)$ and $\overrightarrow{Y} = (y_1, y_2, y_3) \in \mathbb{E}^3$. The norm of $\overrightarrow{X}$ is given by $\left\|\overrightarrow{X}\right\| = \sqrt{\left|\langle \overrightarrow{X}, \overrightarrow{X} \rangle\right|}$ and the vector product is given by

$$\overrightarrow{X} \times \overrightarrow{Y} = \begin{bmatrix} \overrightarrow{e_1} & \overrightarrow{e_2} & \overrightarrow{e_3} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$$

where $\{\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}\}$ is the canonical basis of $\mathbb{E}^3$.

Let I be an interval of $\mathbb{R}$ and let $\gamma : I \to \mathbb{E}^3$ be a regular space curve, that is, $\dot{\gamma}(t) \neq 0$ for all $t \in I$, where $\dot{\gamma}(t) = \dfrac{d\gamma(t)}{dt}$. We say that γ is non-degenerate condition if $\dot{\gamma}(t) \times \ddot{\gamma}(t) \neq 0$ for all $t \in I$.

If we take general parameter t, then the tangent vector, the principal normal vector and the binormal vector are given by

$$\begin{aligned} T(t) &= \frac{\dot{\gamma}(t)}{\left|\dot{\gamma}(t)\right|}, \\ N(t) &= B(t) \times T(t) \\ B(t) &= \frac{\dot{\gamma}(t) \times \ddot{\gamma}(t)}{\left|\dot{\gamma}(t) \times \ddot{\gamma}(t)\right|} \end{aligned}$$

Then $\{T(t), N(t), B(t)\}$ is a moving frame of $\gamma(t)$ and we have the Frenet-Serret formula:

$$\begin{bmatrix} \dot{T}(t) \\ \dot{N}(t) \\ \dot{B}(t) \end{bmatrix} = \begin{bmatrix} 0 & \left|\dot{\gamma}(t)\right| \kappa(t) & 0 \\ -\left|\dot{\gamma}(t)\right| \kappa(t) & 0 & \left|\dot{\gamma}(t)\right| \tau(t) \\ 0 & -\left|\dot{\gamma}(t)\right| \tau(t) & 0 \end{bmatrix} \begin{bmatrix} T(t) \\ N(t) \\ B(t) \end{bmatrix}$$

where

$$\begin{aligned} \kappa(t) &= \frac{\left|\dot{\gamma}(t) \times \ddot{\gamma}(t)\right|}{\left|\dot{\gamma}(t)\right|^3}, \\ \tau(t) &= \frac{\det\left(\dot{\gamma}(t), \ddot{\gamma}(t), \dddot{\gamma}(t)\right)}{\left|\dot{\gamma}(t) \times \ddot{\gamma}(t)\right|^2}. \end{aligned}$$

A framed curve in the 3-dimensional Euclidean space is a smooth space curve with a moving frame which may have singular points, in detail see [6]. Let $\gamma : I \longrightarrow \mathbb{R}^3$ be a curve with singular points. The set

$$\Delta_2 = \left\{\mu = (\mu_1, \mu_2) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \langle \mu_i, \mu_j \rangle = \delta_{i,j}, \;\; i, j = 1, 2\right\}$$

is a 3-dimensional smooth manifold. Suppose that $\mu = (\mu_1, \mu_2) \in \Delta_2$. A unit vector is defined by $\nu = \mu_1 \times \mu_2$.

Definition 2.1. We say that $(\gamma, \mu) : I \longrightarrow \mathbb{R}^3 \times \Delta_2$ is a framed curve is $\langle \dot{\gamma}(t), \mu_i(t) \rangle = 0$ for all $t \in I$ and $i = 1, 2$. We also say that $\gamma : I \longrightarrow \mathbb{R}^3$is a framed base curve if there exist $\mu : I \longrightarrow \Delta_2$ such that (γ, μ) is a framed curve [6].

Let $(\gamma, \mu_1, \mu_2) : I \longrightarrow \mathbb{R}^3 \times \Delta_2$ be a framed curve and the $\nu(t) = \mu_1(t) \times \mu_2(t)$. The Frenet-Serret type formula is given by

$$\begin{bmatrix} \dot{\nu}(t) \\ \dot{\mu}_1(t) \\ \dot{\mu}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & -m(t) & -n(t) \\ m(t) & 0 & l(t) \\ n(t) & -l(t) & 0 \end{bmatrix} \begin{bmatrix} \nu(t) \\ \mu_1(t) \\ \mu_2(t) \end{bmatrix} \tag{2.1}$$

where, $l(t) = \langle \dot{\mu}_1(t), \mu_2(t) \rangle$, $m(t) = \langle \dot{\mu}_1(t), \nu(t) \rangle$ and $n(t) = \langle \dot{\mu}_2(t), \nu(t) \rangle$. Moreover, there exists a smooth mapping $\alpha : I \longrightarrow \mathbb{R}$ such that

$$\dot{\gamma}(t) = \alpha(t)\nu(t).$$

In addition, t_0 is a singular point of the framed curve γ if and only if $\alpha(t_0) = 0$.

Definition 2.2. We say that $\gamma : I \longrightarrow \mathbb{R}^3$is a Frenet-type framed base curve if there exist a regular spherical curve $\tau : I \longrightarrow S^2$and a smooth function $\alpha : I \longrightarrow \mathbb{R}$ such that $\dot{\gamma}(t) = \alpha(t)\mathcal{T}(t)$ for all $t \in I$. Then we call $\mathcal{T}(t)$ a unit tangent vector and $\alpha(t)$ a smooth function of $\gamma(t)$.

Clearly, t_0 is a singular point of γ if and only if $\alpha(t_0) = 0$. We define a unit principal normal vector $\mathcal{N}(t) = \frac{\dot{\mathcal{T}}(t)}{\|\mathcal{T}(t)\|}$and a unit binormal vector $\mathcal{B}(t) = \mathcal{T}(t) \times \mathcal{N}(t)$ of $\gamma(t)$. Then we have an orthonormal frame $\{\mathcal{T}(t), \mathcal{N}(t), \mathcal{B}(t)\}$ along $\gamma(t)$, which is called the Frenet-type frame along $\gamma(t)$. Then we have the following Frenet Serret type formula:

$$\begin{bmatrix} \dot{\mathcal{T}}(t) \\ \dot{\mathcal{N}}(t) \\ \dot{\mathcal{B}}(t) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(t) & 0 \\ -\kappa(t) & 0 & \tau(t) \\ 0 & -\tau(t) & 0 \end{bmatrix} \begin{bmatrix} \mathcal{T}(t) \\ \mathcal{N}(t) \\ \mathcal{B}(t) \end{bmatrix} \tag{2.2}$$

where

$$\kappa(t) = \left\| \dot{\mathcal{T}}(t) \right\|,$$

$$\tau(t) = \frac{\det(\mathcal{T}(t), \dot{\mathcal{T}}(t), \ddot{\mathcal{T}}(t))}{\left\| \dot{\mathcal{T}}(t) \right\|^2}.$$

We call $\kappa(t)$ a curvature and $\tau(t)$ a torsion of γ. Note that the curvature $\kappa(t)$ and torsion $\tau(t)$ are depend on a choice of parametrization. The proofs of the following theorems are given in [6].

Theorem 2.1. *(The Existence Theorem): Let $(l, m, n, \alpha) : I \longrightarrow \mathbb{R}^4$ be a smooth mapping. There exists a framed curve $(\gamma, \mu) : I \longrightarrow \mathbb{R}^3 \times \Delta_2$ whose associated curvature is (l, m, n, α).*

Theorem 2.2. *(The Uniqueness Theorem): Let* $(\gamma,\mu),(\overline{\gamma},\overline{\mu})$ *be framed curves whose curvatures* (l,m,n,α) *and* $(\bar{l},\overline{m},\overline{n},\overline{\alpha})$ *coincide. Then* (γ,μ) *and* $(\overline{\gamma},\overline{\mu})$ *are congruent as framed curves.*

3. Frenet-Type Framed Classical Bertrand Curves in Euclidean 3-space

In this section, Frenet-type framed Bertrand curves are analysed in the classical sense, that is, according to the linear dependence of the principal normal vector fields at the opposite points of the space curve pairs.

Definition 3.1. Let $(\gamma,\mathcal{N},\mathcal{B})$ and $\left(\overline{\gamma},\overline{\mathcal{N}},\overline{\mathcal{B}}\right):I\longrightarrow\mathbb{R}^3\times\Delta_2$ be Frenet-type framed curves in $\mathbb{E}^3$ with curvatures (κ,τ) and $(\overline{\kappa},\overline{\tau})$ respectively. If there exist $\mathcal{N}(t)=\overline{\mathcal{N}}(t)$ for all $t\in I$, then γ is called Frenet-type Framed classical Bertrand curves and $\overline{\gamma}$ is called Frenet-type Framed Bertrand mate of γ.

Theorem 3.1. *Let* $(\gamma,\mathcal{N},\mathcal{B}):I\longrightarrow\mathbb{R}^3\times\Delta_2$ *be a Frenet-type classical Bertrand curve in Euclidean* 3*-space* $\mathbb{E}^3$ *and* $\left(\overline{\gamma},\overline{\mathcal{N}},\overline{\mathcal{B}}\right)$ *be a Frenet-type Bertrand mate curve of* γ *in* $\mathbb{E}^3$*. The curvatures and Frenet vectors of* γ *and* $\overline{\gamma}$ *are related as follows:*

$$
\begin{aligned}
\overline{\mathcal{T}}(t) &= \left(\frac{\alpha(t)-\lambda\kappa(t)}{\overline{\alpha}(t)}\right)\mathcal{T}(t)+\frac{\lambda\tau(t)}{\overline{\alpha}(t)}\mathcal{B}(t),\\
\overline{\mathcal{N}}(t) &= \mathcal{N}(t),\\
\overline{\mathcal{B}}(t) &= \frac{\left(\kappa(t)\left(c_1^2-1\right)-c_1c_2\tau(t)\right)}{\overline{\tau}(t)}\mathcal{T}(t)+\frac{\left(c_1c_2\kappa(t)-\tau(t)\left(c_2^2-1\right)\right)}{\overline{\tau}(t)}\mathcal{B}(t)
\end{aligned}
\tag{3.1}
$$

and

$$
\begin{aligned}
\overline{\kappa}(t) &= \frac{\alpha(t)\kappa(t)-\lambda\left(\kappa^2(t)+\tau^2(t)\right)}{\overline{\alpha}(t)},\\
\overline{\tau}(t) &= \left(\frac{\overline{\alpha}^2(t)\left(\kappa^2(t)+\tau^2(t)\right)-\left(\alpha(t)\kappa(t)-\lambda\left(\kappa^2(t)+\tau^2(t)\right)\right)^2}{\overline{\alpha}^2(t)}\right)^{\frac{1}{2}}.
\end{aligned}
\tag{3.2}
$$

where $\lambda\in\mathbb{R}_0$, $\overline{\alpha}^2(t)=(\alpha(t)-\lambda\kappa(t))^2+\lambda^2\tau^2(t)$ *and* $c_1,c_2\in\mathbb{R}_0$

$$c_1=\tfrac{\alpha(t)-\lambda\kappa(t)}{\overline{\alpha}(t)},\quad c_2=\tfrac{\lambda\tau(t)}{\overline{\alpha}(t)}.$$

Proof. Assume that there exists the Frenet-type classical Bertrand curve γ in $\mathbb{E}^3$ and its Frenet-type Bertrand mate curve $\overline{\gamma}$ in $\mathbb{E}^3$. Then $\overline{\gamma}$ can be parametrized by

$$\overline{\gamma}(t)=\gamma(t)+\lambda(t)\mathcal{N}(t) \tag{3.3}$$

Differentiating equation (3.3) with respect to t and using the equations (2.2) Frenet-type frame, we obtain

$$\overline{\alpha}(t)\overline{\mathcal{T}}(t)=(\alpha(t)-\lambda(t)\kappa(t))\mathcal{T}(t)+\dot{\lambda}(t)\mathcal{N}(t)+\lambda(t)\tau(t)\mathcal{B}(t), \tag{3.4}$$

since γ is a Frenet-type framed curve $\dot{\gamma}(t)=\alpha(t)\mathcal{T}(t)$, where $\alpha : I \longrightarrow \mathbb{R}$ is a smooth function. By taking the inner product of the last relation with $\mathcal{N}(t)=\overline{\mathcal{N}}(t)$, we have

$$\dot{\lambda}(t)=0.$$

Thus λ =constant$\neq 0$. Substituting the last relation in the equation (3.4), we find

$$\overline{\alpha}(t)\overline{\mathcal{T}}(t)=(\alpha(t)-\lambda\kappa(t))\mathcal{T}(t)+\lambda\tau(t)\mathcal{B}(t). \tag{3.5}$$

By taking the inner product of equation (3.5) with itself, we obtain

$$\langle\overline{\alpha}(t)\overline{\mathcal{T}}(t),\overline{\alpha}(t)\overline{\mathcal{T}}(t)\rangle=(\overline{\alpha}(t))^2=(\alpha(t)-\lambda\kappa(t))^2+(\lambda\tau(t))^2 .$$

If we write instead in (3.5)

$$A(t)=\frac{\alpha(t)-\lambda\kappa(t)}{\overline{\alpha}(t)} \text{ and } B(t)=\frac{\lambda\tau(t)}{\overline{\alpha}(t)}$$

we get

$$\overline{\mathcal{T}}(t)=A(t)\mathcal{T}(t)+B(t)\mathcal{B}(t) \tag{3.6}$$

Differentiating equation (3.6) with respect to t and using the equations (2.2) Frenet-type frame, we obtain

$$\overline{\kappa}(t)\overline{\mathcal{N}}(t)=\dot{A}(t)\mathcal{T}(t)+(A(t)\kappa(t)-B(t)\tau(t))\mathcal{N}(t)+\dot{B}(t)\mathcal{B}(t) \tag{3.7}$$

By taking the inner product of the last relation with $\mathcal{N}(t)=\overline{\mathcal{N}}(t)$, we get

$$\overline{\kappa}(t)=\frac{\alpha(t)\kappa(t)-\lambda(\kappa^2(t)+\tau^2(t))}{\overline{\alpha}(t)}$$

If we arrange equation (3.7), we get

$$\overline{\mathcal{N}}(t)=C(t)\mathcal{T}(t)+\mathcal{N}(t)+E(t)\mathcal{B}(t)$$

where, $C(t)=\dfrac{\dot{A}(t)}{\overline{\kappa}(t)}$ and $E(t)=\dfrac{\dot{B}(t)}{\overline{\kappa}(t)}$. By using $\mathcal{N}(t)=\overline{\mathcal{N}}(t)$, we get $C(t)=0$ and $E(t)=0$, then

$$\begin{aligned} A(t)= & \tfrac{\alpha(t)-\lambda\kappa(t)}{\overline{\alpha}(t)} & =\text{constant}=c_1\neq 0\\ B(t)= & \tfrac{\lambda\tau(t)}{\overline{\alpha}(t)} & =\text{constant}=c_2\neq 0\end{aligned} \tag{3.8}$$

Moreover, we can say that from equations (3.8)

$$\overline{\alpha}(t)=\lambda\kappa(t)+\mu\tau(t)$$

where $\mu = \frac{c_1 \lambda}{c2}$. So, we have

$$\mathcal{N}(t) = \overline{\mathcal{N}}(t) \tag{3.9}$$

Differentiating (3.8) with respect to t and using the equations (2.2) Frenet-type frame, we obtain

$$-\overline{\kappa}(t)\overline{\mathcal{T}}(t) + \overline{\tau}(t)\overline{\mathcal{B}}(t) = -\kappa(t)\mathcal{T}(t) + \tau(t)\mathcal{B}(t). \tag{3.10}$$

By taking the inner product of equation (3.10) with itself, we get

$$\overline{\kappa}^2(t) + \overline{\tau}^2(t) = \kappa^2(t) + \tau^2(t). \tag{3.11}$$

From the equation (3.11), we have

$$\overline{\tau}(t) = \left(\frac{\overline{\alpha}^2(t)\left(\kappa^2(t) + \tau^2(t)\right) - \left(\alpha(t)\kappa(t) - \lambda\left(\kappa^2(t) + \tau^2(t)\right)\right)^2}{\overline{\alpha}^2(t)} \right)^{\frac{1}{2}}.$$

Using the equation (3.10) and the equation (3.6), we have

$$\overline{\mathcal{B}}(t) = \frac{\left(\kappa(t)\left(c_1^2 - 1\right) - c_1 c_2 \tau(t)\right)}{\overline{\tau}(t)}\mathcal{T}(t) + \frac{\left(c_1 c_2 \kappa(t) - \tau(t)\left(c_2^2 - 1\right)\right)}{\overline{\tau}(t)}\mathcal{B}(t).$$

Conversely, let γ be a Frenet-type framed Bertrand curve with curvatures $\overline{\alpha}(t)$, $\overline{\kappa}(t)$ and $\overline{\tau}(t)$ that satisfy equation (3.2). Then we can defined a curve $\overline{\gamma}$ as

$$\overline{\gamma}(t) = \gamma(t) + \lambda(t)\mathcal{N}(t). \tag{3.12}$$

We have known that λ is a non-zero constant. Then from differentiating (3.12) with respect to t and using the equations (2.2) Frenet-type frame, we obtain

$$\overline{\alpha}(t)\overline{\mathcal{T}}(t) = (\alpha(t) - \lambda\kappa(t))\mathcal{T}(t) + \lambda\tau(t)\mathcal{B}(t). \tag{3.13}$$

By taking the inner product equation (3.13) with $\mathcal{N}(t) = \overline{\mathcal{N}}(t)$, we get

$$\langle \overline{\mathcal{T}}, \mathcal{N} \rangle = 0$$

which means that $\mathcal{N}$ is coplanar with $\overline{\mathcal{N}}$ and $\overline{\mathcal{B}}$. Then we prove that

$$\langle \mathcal{N}, \overline{\mathcal{N}} \rangle = 1$$

We assume that

$$\mathcal{N}(t) = \sigma\overline{\mathcal{N}}(t) + \rho\overline{\mathcal{B}}(t) \qquad \sigma,\ \rho \in \mathbb{R}. \tag{3.14}$$

Differentiating (3.14) with respect to t and using the equations (2.2) Frenet-type frame, we obtain

$$-\kappa(t)\mathcal{T}(t) + \tau(t)\mathcal{B}(t) = -\sigma\overline{\kappa}(t)\overline{\mathcal{T}}(t) - \rho\overline{\tau}(t)\overline{\mathcal{N}}(t) + \sigma\overline{\tau}(t)\overline{\mathcal{B}}(t). \tag{3.15}$$

By taking the inner product equation (3.15) with equation (3.13), we find

$$-\sigma\overline{\alpha}(t)\overline{\kappa}(t) = -\alpha(t)\kappa(t) + \lambda\left(\kappa^2(t) + \tau^2(t)\right) \tag{3.16}$$

From equation (3.16) and equation (3.2) we easily obtain $\sigma = 1$. Then from equation (3.14), we get

$$\langle \mathcal{N}, \overline{\mathcal{N}} \rangle = \sigma = 1.$$

This completes the proof. □

In this section, Frenet-type framed Bertrand curves are analysed in the classical sense, that is, according to the linear dependence of the principal normal vector fields at the opposite points of the space curve pairs.

Definition 3.2. Let $(\gamma, \mathcal{N}, \mathcal{B})$ and $\left(\overline{\gamma}, \overline{\mathcal{N}}, \overline{\mathcal{B}}\right) : I \longrightarrow \mathbb{R}^3 \times \Delta_2$ be Frenet-type framed curves in $\mathbb{E}^3$ with curvatures (κ, τ) and $(\overline{\kappa}, \overline{\tau})$ respectively. If there exist $\mathcal{N}(t) = \overline{\mathcal{N}}(t)$ for all $t \in I$, then γ is called Frenet-type Framed classical Bertrand curves and $\overline{\gamma}$ is called Frenet-type Framed Bertrand mate of γ.

Theorem 3.2. *Let $(\gamma, \mathcal{N}, \mathcal{B}) : I \longrightarrow \mathbb{R}^3 \times \Delta_2$ be a Frenet-type classical Bertrand curve in Euclidean 3-space $\mathbb{E}^3$ and $\left(\overline{\gamma}, \overline{\mathcal{N}}, \overline{\mathcal{B}}\right)$ be a Frenet-type Bertrand mate curve of γ in $\mathbb{E}^3$. The curvatures and Frenet vectors of γ and $\overline{\gamma}$ are related as follows:*

$$\begin{aligned}
\overline{\mathcal{T}}(t) &= \left(\frac{\alpha(t) - \lambda\kappa(t)}{\overline{\alpha}(t)}\right)\mathcal{T}(t) + \frac{\lambda\tau(t)}{\overline{\alpha}(t)}\mathcal{B}(t),\\
\overline{\mathcal{N}}(t) &= \mathcal{N}(t),\\
\overline{\mathcal{B}}(t) &= \frac{\left(\kappa(t)\left(c_1^2-1\right) - c_1c_2\tau(t)\right)}{\overline{\tau}(t)}\mathcal{T}(t) + \frac{\left(c_1c_2\kappa(t) - \tau(t)\left(c_2^2-1\right)\right)}{\overline{\tau}(t)}\mathcal{B}(t)
\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}
\overline{\kappa}(t) &= \frac{\alpha(t)\kappa(t) - \lambda\left(\kappa^2(t) + \tau^2(t)\right)}{\overline{\alpha}(t)},\\
\overline{\tau}(t) &= \left(\frac{\overline{\alpha}^2(t)\left(\kappa^2(t) + \tau^2(t)\right) - \left(\alpha(t)\kappa(t) - \lambda\left(\kappa^2(t) + \tau^2(t)\right)\right)^2}{\overline{\alpha}^2(t)}\right)^{\frac{1}{2}}.
\end{aligned} \tag{3.2}$$

where $\lambda \in \mathbb{R}_0$, $\overline{\alpha}^2(t) = (\alpha(t) - \lambda\kappa(t))^2 + \lambda^2\tau^2(t)$ and $c_1, c_2 \in \mathbb{R}_0$

$$c_1 = \frac{\alpha(t) - \lambda\kappa(t)}{\overline{\alpha}(t)}, \quad c_2 = \frac{\lambda\tau(t)}{\overline{\alpha}(t)}.$$

Proof. Assume that there exists the Frenet-type classical Bertrand curve γ in $\mathbb{E}^3$ and its Frenet-type Bertrand mate curve $\overline{\gamma}$ in $\mathbb{E}^3$. Then $\overline{\gamma}$ can be parametrized by

$$\overline{\gamma}(t) = \gamma(t) + \lambda(t)\mathcal{N}(t) \tag{3.3}$$

Differentiating equation (3.3) with respect to t and using the equations (2.2) Frenet-type frame, we obtain

$$\overline{\alpha}(t)\overline{\mathcal{T}}(t) = (\alpha(t) - \lambda(t)\kappa(t))\mathcal{T}(t) + \dot{\lambda}(t)\mathcal{N}(t) + \lambda(t)\tau(t)\mathcal{B}(t), \tag{3.4}$$

since γ is a Frenet-type framed curve $\dot{\gamma}(t) = \alpha(t)\mathcal{T}(t)$, where $\alpha : I \longrightarrow \mathbb{R}$ is a smooth function. By taking the inner product of the last relation with $\mathcal{N}(t) = \overline{\mathcal{N}}(t)$, we have

$$\dot{\lambda}(t) = 0.$$

Thus λ =constant$\neq 0$. Substituting the last relation in the equation (3.4), we find

$$\overline{\alpha}(t)\overline{\mathcal{T}}(t) = (\alpha(t) - \lambda\kappa(t))\mathcal{T}(t) + \lambda\tau(t)\mathcal{B}(t). \tag{3.5}$$

By taking the inner product of equation (3.5) with itself, we obtain

$$\langle\overline{\alpha}(t)\overline{\mathcal{T}}(t), \overline{\alpha}(t)\overline{\mathcal{T}}(t)\rangle = (\overline{\alpha}(t))^2 = (\alpha(t) - \lambda\kappa(t))^2 + (\lambda\tau(t))^2.$$

If we write instead in (3.5)

$$A(t) = \frac{\alpha(t) - \lambda\kappa(t)}{\overline{\alpha}(t)} \text{ and } B(t) = \frac{\lambda\tau(t)}{\overline{\alpha}(t)}$$

we get

$$\overline{\mathcal{T}}(t) = A(t)\mathcal{T}(t) + B(t)\mathcal{B}(t) \tag{3.6}$$

Differentiating equation (3.6) with respect to t and using the equations (2.2) Frenet-type frame, we obtain

$$\overline{\kappa}(t)\overline{\mathcal{N}}(t) = \dot{A}(t)\mathcal{T}(t) + (A(t)\kappa(t) - B(t)\tau(t))\mathcal{N}(t) + \dot{B}(t)\mathcal{B}(t) \tag{3.7}$$

By taking the inner product of the last relation with $\mathcal{N}(t) = \overline{\mathcal{N}}(t)$, we get

$$\overline{\kappa}(t) = \frac{\alpha(t)\kappa(t) - \lambda(\kappa^2(t) + \tau^2(t))}{\overline{\alpha}(t)}$$

If we arrange equation (3.7), we get

$$\overline{\mathcal{N}}(t) = C(t)\mathcal{T}(t) + \mathcal{N}(t) + E(t)\mathcal{B}(t)$$

where, $C(t) = \dfrac{\dot{A}(t)}{\overline{\kappa}(t)}$ and $E(t) = \dfrac{\dot{B}(t)}{\overline{\kappa}(t)}$. By using $\mathcal{N}(t) = \overline{\mathcal{N}}(t)$, we get $C(t) = 0$ and $E(t) = 0$, then

$$\begin{aligned} A(t) = & \tfrac{\alpha(t)-\lambda\kappa(t)}{\overline{\alpha}(t)} & = \text{constant} = c_1 \neq 0 \\ B(t) = & \tfrac{\lambda\tau(t)}{\overline{\alpha}(t)} & = \text{constant} = c_2 \neq 0 \end{aligned} \tag{3.8}$$

Moreover, we can say that from equations (3.8)

$$\overline{\alpha}(t) = \lambda\kappa(t) + \mu\tau(t)$$

where $\mu = \frac{c_1\lambda}{c2}$. So, we have

$$\mathcal{N}(t) = \overline{\mathcal{N}}(t) \tag{3.9}$$

Differentiating (3.8) with respect to t and using the equations (2.2) Frenet-type frame, we obtain

$$-\overline{\kappa}(t)\overline{\mathcal{T}}(t) + \overline{\tau}(t)\overline{\mathcal{B}}(t) = -\kappa(t)\mathcal{T}(t) + \tau(t)\mathcal{B}(t). \tag{3.10}$$

By taking the inner product of equation (3.10) with itself, we get

$$\overline{\kappa}^2(t) + \overline{\tau}^2(t) = \kappa^2(t) + \tau^2(t). \tag{3.11}$$

From the equation (3.11), we have

$$\overline{\tau}(t) = \left(\frac{\overline{\alpha}^2(t)\left(\kappa^2(t) + \tau^2(t)\right) - \left(\alpha(t)\kappa(t) - \lambda\left(\kappa^2(t) + \tau^2(t)\right)\right)^2}{\overline{\alpha}^2(t)}\right)^{\frac{1}{2}}.$$

Using the equation (3.10) and the equation (3.6), we have

$$\overline{\mathcal{B}}(t) = \frac{\left(\kappa(t)\left(c_1^2 - 1\right) - c_1c_2\tau(t)\right)}{\overline{\tau}(t)}\mathcal{T}(t) + \frac{\left(c_1c_2\kappa(t) - \tau(t)\left(c_2^2 - 1\right)\right)}{\overline{\tau}(t)}\mathcal{B}(t).$$

Conversely, let γ be a Frenet-type framed Bertrand curve with curvatures $\overline{\alpha}(t)$, $\overline{\kappa}(t)$ and $\overline{\tau}(t)$ that satisfy equation (3.2). Then we can defined a curve $\overline{\gamma}$ as

$$\overline{\gamma}(t) = \gamma(t) + \lambda(t)\mathcal{N}(t). \tag{3.12}$$

We have known that λ is a non-zero constant. Then from differentiating (3.12) with respect to t and using the equations (2.2) Frenet-type frame, we obtain

$$\overline{\alpha}(t)\overline{\mathcal{T}}(t) = (\alpha(t) - \lambda\kappa(t))\mathcal{T}(t) + \lambda\tau(t)\mathcal{B}(t). \tag{3.13}$$

By taking the inner product equation (3.13) with $\mathcal{N}(t) = \overline{\mathcal{N}}(t)$, we get

$$\langle\overline{\mathcal{T}}, \mathcal{N}\rangle = 0$$

which means that $\mathcal{N}$ is coplanar with $\overline{\mathcal{N}}$ and $\overline{\mathcal{B}}$. Then we prove that

$$\langle\mathcal{N}, \overline{\mathcal{N}}\rangle = 1$$

We assume that

$$\mathcal{N}(t) = \sigma\overline{\mathcal{N}}(t) + \rho\overline{\mathcal{B}}(t) \qquad \sigma,\ \rho \in \mathbb{R}. \tag{3.14}$$

Differentiating (3.14) with respect to t and using the equations (2.2) Frenet-type frame, we obtain

$$-\kappa(t)\mathcal{T}(t) + \tau(t)\mathcal{B}(t) = -\sigma\overline{\kappa}(t)\overline{\mathcal{T}}(t) - \rho\overline{\tau}(t)\overline{\mathcal{N}}(t) + \sigma\overline{\tau}(t)\overline{\mathcal{B}}(t). \tag{3.15}$$

By taking the inner product equation (3.15) with equation (3.13), we find

$$-\sigma\overline{\alpha}(t)\overline{\kappa}(t) = -\alpha(t)\kappa(t) + \lambda\left(\kappa^2(t) + \tau^2(t)\right) \tag{3.16}$$

From equation (3.16) and equation (3.2) we easily obtain $\sigma = 1$. Then from equation (3.14), we get

$$\langle \mathcal{N}, \overline{\mathcal{N}} \rangle = \sigma = 1.$$

This completes the proof. □

Acknowledgements

We would like to thank the Editors and referees for their positive approach and valuable comments during the review process of the paper.

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