

ROUGH STATISTICAL CONVERGENCE OF SEQUENCES IN A PARTIAL METRIC SPACE

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Abstract. In this paper, using the concept of natural density, we have introduced the notion of rough statistical convergence which is an extension of the notion of rough convergence in a partial metric space. We have defined the set of rough statistical limit points of a sequence in a partial metric space and proved that this set is closed and bounded. Finally, we have found out the relationship between the set of statistical cluster points and the set of rough statistical limit points of sequences in a partial metric space.

Keywords: Natural density, statistical convergence, rough convergence, partial metric spaces, rough limit sets.

1. Introduction

The idea of rough convergence of sequences was first introduced in a normed linear space by H. X. Phu [16] in 2001 and then he further studied the same on an infinite dimensional normed linear space [17]. The formal definition given by him is as follows: if $\{x_n\}$ is a sequence in a normed linear space $(X, \|\cdot\|)$ and r is a non-negative real number, then $\{x_n\}$ is said to be r -convergent to x , denoted by $x_n \xrightarrow{r} x$, if $\forall \varepsilon > 0, \exists N \in \mathbb{N} : n \geq N \longrightarrow \|x_n - x\| < r + \varepsilon$, or equivalently, if $\limsup_{n \rightarrow \infty} \|x_n - x\| < r$. This is the idea of rough convergence with r as roughness degree. For $r = 0$ we have the ordinary convergence. The main difference

Received: March 12, 2024, revised: December 09, 2024, accepted: December 12, 2024

Communicated by Dimitris Georgiou

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2020 *Mathematics Subject Classification.* Primary 40A05; Secondary 40A99

of ordinary and rough convergence is that if a sequence in a normed linear space converges, then its limit is unique but for rough convergence with roughness degree $r > 0$ the limit may be infinite. The set of all r -limits is denoted by $LIM^r x_n$. Phu discussed about the idea of rough limit sets and also some basic properties of rough limit sets such as boundedness, closedness and convexity etc. Also he introduced the idea of rough Cauchy sequences. Later works on rough convergence in many directions were carried out by many authors [1, 2, 4, 7, 11, 12, 13].

We intended to study the idea of rough convergence in a more generalized form. So, it is required to discuss in brief the ideas of statistical convergence.

Statistical convergence is a generalization of the ordinary convergence. The concept of statistical convergence was introduced by H. Fast [8] and H. Steinhaus [18] in the year 1951. The idea of statistical convergence has been dependent on the structure of subsets of the natural numbers by T. Salat [19] as follows: if $B \subset \mathbb{N}$, then B_n will denote the set $\{k \in B : k \leq n\}$ and $|B_n|$ stands for the cardinality of B_n . The natural density of B is denoted by $d(B)$ and defined by $d(B) = \lim_{n \rightarrow \infty} \frac{|B_n|}{n}$, if the limit exists. A real sequence $\{\xi_n\}$ is said to be statistically convergent to ξ if for every $\varepsilon > 0$ the set $B(\varepsilon) = \{k \in \mathbb{N} : |\xi_n - \xi| \geq \varepsilon\}$ has natural density zero. In this case, ξ is called the statistical limit of the sequence $\{\xi_n\}$ and we write $st - LIM^r \xi_n = \xi$. We have $d(B^c) = 1 - d(B)$, where $B^c = \mathbb{N} \setminus B$ is the complement of B . If $B_1 \subset B_2$, then $d(B_1) \leq d(B_2)$. Also, in [9, 10] and their references we can find more results on statistical convergence.

In 1994, partial metric spaces were introduced by S. Matthews [14] as a generalization of metric spaces. In our present work we discuss the idea of rough statistical convergence of sequences in partial metric spaces. We have given the idea of statistical boundedness in a partial metric space. We have also defined the set of rough statistical limit points and found out several properties of this set like boundedness and closedness etc.

2. Preliminaries

Definition 2.1. [6] A partial metric on a nonempty set X is a function $\rho : X \times X \rightarrow [0, \infty)$ such that for all $x, y, z \in X$:

- (p1) $0 \leq \rho(x, x) \leq \rho(x, y)$ (nonnegativity and small self-distances),
- (p2) $x = y \iff \rho(x, x) = \rho(x, y) = \rho(y, y)$ (indistancy both implies equality),
- (p3) $\rho(x, y) = \rho(y, x)$ (symmetry),
- (p4) $\rho(x, y) \leq \rho(x, z) + \rho(z, y) - \rho(z, z)$ (triangularity).

Then (X, ρ) is said to be a partial metric space, where X is a nonempty set and ρ is a partial metric on X .

Properties and examples of partial metric spaces have been thoroughly discussed in [6].

Definition 2.2. [6] In a partial metric space (X, ρ) , for $r > 0$ and $x \in X$ we define the open and closed ball of radius r and center x respectively are as follows:

$$\begin{aligned} B_r^\rho(x) &= \{y \in X : \rho(x, y) < \rho(x, x) + r\}, \\ \overline{B}_r^\rho(x) &= \{y \in X : \rho(x, y) \leq \rho(x, x) + r\}. \end{aligned}$$

Definition 2.3. [6] Let (X, ρ) be a partial metric space. A subset U of X is said to be bounded in X if there exists a positive real number M such that $\sup \{\rho(x, y) : x, y \in U\} < M$.

Definition 2.4. [6] Let (X, ρ) be a partial metric space and $\{x_n\}$ be a sequence in X . Then $\{x_n\}$ is said to converge to $x \in X$ if and only if $\lim_{n \rightarrow \infty} \rho(x_n, x) = \rho(x, x)$; i.e. if for each $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that $|\rho(x_n, x) - \rho(x, x)| < \varepsilon$ for all $n \geq k$.

Definition 2.5. [15] Let (X, ρ) be a partial metric space. Then the sequence $\{x_n\}$ is said to be statistically convergent to x if for every $\varepsilon > 0$,

$$d(n \in \mathbb{N} : |\rho(x_n, x) - \rho(x, x)| \geq \varepsilon) = 0.$$

Definition 2.6. [15] Let (X, ρ) be a partial metric space. Then the sequence $\{x_n\}$ is called statistically Cauchy if for every $\varepsilon > 0$ there is a positive integer m and $l \geq 0$ such that

$$d(n \in \mathbb{N} : |\rho(x_n, x_m) - l| \geq \varepsilon) = 0.$$

Definition 2.7. [5] Let (X, ρ) be a partial metric space. A sequence $\{x_n\}$ in X is said to be rough convergent (or r -convergent) to x of roughness degree r for some non-negative real number r if for every $\varepsilon > 0$ there exists a natural number k such that $|\rho(x_n, x) - \rho(x, x)| < r + \varepsilon$ holds for all $n \geq k$.

3. Rough statistical convergence in partial metric spaces

Definition 3.1. A sequence $\{x_n\}$ in a partial metric space (X, ρ) is said to be rough statistically convergent (or in short r -statistically convergent or r -st convergent) to x of roughness degree r for some non-negative real number r if for every $\varepsilon > 0$,

$$d(\{n \in \mathbb{N} : |\rho(x_n, x) - \rho(x, x)| \geq r + \varepsilon\}) = 0.$$

We denote it by the notation $x_n \xrightarrow{r-st} x$ in (X, ρ) . When $r = 0$, then the rough statistical convergence becomes the statistical convergence in any partial metric space (X, ρ) . Let a sequence $\{x_n\}$ be rough statistically convergent to x , then x is said to be a rough statistical limit point of $\{x_n\}$. And the set of all rough statistical limit points of a sequence $\{x_n\}$ is said to be the r -st limit set. We will denote it by $st-LIM^r x_n$. Hence $st-LIM^r x_n = \left\{x \in X : x_n \xrightarrow{r-st} x\right\}$.

Theorem 3.1. *Every rough convergent sequence in a partial metric space (X, ρ) is rough statistically convergent in (X, ρ) .*

Proof. Let $\{x_n\}$ be a sequence in (X, ρ) and rough convergent to x . Then for an arbitrary $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $|\rho(x_n, x) - \rho(x, x)| < r + \varepsilon$, $\forall n \geq k$. Now, the set $A = \{n \in \mathbb{N} : |\rho(x_n, x) - \rho(x, x)| \geq r + \varepsilon\} \subset \{1, 2, 3, \dots, (k-1)\}$ is a finite set. So, $d(A) = 0$. Hence $\{x_n\}$ is rough statistically convergent in (X, ρ) . $\square$

Remark 3.1. The converse of the above theorem may not be true i.e. rough statistically convergent sequence may not be rough convergent in (X, ρ) .

Example 3.1. Let $X = \mathbb{R}^+$ and $\rho : X \times X \rightarrow \mathbb{R}^+$ be given by $\rho(x, y) = \max\{x, y\}$ for all $x, y \in X$. Then (X, ρ) is a partial metric space. Let us took a sequence $\{x_n\}$ which is defined by

$$x_n = \begin{cases} k, & \text{if } n = k^2, \\ 2, & \text{if } n \neq k^2 \text{ and } n \text{ is even,} \\ 0, & \text{if } n \neq k^2 \text{ and } n \text{ is odd.} \end{cases}$$

Let $\varepsilon > 0$ be given.

Then $A_1 = \{n \in \mathbb{N} : |\rho(x_n, 0) - \rho(0, 0)| \geq 1 + \varepsilon\} \subset \{1^2, 2^2, 3^2, 4^2, 5^2, \dots\} = P$ (say).

Similarly, the sets $A_2 = \{n \in \mathbb{N} : |\rho(x_n, 1) - \rho(1, 1)| \geq 1 + \varepsilon\} \subset P$ and $A_3 = \{n \in \mathbb{N} : |\rho(x_n, 2) - \rho(2, 2)| \geq 1 + \varepsilon\} \subset P$.

Since $d(P) = 0$, so $d(A_1) = d(A_2) = d(A_3) = 0$.

So, $\{x_n\}$ is rough statistical convergent to 0, 1 and 2 of roughness degree 1.

Again, for any k such that $1 < k < 2$, the set $A_k = \{n \in \mathbb{N} : |\rho(x_n, k) - \rho(k, k)| \geq 1 + \varepsilon\} \subset \{1^2, 2^2, 3^2, \dots\} = P$.

If $k > 2$, then the set $A_k \subset P$. So, the statistical rough limit set of roughness degree 1 is $\{0\} \cup \{k \in \mathbb{R}^+ : k \geq 1\}$ i.e. $st-LIM^r x_n = \{0\} \cup [1, \infty)$. But

$$|\rho(x_n, 2) - \rho(2, 2)| = \begin{cases} |\rho(k, 2) - \rho(2, 2)| = |k - 2|, & \text{if } n = k^2, \\ |\rho(2, 2) - \rho(2, 2)| = 0, & \text{if } n \neq k^2 \text{ and } n \text{ is even,} \\ |\rho(0, 2) - \rho(2, 2)| = 0, & \text{if } n \neq k^2 \text{ and } n \text{ is odd.} \end{cases}$$

So, when $n = k^2$, there does not exist any positive integer n_0 such that the condition $|\rho(x_n, 2) - \rho(2, 2)| < r + \varepsilon$ for all $n \geq n_0$ holds, since $|k - 2| \rightarrow \infty$ as $k^2 \rightarrow \infty$.

Hence $\{x_n\}$ is not rough convergent to 2 of any roughness degree $r > 0$. Similarly, it can be shown that $\{x_n\}$ is not rough convergent to 0 or any number in $[1, \infty)$ i.e. $LIM^r x_n = \phi$.

Definition 3.2. [5] The diameter of a set B in a partial metric space (X, ρ) is defined by

$$diam(B) = \sup \{\rho(x, y) : x, y \in B\}.$$

Theorem 3.2. *Let (X, ρ) be a partial metric space and a be a positive real number such that $\rho(x, x) = a$ for all x in X . Then for a sequence $\{x_n\}$, we have $diam(st-LIM^r x_n) \leq (2r + 2a)$.*

Proof. Let $\text{diam}(st - LIM^r x_n) > 2r + 2a$. Then there exist elements $y, z \in st - LIM^r x_n$ such that $\rho(y, z) > 2r + 2a$. Let us take $\varepsilon \in (0, \frac{\rho(y, z)}{2} - r - a)$. Since $y, z \in st - LIM^r x_n$, we have $d(M_1) = 0$ and $d(M_2) = 0$, where $M_1 = \{n \in \mathbb{N} : |\rho(x_n, y) - \rho(y, y)| \geq r + \varepsilon\}$ and $M_2 = \{n \in \mathbb{N} : |\rho(x_n, z) - \rho(z, z)| \geq r + \varepsilon\}$. Using the property of natural density, we get $d(M_1^c \cap M_2^c) = 1$. So, for all $n \in M_1^c \cap M_2^c$, we have

$$\begin{aligned} \rho(y, z) &\leq \rho(y, x_n) + \rho(x_n, z) - \rho(x_n, x_n) \\ &= \{\rho(x_n, y) - \rho(y, y)\} + \{\rho(x_n, z) - \rho(z, z)\} - \rho(x_n, x_n) + \rho(y, y) + \rho(z, z) \\ &< 2(r + \varepsilon) - a + a + a \\ &= 2r + 2\varepsilon + a \\ &< 2r + \rho(y, z) - 2r - 2a + a \\ &= \rho(y, z) - a, \text{ which is a contradiction.} \end{aligned}$$

Hence we must have $\text{diam}(st - LIM^r x_n) \leq 2r + 2a$. $\square$

Theorem 3.3. *If a sequence $\{x_n\}$ statistically converges to x in a partial metric space (X, ρ) , then $\{y \in \overline{B_r^\rho(x)} : \rho(x, x) = \rho(y, y)\} \subseteq st - LIM^r x_n$.*

Proof. Let $\varepsilon > 0$ and a sequence $\{x_n\}$ statistically converges to x in a partial metric space (X, ρ) . Then $d(A) = 0$, where $A = \{n \in \mathbb{N} : |\rho(x_n, x) - \rho(x, x)| \geq \varepsilon\}$. Let $y \in \overline{B_r^\rho(x)}$ such that $\rho(x, x) = \rho(y, y)$. Then $\rho(x, y) \leq \rho(x, x) + r$ such that $\rho(x, x) = \rho(y, y)$. For $n \in A^c$,

$$\begin{aligned} \rho(x_n, y) &\leq \rho(x_n, x) + \rho(x, y) - \rho(x, x) \\ &\leq \{\rho(x_n, x) - \rho(x, x)\} + \rho(x, y) \\ &< \varepsilon + \{\rho(x, x) + r\} \\ &= \rho(x, x) + (r + \varepsilon) \end{aligned}$$

Therefore,

$$\begin{aligned} \rho(x_n, y) - \rho(y, y) &< \rho(x, x) - \rho(y, y) + (r + \varepsilon) \\ &= (r + \varepsilon), \text{ since } \rho(x, x) = \rho(y, y). \end{aligned}$$

So, by (p1) axiom, $|\rho(x_n, y) - \rho(y, y)| = \rho(x_n, y) - \rho(y, y) < (r + \varepsilon)$, for every $n \in A^c$. So, $\{n \in \mathbb{N} : |\rho(x_n, y) - \rho(y, y)| \geq r + \varepsilon\} \subset A$ and hence $d(\{n \in \mathbb{N} : |\rho(x_n, y) - \rho(y, y)| \geq r + \varepsilon\}) = 0$. So, $y \in st - LIM^r x_n$. $\square$

Theorem 3.4. *Let $\{x_n\}$ be a r -statistical convergent sequence in (X, ρ) and $\{y_n\}$ be a convergent sequence in $st - LIM^r x_n$ converging to y . Then y must belong to $st - LIM^r x_n$.*

Proof. If $st - LIM^r x_n = \phi$, then there is nothing to prove. So, we can assume that $st - LIM^r x_n \neq \phi$. Let $\{y_n\}$ be a sequence in $st - LIM^r x_n$ such that $\{y_n\} \rightarrow y$. Let

$\varepsilon > 0$ be given. Since $\{y_n\} \rightarrow y$, there exists $n_{\frac{\varepsilon}{2}} \in \mathbb{N}$ such that $|\rho(y_n, y) - \rho(y, y)| < \frac{\varepsilon}{2}$ for all $n \geq n_{\frac{\varepsilon}{2}}$. Now, choose an $n_0 \in \mathbb{N}$ such that $n_0 > n_{\frac{\varepsilon}{2}}$. Then we can write $|\rho(y_{n_0}, y) - \rho(y, y)| < \frac{\varepsilon}{2}$. On the other hand, because $\{y_n\} \subset st - LIM^r x_n$, we have $y_{n_0} \in st - LIM^r x_n$ and

$$(3.1) \quad d(\{n \in \mathbb{N} : |\rho(x_n, y_{n_0}) - \rho(y_{n_0}, y_{n_0})| \geq r + \frac{\varepsilon}{2}\}) = 0$$

Now, we show

$$(3.2) \quad \{n \in \mathbb{N} : |\rho(x_n, y) - \rho(y, y)| < r + \varepsilon\} \supseteq \{n \in \mathbb{N} : |\rho(x_n, y_{n_0}) - \rho(y_{n_0}, y_{n_0})| < r + \frac{\varepsilon}{2}\}$$

Let $k \in \{n \in \mathbb{N} : |\rho(x_n, y_{n_0}) - \rho(y_{n_0}, y_{n_0})| < r + \frac{\varepsilon}{2}\}$.

Then we have $|\rho(x_k, y_{n_0}) - \rho(y_{n_0}, y_{n_0})| < r + \frac{\varepsilon}{2}$ and hence

$$\begin{aligned} \rho(x_k, y) - \rho(y, y) &\leq \rho(x_k, y_{n_0}) + \rho(y_{n_0}, y) - \rho(y_{n_0}, y_{n_0}) - \rho(y, y) \\ &\leq |\rho(x_k, y_{n_0}) - \rho(y_{n_0}, y_{n_0}) + \rho(y_{n_0}, y) - \rho(y, y)| \\ &\leq |\rho(x_k, y_{n_0}) - \rho(y_{n_0}, y_{n_0})| + |\rho(y_{n_0}, y) - \rho(y, y)| \\ &< (r + \frac{\varepsilon}{2}) + \frac{\varepsilon}{2} \\ &= r + \varepsilon \end{aligned}$$

So, by (p1) axiom, $|\rho(x_k, y) - \rho(y, y)| < r + \varepsilon$. Therefore, $k \in \{n \in \mathbb{N} : |\rho(x_n, y) - \rho(y, y)| < r + \varepsilon\}$, which proves (3.2). From (3.1), we can say that the set on the right-hand side of (3.2) has a natural density of 1 and so, the natural density of the set on the left-hand side of (3.2) is equal to 1. Hence $d(\{n \in \mathbb{N} : |\rho(x_n, y) - \rho(y, y)| \geq r + \varepsilon\}) = 0$. This proves that $y \in st - LIM^r x_n$. $\square$

Corollary 3.1. *Let $\{x_n\}$ be a r -statistical convergent sequence in a partial metric space (X, ρ) . Then $st - LIM^r x_n$ is a closed set for any degree of roughness $r \geq 0$.*

Proof. Since the partial metric space (X, ρ) is first countable [5], the result follows directly from Theorem 3.4. $\square$

Definition 3.3. [3] A sequence $\{x_n\}$ in a partial metric space (X, ρ) is said to be statistically bounded if for any fixed $u \in X$ there exists a positive real number M such that

$$d(\{n \in \mathbb{N} : \rho(x_n, u) \geq M\}) = 0.$$

Remark 3.2. In [15], the definition of boundedness of a sequence in a partial metric space is given as follows:

$\{x_n\}$ is bounded if there exists $M > 0$ such that $\rho(x_n, x_m) \leq M, \forall n, m \in \mathbb{N}$.

This definition is equivalent to Definition 3.3.

Theorem 3.5. *Let (X, ρ) be a partial metric space and a be a positive real number such that $\rho(x, x) = a, \forall x \in X$. Then a sequence $\{x_n\}$ is statistically bounded in (X, ρ) if and only if there exists a non-negative real number r such that $st - LIM^r x_n \neq \phi$.*

Proof. Let $u \in X$ be a fixed element in X . Since the sequence $\{x_n\}$ is statistically bounded, there exists a positive real number M such that $d(\{n \in \mathbb{N} : \rho(x_n, u) \geq M\}) = 0$. Let $\varepsilon > 0$ be arbitrary and $r = M + a$. Now, we show that

$$(3.3) \quad \{n \in \mathbb{N} : |\rho(x_n, u) - \rho(u, u)| < r + \varepsilon\} \supset \{n \in \mathbb{N} : \rho(x_n, u) < M\}$$

Let $i \in \{n \in \mathbb{N} : \rho(x_n, u) < M\}$. Then $\rho(x_i, u) < M$. Now, $\rho(x_i, u) - \rho(u, u) \leq |\rho(x_i, u) - \rho(u, u)| \leq |\rho(x_i, u)| + |\rho(u, u)| < M + a = r < r + \varepsilon$. So, $i \in \{n \in \mathbb{N} : |\rho(x_n, u) - \rho(u, u)| < r + \varepsilon\}$. So, (3.3) holds and hence $\{n \in \mathbb{N} : |\rho(x_n, u) - \rho(u, u)| \geq r + \varepsilon\} \subset \{n \in \mathbb{N} : \rho(x_n, u) \geq M\}$. This implies that $d(\{n \in \mathbb{N} : |\rho(x_n, u) - \rho(u, u)| \geq r + \varepsilon\}) = 0$. Therefore, $u \in st - LIM^r x_n$ i.e. $st - LIM^r x_n \neq \phi$.

Conversely, suppose that $st - LIM^r x_n \neq \phi$. So, let u be a r -limit of $\{x_n\}$. Therefore, for $\varepsilon > 0$, $d(\{n \in \mathbb{N} : |\rho(x_n, u) - \rho(u, u)| \geq r + \varepsilon\}) = 0$. Let $K = \{n \in \mathbb{N} : |\rho(x_n, u) - \rho(u, u)| \geq r + \varepsilon\}$ and $M = r + a + \varepsilon$. Then $d(K) = 0$ and if $n \in K^c$, then

$$\begin{aligned} \rho(x_n, u) &= |\rho(x_n, u) - \rho(u, u) + \rho(u, u)| \\ &\leq |\rho(x_n, u) - \rho(u, u)| + |\rho(u, u)| \\ &< r + \varepsilon + a \\ &< M \end{aligned}$$

So, $n \in \{n \in \mathbb{N} : \rho(x_n, u) < M\}$. This implies $K^c \subset \{n \in \mathbb{N} : \rho(x_n, u) < M\}$. So, $\{n \in \mathbb{N} : \rho(x_n, u) \geq M\} \subset K$. Since $d(K) = 0$, $d(\{n \in \mathbb{N} : \rho(x_n, u) \geq M\}) = 0$. Hence $\{x_n\}$ is statistically bounded. $\square$

Theorem 3.6. *Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that $d(\{n_1, n_2, \dots\}) = 1$, then $st - LIM^r x_n \subseteq st - LIM^r x_{n_k}$.*

Proof. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ and $x \in st - LIM^r x_n$ and let $\varepsilon > 0$. So the set $A = \{n \in \mathbb{N} : |\rho(x_n, x) - \rho(x, x)| \geq r + \varepsilon\}$ has density zero. So, $d(A^c) = 1$. Since the set $K = \{n_1, n_2, \dots\}$ has density 1, $A^c \cap K \neq \phi$. For if $A^c \cap K = \phi$, then $K \subset A$ and so $d(K) = 0$, since $d(A) = 0$. But $d(K) = 1$, then $A^c \cap K \neq \phi$. Let $n_q \in A^c \cap K$. Then $|\rho(x_{n_q}, x) - \rho(x, x)| < r + \varepsilon$ i.e. $n_q \in \{n_k \in K : |\rho(x_{n_k}, x) - \rho(x, x)| < r + \varepsilon\}$. So, $\{n_k \in K : |\rho(x_{n_k}, x) - \rho(x, x)| \geq r + \varepsilon\} \subset A \cup K^c$. This implies that $d(\{n_k \in K : |\rho(x_{n_k}, x) - \rho(x, x)| \geq r + \varepsilon\}) = 0$, since $d(A \cup K^c) \leq d(A) + d(K^c) = 0 + 0 = 0$. Therefore, $x \in st - LIM^r x_{n_k}$. Hence $st - LIM^r x_n \subseteq st - LIM^r x_{n_k}$. $\square$

Theorem 3.7. *Let $\{x_n\}$ and $\{y_n\}$ be two sequences in (X, ρ) such that $\rho(x_n, y_n) \rightarrow 0$ as $n \rightarrow \infty$. If $\{x_n\}$ is r -statistically convergent to x and $\rho(x_n, x_n) \rightarrow 0$*

as $n \rightarrow \infty$, then $\{y_n\}$ is r -statistically convergent to x . Conversely, if $\{y_n\}$ is r -statistically convergent to y and $\rho(y_n, y_n) \rightarrow 0$ as $n \rightarrow \infty$, then $\{x_n\}$ is r -statistically convergent to y .

Proof. Let $\{x_n\}$ be r -statistically convergent to x and let $\varepsilon > 0$ be arbitrary. So, for $\varepsilon > 0$, $d(A) = 0$, where $A = \{n \in \mathbb{N} : |\rho(x_n, x) - \rho(x, x)| \geq r + \frac{\varepsilon}{3}\}$. Again, since $\rho(x_n, y_n) \rightarrow 0$ as $n \rightarrow \infty$, for $\varepsilon > 0$, $\exists k_1 \in \mathbb{N}$ such that

$$(3.4) \quad \rho(x_n, y_n) \leq \frac{\varepsilon}{3}, \text{ when } n \geq k_1.$$

Since $\rho(x_n, x_n) \rightarrow 0$ as $n \rightarrow \infty$, $\exists k_2 \in \mathbb{N}$ such that

$$(3.5) \quad \rho(x_n, x_n) \leq \frac{\varepsilon}{3}, \text{ when } n \geq k_2.$$

Let $k = \max\{k_1, k_2\}$. Then for $n \geq k$, (3.4) and (3.5) both hold.

We can write $\rho(y_n, x) \leq \rho(y_n, x_n) + \rho(x_n, x) - \rho(x_n, x_n)$. So, $\rho(y_n, x) - \rho(x, x) \leq \rho(y_n, x_n) + \rho(x_n, x) - \rho(x_n, x_n) - \rho(x, x)$.

Since $d(A) = 0$, $d(A^c) = 1$. Since $d(\{1, 2, \dots, k\}) = 0$, so $d(\{1, 2, \dots, k\}^c) = 1$. So, $A^c \cap \{1, 2, \dots, k\}^c \neq \emptyset$. Therefore, if $n \in A^c \cap \{1, 2, \dots, k\}^c$, then

$$\begin{aligned} |\rho(y_n, x) - \rho(x, x)| &= |\rho(y_n, x) - \rho(x, x)| \\ &\leq |\rho(y_n, x_n) + \rho(x_n, x) - \rho(x_n, x_n) - \rho(x, x)| \\ &\leq |\rho(y_n, x_n)| + |\rho(x_n, x) - \rho(x, x)| + |\rho(x_n, x_n)| \\ &< \frac{\varepsilon}{3} + (r + \frac{\varepsilon}{3}) + \frac{\varepsilon}{3} \\ &= r + \varepsilon \end{aligned}$$

This implies that $A^c \cap \{1, 2, \dots, k\}^c \subset \{n \in \mathbb{N} : |\rho(y_n, x) - \rho(x, x)| < r + \varepsilon\}$ and hence $\{n \in \mathbb{N} : |\rho(y_n, x) - \rho(x, x)| \geq r + \varepsilon\} \subset (A^c \cap \{1, 2, \dots, k\}^c)^c = A \cup \{1, 2, \dots, k\}$. Since $d(A \cup \{1, 2, \dots, k\}) \leq d(A) + d(\{1, 2, \dots, k\}) = 0 + 0 = 0$, it follows that $d(\{n \in \mathbb{N} : |\rho(y_n, x) - \rho(x, x)| \geq r + \varepsilon\}) = 0$. Therefore, $\{y_n\}$ is r -statistical convergent to x .

Converse part is similar. $\square$

Theorem 3.8. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in (X, ρ) such that $\rho(x_n, y_n) \rightarrow 0$ as $n \rightarrow \infty$. If $\{x_n\}$ is r -statistically convergent to x and a positive number c such that $\rho(x_n, x_n) \leq c$ for all n (i.e. self-distance of the sequence is less or equal to c), then $\{y_n\}$ is $(r + c)$ -statistically convergent to x . Conversely, if $\{y_n\}$ is r -statistically convergent to y and a positive number d such that $\rho(y_n, y_n) \leq d$ for all n , then $\{x_n\}$ is $(r + d)$ -statistically convergent to y .

Proof. The proof is parallel to the proof of the above theorem and so is omitted. $\square$

Definition 3.4. Let (X, ρ) be a partial metric space. Then $c \in X$ is called a statistical cluster point of a sequence $\{x_n\}$ in (X, ρ) if for every $\varepsilon > 0$, $d(\{n \in \mathbb{N} : |\rho(x_n, c) - \rho(c, c)| < \varepsilon\}) \neq 0$.

Theorem 3.9. *Let (X, ρ) be a partial metric space and a be a real constant such that $\rho(x, x) = a, \forall x \in X$. Let $\{x_n\}$ be a sequence in (X, ρ) . If c is a cluster point of $\{x_n\}$, then $st - LIM^r x_n \subset \overline{B_r^\rho}(c)$ for some $r > 0$.*

Proof. If possible suppose that $y \in st - LIM^r x_n$ but $y \notin \overline{B_r^\rho}(c) = \{y \in X : \rho(c, y) \leq \rho(c, c) + r\} = \{y \in X : \rho(c, y) \leq a + r\}$. So, $a + r < \rho(c, y)$. Let $\varepsilon' = \rho(c, y) - (a + r)$, so that $\rho(c, y) = \varepsilon' + a + r$, where $\varepsilon' > 0$. Choose $\varepsilon = \frac{\varepsilon'}{2}$ and so we can write $\rho(c, y) = 2\varepsilon + a + r$. Then $B_{r+\varepsilon}(y) \cap B_\varepsilon(c) = \phi$. [For, if $q \in B_{r+\varepsilon}(y) \cap B_\varepsilon(c)$, then it would imply that $\rho(q, y) < \rho(y, y) + r + \varepsilon = a + r + \varepsilon$ and $\rho(q, c) < \rho(c, c) + \varepsilon = a + \varepsilon$ which in turn implies that $\rho(c, y) \leq \rho(c, q) + \rho(q, y) - \rho(q, q) < \{a + \varepsilon\} + \{a + r + \varepsilon\} - a = a + r + 2\varepsilon = \rho(c, y)$, a contradiction]. Therefore, $B_{r+\varepsilon}(y) \cap B_\varepsilon(c) = \phi$. But since $y \in st - LIM^r x_n$, for $\varepsilon > 0$, $d(A_1) = 0$, where $A_1 = \{n \in \mathbb{N} : |\rho(x_n, y) - \rho(y, y)| \geq r + \varepsilon\}$. Again, since c is a cluster point of $\{x_n\}$, for the same $\varepsilon > 0$, $d(A_2) \neq 0$, where $A_2 = \{n \in \mathbb{N} : |\rho(x_n, c) - \rho(c, c)| < \varepsilon\}$. Now, let $k \in A_1^c \cap A_2$, then $|\rho(x_k, c) - \rho(c, c)| < \varepsilon$. This implies that $\rho(x_k, c) - \rho(c, c) < \varepsilon$. So, $\rho(c, x_k) < \rho(c, c) + \varepsilon$. Hence $x_k \in B_\varepsilon(c)$. Also, $|\rho(x_k, y) - \rho(y, y)| < r + \varepsilon$. So, $\rho(x_k, y) - \rho(y, y) < r + \varepsilon$. This implies that $\rho(y, x_k) < \rho(y, y) + r + \varepsilon$. Hence $x_k \in B_{r+\varepsilon}(y)$. So, $x_k \in B_{r+\varepsilon}(y) \cap B_\varepsilon(c)$, which is a contradiction. Hence $y \in \overline{B_r^\rho}(c)$. $\square$

Acknowledgements

The first author is thankful to The University of Burdwan for the grant of Senior Research Fellowship (State Funded) during the preparation of this paper. Both authors are also thankful to DST, Govt. of India for providing the FIST project to the Dept. of Mathematics, B.U.

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