FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 40, No 2 (2025), 319–332
https://doi.org/10.22190/FUMI240319023B
Original Scientific Paper

EXISTENCE RESULTS FOR A HYBRID SYSTEM OF MIXED DIFFERENTIAL EQUATIONS WITH SEQUENTIAL FRACTIONAL DERIVATIVES

Hammou Benmehidi[1], Mehmet Zeki Sarikaya[2] and Zoubir Dahmani[3]

[1] **Department of Biology, University of Tiaret, Algeria**
[2] **Department of Mathematics, Faculty of Science and Arts, Düzce University Düzce, Turkey**
[3] **Department of Mathematics, University of Blida 1, Algeria**

ORCID IDs: Hammou Benmehidi https://orcid.org/0009-0008-6685-8831
Mehmet Zeki Sarikaya https://orcid.org/0000-0002-6165-9242
Zoubir Dahmani https://orcid.org/0000-0003-4659-0723

Abstract. In this paper, we focus on the study of a hybrid system of sequential type that incorporates both Caputo and Hadamard fractional derivatives. Our approach leverages the fixed point principle to derive novel results concerning the existence and uniqueness of solutions to this system. Additionally, we establish further results by employing Schaefer's fixed point theorem, which allows us to extend the applicability of our findings. To illustrate the practical relevance and application of our theoretical results, we also provide a detailed example at the conclusion of the paper. At the end, an example is given.
Keywords: Caputo fractional derivatives, Hadamard fractional derivatives, Schaefer's fixed point theorem.

1. Introduction

In recent years, fractional calculus has established itself as an essential mathematical framework for the modeling and analysis of complex phenomena across a wide array of scientific disciplines, including but not limited to physics, engineering, biology, and finance. Unlike classical calculus, which is limited to integer-order derivatives,

Received March 19, 2024, accepted: August 19, 2024
Communicated by Marko Petković
Corresponding Author: Mehmet Zeki Sarikaya. E-mail addresses: hammou.benmehidi@univ-tiaret.dz (H. Benmehidi), sarikayamz@gmail.com (M. Z. Sarikaya), zzdahmani@yahoo.fr (Z. Dahmani)
2020 *Mathematics Subject Classification.* Primary 30C45, 39B72; Secondary 39B85

fractional calculus extends the concept of differentiation and integration to non-integer orders. This extension allows for the modeling of systems with memory effects and long-range dependencies, which are characteristic of many natural and engineered processes. Among the various fractional operators, the Caputo and Hadamard fractional derivatives have gained particular prominence due to their applicability in describing anomalous diffusion, viscoelastic materials, and other processes where the current state depends on the history of the system.

The Caputo derivative is widely used for its suitability in initial value problems, aligning well with physical interpretations and boundary conditions. On the other hand, the Hadamard fractional derivative is particularly useful in dealing with problems on unbounded domains, where scale invariance plays a significant role. By incorporating these fractional derivatives into differential equations, researchers can capture more realistic and nuanced dynamics that are often observed in complex systems but are inadequately addressed by traditional integer-order models.

Simultaneously, the study of hybrid systems that combine both discrete and continuous dynamics has emerged as a critical area of research. Many real-world processes, such as those found in robotics, control theory, and biological systems, exhibit behaviours that involve instantaneous changes (jumps) alongside continuous evolution over time. These hybrid systems require sophisticated mathematical models that can account for the interplay between discrete events and continuous processes. For example, in robotics, hybrid systems can model the transition between different operational modes, while in control theory, they are used to design controllers that switch between different strategies depending on the system's state. For more details, see [2, 6, 7, 16].

Understanding the dynamics, stability, and control of hybrid systems is crucial for optimizing performance and ensuring reliable operation in applications where both types of dynamics are present. Fractional hybrid systems, which combine the concepts of fractional calculus and hybrid dynamics, offer a particularly rich field of study. These systems not only incorporate the memory and hereditary properties of fractional calculus but also accommodate the complex interactions between discrete and continuous components. The development of robust mathematical tools and methods for analysing such systems is essential for advancing technologies that depend on precise modelling and control, such as autonomous vehicles, medical devices, and adaptive control systems. For more details, see [1, 5, 8, 11, 13, 15, 17, 18].

Zhao et al. [19] investigated the existence result for the fractional hybrid differential equations with Riemann-Liouville fractional derivatives given by

$$\begin{cases} \left({}^{RL}D^{r}_{0+}\right)\left(\frac{\tau(\xi)}{F(\xi,\tau(\xi))}\right) = G(\xi,\tau(\xi)), \xi \in [0,T], r \in (0,1), \\ \tau(0) = 0, \end{cases}$$

where ${}^{RL}D^{r}$ is Riemann-Liouville fractional derivative, $F : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}/\{0\}, G : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ are assumed to be continuous.

Also, Hilal and Kajouni [10] studied the following Caputo hybrid problem:

$$\begin{cases} \left({}^{C}D_{0+}^{r}\right)\left(\frac{\tau(\xi)}{F(\xi,\tau(\xi))}\right) = G(\xi,\tau(\xi)), \xi \in [0,L], r \in (0,1), \\ a_1 \frac{\tau(0)}{F(0,\tau(0))} + a_2 \frac{\tau(L)}{F(L,\tau(L))} = d, \end{cases}$$

where, $F : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}/\{0\}, G : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ are assumed to be continuous and $a_1 + a_2 \neq 0$.

Then, in [4], the authors considered the following sequential hybrid problem:

$$\begin{cases} \left({}^{C}D_{1-}^{q}\,{}^{RL}D_{0+}^{r}\right)\left(\dfrac{\chi(\tau)}{\hbar(\tau,\chi(\tau),\theta(\tau))}\right) = \psi(\tau,\chi(\tau),\theta(\tau)), \tau \in [0,1], q \in (1,2], r \in (0,1], \\ \left({}^{C}D_{1-}^{q}\,{}^{RL}D_{0+}^{p}\right)\left(\dfrac{\theta(\xi)}{\chi(\tau,\chi(\tau),\vartheta(\tau))}\right) = \varphi(\tau,\chi(\tau),\theta(\tau)), \quad 0 < p \leq 1, \\ \chi(0) = \chi'(0) = 0, \quad \chi(1) = \delta\chi(\zeta), \delta \in \mathbb{R}, \zeta \in (0,1), \\ \theta(0) = \vartheta'(0) = 0, \quad \theta(1) = \varepsilon\theta(\xi), \varepsilon \in \mathbb{R}, \xi \in (0,1). \end{cases}$$

where ${}^{C}D^{q}, {}^{RL}D^{\zeta}, \zeta \in \{p, r\}$ are Caputo and Riemann-Liouville fractional derivative, $\hbar, \chi : [0,1] \times \mathbb{R}^2 \longrightarrow \mathbb{R}/\{0\}, \varphi, \psi : [0,1] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ are assumed to be continuous. Recently, in [3] the authors investigated the existence of the solution for the following hybrid fractional differential equation

$$\begin{cases} {}^{c}\mathcal{D}^{\beta}\left({}^{c}\mathcal{D}^{\omega} + \lambda\right)\left(\frac{\varepsilon_1(\xi)}{\kappa_1(\xi,\varepsilon_1(\xi))}\right) = \mathcal{F}\left(\xi, \varepsilon_1(\xi)\right) \text{ a.e. } \xi \in \mathcal{J} = [0,1] \\ \frac{\varepsilon_1(0)}{\kappa_1(0,\varepsilon_1(0))} + \mu\frac{\varepsilon_1(1)}{\kappa_1(1,\varepsilon_1(1))} = \varrho_1 \int_0^1 \mathfrak{G}\left(\iota_1, \varepsilon_1\left(\iota_1\right)\right) d\iota_1 \\ {}^{c}\mathcal{D}^{\omega}\left(\frac{\varepsilon_1(0)}{\kappa_1(0,\varepsilon_1(0))}\right) + \varphi_1 {}^{c}\mathcal{D}^{\omega}\left(\frac{\varepsilon_1(1)}{\kappa_1(1,\varepsilon_1(1))}\right) = e_2 \int_0^1 \mathfrak{h}\left(\iota_1, \varepsilon_1\left(\iota_1\right)\right) d\iota_1 \\ {}^{c}\mathcal{D}^{2\omega}\left(\frac{\varepsilon_1(0)}{\kappa_1(0,\varepsilon_1(0))}\right) + \varphi_1 {}^{c}\mathcal{D}^{2\omega}\left(\frac{\varepsilon_1(1)}{\kappa_1(1,\varepsilon_1(1))}\right) = \varrho_3 \int_0^1 \mathfrak{l}\left(\iota_1, \varepsilon_1\left(\iota_1\right)\right) d\xi_1, \end{cases}$$

where $0 < \omega < 1, 1 < \beta \leq 2, \lambda, \varphi_1, \varrho_i, \in \mathcal{E}^*$ for $i = 1,2,3$, with $\varphi_1 = -1$. ${}^{c}\mathcal{D}^{\omega}, {}^{c}\mathcal{D}^{\beta}$ are the Caputo's fractional derivatives, and $\kappa_1 : [0,1] \times \mathcal{E} \to \mathcal{E}\{0\}, \omega \in \mathcal{C}([0,1] \times \mathcal{E}, \mathcal{E})$ and $\mathfrak{G}, \mathfrak{h}, \ell : [0,1] \times \mathcal{E} \to \mathcal{E}$ are a given continuous functions.

Motivated by the above research papers, in this article we shall study the existence of the solutions of the following differential system of sequential type:

$$\begin{cases} \left({}^{C}D^{\alpha_1}\,{}^{H}D^{\beta_1}\right)\left(\dfrac{\tau(\xi)}{\eta(\xi,\tau(\xi),\varsigma(\xi))}\right) = F(\xi,\tau(\xi),\varsigma(\xi)), \qquad \xi \in [a,b], a > 0 \\ \left({}^{H}D^{\beta_2}\,{}^{C}D^{\alpha_2}\right)\left(\dfrac{\varsigma(\xi)}{\vartheta(\xi,\tau(\xi),\varsigma(\xi))}\right) = G(\xi,\tau(\xi),\varsigma(\xi)), \\ {}^{H}D^{\beta_1}\left(\frac{\tau(\xi)}{\eta(\xi,\tau(\xi),\varsigma(\xi))}\right)\Big|_{\xi=a} = 0, \quad {}^{C}D^{\alpha_2}\left(\frac{\varsigma(\xi)}{\vartheta(\xi,\tau(\xi),\varsigma(\xi))}\right)\Big|_{\xi=a} = 0, \\ \tau(b) = \theta_1\eta\left(b,\tau(b),\varsigma(b)\right), \quad \varsigma(b) = \theta_2\vartheta\left(b,\tau(b),\varsigma(b)\right), \quad \theta_1, \theta_2 \in \mathbb{R}. \end{cases} \tag{1.1}$$

where ${}^{C}D^{\alpha_i}$, ${}^{H}D^{\beta_i}, i = 1,2$ are the Caputo and Hadamard fractional derivatives of orders α_i and β_i, respectively with, $0 < \alpha_i, \beta_i \leq 1$, $\eta, \vartheta : [a,b] \times \mathbb{R}^2 \longrightarrow \mathbb{R}/\{0\}$ are continuous functions, $\theta_i \in \mathbb{R}, (i = 1,2)$ and $F, G : [a,b] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ are two functions.

2. Preliminaries

In this section, we review some of the basic concepts of fractional calculus, which we will use throughout in this stady. For more details, see the following references [12, 14, 17].

Definition 2.1. [17] The Riemann-Liouville fractional integral of order $\rho > 0$ for a continuous function is defined by

$$^{R}I^{\rho}u(\xi) = \frac{1}{\Gamma(\rho)}\int_a^{\xi}(\xi - s)^{\rho-1}u(s)ds.$$

provided the right-hand side exists on (a, ∞).

Definition 2.2. [17] The Riemann-Liouville fractional derivative of order $\rho > 0$ of a continuous function is defined by

$$\begin{aligned} {}^{\mathrm{R}}D^{\rho}u(t) &= D^{n}I^{n-\rho}u(t) \\ &= \frac{1}{\Gamma(n-\rho)}\left(\frac{d}{dt}\right)^{n}\int_a^t (t-s)^{n-\rho-1}u(s)ds, \quad n-1 < \rho < n, \end{aligned}$$

where $n = [\rho] + 1$ denotes the integer part of real number ρ and $D = d/d\xi$, provided the right-hand side is point-wise defined on (a, ∞).

Definition 2.3. [14] The Hadamard fractional integral of order θ for a continuous function $\psi : [a, \infty) \longrightarrow \mathbb{R}$ is defined as

$$^{H}I^{\theta}\psi(\xi) = \frac{1}{\Gamma(\theta)}\int_a^{\xi}\left(\ln\frac{\xi}{\tau}\right)^{\theta-1}\frac{\psi(\tau)}{\tau}\,\mathrm{d}\tau, \quad \theta > 0.$$

Definition 2.4. [17] For an at least n-times differentiable function $G : (0, \infty) \rightarrow \mathbb{R}$, the Hadamard-Caputo derivative of fractional order $q > 0$ is defined as

$$^{H}D^{q}G(\xi) = \frac{1}{\Gamma(n-q)}\int_0^{\xi}\left(\log\frac{\xi}{s}\right)^{n-q-1}\delta^{n}G(s)\frac{ds}{s}, \ n-1 < q < n, \ n = [q] + 1,$$

where $\delta = \xi\frac{d}{d\xi}$ and $\log(\cdot) = \log_e(\cdot)$

Lemma 2.1. *[12] Let* $AC_{\delta}^{n}[a, b] = \left\{G : [a, b] \rightarrow \mathbb{C} : \delta^{n-1}G(\xi) \in AC[a, b]\right\}$ *and* $u \in AC_{\delta}^{n}[a, b]$ *or* $C_{\delta}^{n}[a, b]$ *and* $q \in \mathbb{C}$. *Then, the following formula holds*

$$^{H}I^{q}\left(^{H}D^{q}\right)u(\xi) = u(\xi) - \sum_{k=0}^{n-1}c_k(\log(\xi/a))^{k},$$

where $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$ $(n = [q] + 1)$.

Lemma 2.2. *[12] Let $\alpha > 0$. Then*

$$^{R}I^{\alpha}D^{\alpha}\tau(\xi) = \tau(\xi) + \sum_{i=0}^{n-1} c_i(\xi - a)^i, \quad n = [\alpha] + 1.$$

In the following lemma, we establish the existence of a solution to the linear problem. Thus, we proceed to prove the lemma below.

Lemma 2.3. *Let $\phi, \psi \in C\left([a, b], \mathbb{R}\right)$. Then, the solution of the system*

$$\begin{cases} \left({}^{C}D^{\alpha_1}\,{}^{H}D^{\beta_1}\right)\left(\dfrac{\tau(\xi)}{\eta(\xi, \tau(\xi), \varsigma(\xi))}\right) = \phi(\xi), \ \xi \in [a, b], \\ \left({}^{H}D^{\beta_2}\,{}^{C}D^{\alpha_2}\right)\left(\dfrac{\varsigma(\xi)}{\vartheta(\xi, \tau(\xi), \varsigma(\xi))}\right) = \psi(\xi), \end{cases} \tag{2.1}$$

subject with the conditions

$$\begin{cases} {}^{H}D^{\beta_1}\left(\frac{\tau(\xi)}{\eta(\xi,\tau(\xi),\varsigma(\xi))}\right)\Big|_{\xi=a} = 0, \quad {}^{C}D^{\alpha_2}\left(\frac{\varsigma(\xi)}{\vartheta(\xi,\tau(\xi),\varsigma(\xi))}\right)\Big|_{\xi=a} = 0, \\ \tau(b) = \theta_1\eta\left(b, \tau(b), \varsigma(b)\right), \quad \varsigma(b) = \theta_2\vartheta\left(b, \tau(b), \varsigma(b)\right), \quad \theta_1, \theta_2 \in \mathbb{R} \end{cases}$$

is given by:

$$\tau(\xi) = \eta(\xi, \tau(\xi), \varsigma(\xi))\left(\theta_1 -^{H} I_{a^+}^{\beta_1}({}^{R}I_{a^+}^{\alpha_1}\phi)(b) +^{H} I_{a^+}^{\beta_1}({}^{R}I_{a^+}^{\alpha_1}\phi)(\xi)\right)$$

and

$$\varsigma(\xi) = \vartheta(\xi, \tau(\xi), \varsigma(\xi))\left(\theta_2 -^{R} I_{a^+}^{\alpha_2}\left({}^{H}I_{a^+}^{\beta_2}\psi\right)(b) +^{R} I_{a^+}^{\alpha_2}\left({}^{H}I_{a^+}^{\beta_2}\psi\right)(\xi)\right).$$

Proof. Applying the operator ${}^{R}I_{a^+}^{\alpha_1}$ to the first equation of (2.1) and using Lemma 2.2, we get

$$^{H}D^{\beta_1}\left(\frac{\tau(\xi)}{\eta(\xi, \tau(\xi), \varsigma(\xi))}\right) = a_0 +^{R} I_{a^+}^{\alpha_1}\phi(\xi), \quad a_0 \in \mathbb{R}.$$

Using the condition ${}^{H}D^{\beta_1}\left(\frac{\tau(\xi)}{\eta(\xi,\tau(\xi),\varsigma(\xi))}\right)\Big|_{\xi=a} = 0$, we get $a_0 = 0$, and consequently,

$$^{H}D^{\beta_1}\left(\frac{\tau(\xi)}{\eta(\xi, \tau(\xi), \varsigma(\xi))}\right) =^{R} I_{a^+}^{\alpha_1}\phi(\xi). \tag{2.2}$$

Applying the integral ${}^{H}I_{a^+}^{\beta_1}$ on (2.2) and using Lemma 2.1, we obtain

$$\frac{\tau(\xi)}{\eta(\xi, \tau(\xi), \varsigma(\xi))} = b_0 +^{H} I_{a^+}^{\beta_1}({}^{R}I_{a^+}^{\alpha_1}\phi)(\xi).$$

Hence, we obtain

$$\tau(\xi) = \eta(\xi, \tau(\xi), \varsigma(\xi)) \times \left(b_0 +^H I_{a^+}^{\beta_1} ({}^R I_{a^+}^{\alpha_1} \phi)(\xi)\right). \tag{2.3}$$

Using $\tau(b) = \theta_1 \eta(b, \tau(b), \varsigma(b))$, we get

$$\theta_1 \eta(b, \tau(b), \varsigma(b)) = \eta(b, \tau(b), \varsigma(b)) \left(b_0 +^H I_{a^+}^{\beta_1} ({}^R I_{a^+}^{\alpha_1} \phi)(b)\right),$$

from where we find

$$b_0 = \theta_1 -^H I_{a^+}^{\beta_1} ({}^R I_{a^+}^{\alpha_1} \phi)(b). \tag{2.4}$$

For the second equation of (2.1), by applying the operator ${}^H I_{a^+}^{\beta_2}$ and exploiting Lemma 2.1, we get

$${}^C D^{\alpha_2} \left(\frac{\varsigma(\xi)}{\vartheta(\xi, \tau(\xi), \varsigma(\xi))}\right) = c_0 +^H I_{a^+}^{\beta_2} \psi(\xi), \ c_0 \in \mathbb{R}.$$

Thanks to condition ${}^C D^{\alpha_2} \left(\frac{\varsigma(\xi)}{\vartheta(\xi, \tau(\xi), \varsigma(\xi))}\right)\Big|_{\xi=a} = 0$ implies $c_0 = 0$. Therefore, we can write

$${}^C D^{\alpha_2} \left(\frac{\varsigma(\xi)}{\vartheta(\xi, \tau(\xi), \varsigma(\xi))}\right) =^H I_{a^+}^{\beta_2} \psi(\xi). \tag{2.5}$$

Taking Riemann-Liouville integral of order α_2 in (2.5), we obtain

$$\left(\frac{\varsigma(\xi)}{\vartheta(\xi, \tau(\xi), \varsigma(\xi))}\right) = d_0 +^R I_{a^+}^{\alpha_2} \left({}^H I_{a^+}^{\beta_2} \psi\right)(\xi), \ d_0 \in \mathbb{R}.$$

Hence, it follows that

$$\varsigma(\xi) = \vartheta(\xi, \tau(\xi), \varsigma(\xi)) \left(d_0 +^R I_{a^+}^{\alpha_2} \left({}^H I_{a^+}^{\beta_2} \psi\right)(\xi)\right). \tag{2.6}$$

Using the condition $\varsigma(b) = \theta_2 \vartheta(b, \tau(b), \varsigma(b))$, we obtain

$$\theta_2 \vartheta(b, \tau(b), \varsigma(b)) = \vartheta(b, \tau(b), \varsigma(b)) \left(d_0 +^R I_{a^+}^{\alpha_2} \left({}^H I_{a^+}^{\beta_2} \psi\right)(b)\right).$$

Then

$$d_0 = \theta_2 -^R I_{a^+}^{\alpha_2} \left({}^H I_{a^+}^{\beta_2} \psi\right)(b). \tag{2.7}$$

By replacing (2.4) in (2.3) and (2.7) in (2.6), the proof is complete.

3. Main Results

Let $\mathcal{C} = \mathcal{C}([a,b],\mathbb{R}), a > 0$, be the set of all continuous functions from $[a,b]$ to $\mathbb{R}$. We introduce the spaces:$X := \{\tau(\xi) : \tau(\xi) \in \mathcal{C}([a,b],\mathbb{R})\}$ with the norm $\|\tau\| = \sup\{|\tau(\xi)|, \xi \in [a,b]\}$. $Y := \{\varsigma(\xi) : \varsigma(\xi) \in \mathcal{C}([a,b],\mathbb{R})\}$ with the norm $\|\varsigma\| = \sup\{|\varsigma(\xi)|, \xi \in [a,b]\}$. Then, the product space $\Big(X \times Y, \|.\|_{X\times Y}\Big)$ is a Banach space endowed with the norm defined as $\|\,(\tau,\varsigma)\,\|_{X\times Y} = \|\tau\| + \|\varsigma\|$. For reasons of simplicity, we may use the notation

$$\Psi_{x,y}(t) = \Psi(t, x(t), y(t)),\ \Psi \in \{f, g, \eta, \vartheta\},$$

$$ {}^{H}I^{\beta RL}_{a^+} I^{\alpha}_{a^+} f_{x,y}(\xi) = \frac{1}{\Gamma(\beta)\Gamma(\alpha)} \int_a^{\xi}\int_a^{\tau} \left(\log\frac{\xi}{\tau}\right)^{\beta-1} (\tau - r)^{\alpha-1} f_{x,y}(r)dr\frac{d\tau}{\tau},$$

and

$$ {}^{RL}I^{\alpha H}_{a^+} I^{\beta}_{a^+} F_{\tau,\varsigma}(\xi) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^{\xi}\int_a^{\tau} (\xi-\tau)^{\alpha-1} \left(\log\frac{\tau}{r}\right)^{\beta-1} F_{\tau,\varsigma}(r)\frac{dr}{r}d\tau,$$

where $\xi \in \{\xi, b\}$. In view of Lemma 2.1, we define the operator $\mathcal{A}$ by:

$$\begin{aligned}\mathcal{A}:\quad & X\times Y \to X \times Y\\ & (x,y)(t) \to \left(\mathcal{A}_1(x,y)(t), \mathcal{A}_2(x,y)(t)\right),\end{aligned}$$

where, $\forall \xi \in [a,b]$,

$$A_1(x,y)(t) := \eta_{x,y}(t)\Big[\theta_1 -^{H} I^{\beta_1}_{a^+}\big(\,{}^{R}I^{\alpha_1}_{a^+} f_{x,y}\big)(b) +^{H} I^{\beta_1}_{a^+}\big(\,{}^{R}I^{\alpha_1}_{a^+} f_{x,y}\big)(t)\Big]$$

and

$$A_2(x,y)(t) := \vartheta_{x,y}(t)\Big[\theta_2 -^{R} I^{\alpha_2}_{a^+}\big(\,{}^{H}I^{\beta_2}_{a^+} g_{x,y}\big)(b) +^{R} I^{\alpha_2}_{a^+}\big(\,{}^{H}I^{\beta_2}_{a^+} g_{x,y}\big)(t)\Big].$$

We impose the following hypotheses:

(H_1) : The functions $F, G : [a,b]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are continuous, and suppose that there exist constants $m_i, n_i, (i = 1,2)$, such that for all $\xi \in [a,b]$ and $\tau_i, \varsigma_i \in \mathbb{R}, (i = 1,2)$

$$|F(\xi,\tau_1,\varsigma_1) - F(\xi,\tau_2,\varsigma_2)| \le m_1|\tau_1-\tau_2| + m_2|\varsigma_1-\varsigma_2|$$

and

$$|G(\xi,\tau_1,\varsigma_1) - G(\xi,\tau_2,\varsigma_2)| \le n_1|\tau_1-\tau_2| + n_2|\varsigma_1-\varsigma_2|.$$

Moreover,

$$M = \sup\{|F(\xi,0,0)| : \xi \in [a,b]\} \text{ and } \widetilde{M} = \sup\{|G(\xi,0,0)| : \xi\in[a,b]\}.$$

(H_2) : Let $\Omega \subset \mathcal{C} \times \mathcal{C}$ be a bounded set, thent there exist constants $\rho_i > 0, (i = 1, 2)$ such that

$$|F(\xi, u(\xi), v(\xi))| \leq \rho_1 \quad \text{and} \quad |G(\xi, u(\xi), v(\xi))| \leq \rho_2, \ \forall (u, v) \in \Omega.$$

(H_3) : We assume that the functions η, ϑ are continuous and there exists $M_\eta, M_\vartheta, \in \mathbb{R}^+$such that

$$|\eta(\xi, u)| \leq M_\eta \ \text{ and } \ |\vartheta(\xi, u)| \leq M_\vartheta$$

To simplify our computations, we put:

$$\begin{aligned}
\Lambda_1 &:= 2 (m_1 + m_2)\, {}^H I^{\beta_1 R} I^{\alpha_1} (1) (b) \\
\Lambda_2 &:= |\theta_1| + 2M\, {}^H I^{\beta_1 R} I^{\alpha_1} (1) (b) \\
\Lambda_3 &:= 2 (n_1 + n_2)\, {}^R I^{\alpha_2 H} I^{\beta_2} (1) (b) \\
\Lambda_4 &:= |\theta_2| + 2\widetilde{M}\, {}^R I^{\alpha_2 H} I^{\beta_2} (1) (b) \\
\Sigma_1 &:= 2M_\eta\ {}^H I^{\beta_1 R} I^{\alpha_1} (1) (b) \\
\Sigma_2 &:= 2M_\vartheta\ {}^R I^{\alpha_2 H} I^{\beta_2} (1) (b).
\end{aligned}$$

Theorem 3.1. *Assume that (H_1) and (H_3) are satisfied. Then the system (1.1) has a unique solution on $[a, b]$ if*

$$(m_1 + m_2) \Sigma_1 + (n_1 + n_2) \Sigma_2 < 1. \tag{3.1}$$

Proof. Define $\mathcal{B}_\varrho = \{(\tau, \varsigma) \in \mathcal{C} \times \mathcal{C} : \|(\tau, \varsigma)\| \leq \varrho\}$ such that

$$\varrho \geq \frac{M_\eta \Lambda_2 + M_\vartheta \Lambda_4}{1 - (M_\eta \Lambda_1 + M_\vartheta \Lambda_3)}.$$

We show that $\mathcal{A}\mathcal{B}_\varrho \subset \mathcal{B}_\varrho$. Let $(\tau, \varsigma) \in \mathcal{B}_\varrho$ and $\xi \in [a, b]$, then we have

$$\begin{aligned}
|A_1(x, y)(t)| &= \left| \eta_{x,y}(t) \left[\theta_1 -^H I_{a^+}^{\beta_1} \left({}^R I_{a^+}^{\alpha_1} f_{x,y} \right)(b) +^H I_{a^+}^{\beta_1} \left({}^R I_{a^+}^{\alpha_1} f_{x,y} \right)(t) \right] \right| \\
&\leq M_\eta \left[|\theta_1| + \left| {}^H I_{a^+}^{\beta_1} \left({}^R I_{a^+}^{\alpha_1} f_{x,y} \right)(b) \right| + \left| {}^H I_{a^+}^{\beta_1} \left({}^R I_{a^+}^{\alpha_1} f_{x,y} \right)(b) \right| \right] \\
&\leq M_\eta \left[|\theta_1| + 2\ {}^H I^{\beta_1 R} I^{\alpha_1} (f_{x,y}) (b) \right] \\
&\leq M_\eta \left[|\theta_1| + 2\ {}^H I^{\beta_1 R} I^{\alpha_1} (|f_{x,y} - f_{0,0}| + |f_{0,0}|) (b) \right] \\
&\leq M_\eta \left[|\theta_1| + 2\ {}^H I^{\beta_1 R} I^{\alpha_1} (m_1 \|x\| + m_2 \|y\| + M) (b) \right] \\
&\leq M_\eta \left[\varrho \Lambda_1 + \Lambda_2 \right].
\end{aligned}$$

Similarly, we can find

$$\begin{aligned}|A_2(x,y)(t)| &= \left|\vartheta_{x,y}(t)\left[\theta_2 - {}^R I_{a^+}^{\alpha_2}\left({}^H I_{a^+}^{\beta_2} g_{x,y}\right)(b) + {}^R I_{a^+}^{\alpha_2}\left({}^H I_{a^+}^{\beta_2} g_{x,y}\right)(t)\right]\right| \\
&\leq M_\vartheta\left[|\theta_2| + \left|{}^R I_{a^+}^{\alpha_2}\left({}^H I_{a^+}^{\beta_2} g_{x,y}\right)(b)\right| + \left|{}^R I_{a^+}^{\alpha_2}\left({}^H I_{a^+}^{\beta_2} g_{x,y}\right)(b)\right|\right] \\
&\leq M_\vartheta\left[|\theta_2| + 2\ {}^R I^{\alpha_2 H} I^{\beta_2}\left(g_{x,y}\right)(b)\right] \\
&\leq M_\vartheta\left[|\theta_2| + 2\ {}^R I^{\alpha_2 H} I^{\beta_2}\left(|g_{x,y} - g_{0,0}| + |f_{0,0}|\right)(b)\right] \\
&\leq M_\vartheta\left[|\theta_2| + 2\ {}^R I^{\alpha_2 H} I^{\beta_2}\left(n_1\|x\| + n_2\|y\| + \widetilde{M}\right)(b)\right] \\
&\leq M_\vartheta\left[\varrho\Lambda_3 + \Lambda_4\right].\end{aligned}$$

Consequently, we have

$$\|\mathcal{A}(x,y)\| \leq \left(M_\eta\Lambda_1 + M_\vartheta\Lambda_3\right)\varrho + M_\eta\Lambda_2 + M_\vartheta\Lambda_4 \leq \varrho,$$

which implies $\mathcal{A}\mathcal{B}_\varrho \subset \mathcal{B}_\varrho$.
Next we show that the operator $\mathcal{A}$ is contractive.
For all $(\tau_1,\varsigma_1),(\tau_2,\varsigma_2) \in X \times Y$, we get

$$\begin{aligned}&\left|\mathcal{A}_1(x_1,y_1)(t) - \mathcal{A}_1(x_2,y_2)(t)\right| \\
&\leq M_\eta\left|\left(\theta_1 - {}^H I^{\beta_1 R} I^{\alpha_1}\left(f_{x_1,y_1}\right)(b) + {}^H I^{\beta_1 R} I^{\alpha_1}\left(f_{x_1,y_1}\right)(t)\right)\right. \\
&\qquad\left. -\left(\theta_1 - {}^H I^{\beta_1 R} I^{\alpha_1}\left(f_{x_2,y_2}\right)(b) + {}^H I^{\beta_1 R} I^{\alpha_1}\left(f_{x_2,y_2}\right)(t)\right)\right| \\
&\leq M_\eta\left(\left|{}^H I^{\beta_1 R} I^{\alpha_1}\left(f_{x_2,y_2} - f_{x_1,y_1}\right)(b)\right| + \left|{}^H I^{\beta_1 R} I^{\alpha_1}\left(f_{x_2,y_2} - f_{x_1,y_1}\right)(t)\right|\right) \\
&\leq 2M_\eta {}^H I^{\beta_1 R} I^{\alpha_1}\left|f_{x_2,y_2} - f_{x_1,y_1}\right|(b) \\
&\leq 2M_\eta\left(m_1|x_1 - x_2| + m_2|y_1 - y_2|\right)^H I^{\beta_1 R} I^{\alpha_1}(1)(b) \\
&\leq \left(m_1 + m_2\right)\Sigma_1\left(|x_1 - x_2| + |y_1 - y_2|\right).\end{aligned}$$

Consequently, we obtain the inequality

$$\|\mathcal{A}_1(x_1,y_1) - \mathcal{A}_1(x_2,y_2)\| \leq \left(m_1 + m_2\right)\Sigma_1\left(\|x_1 - x_2\| + \|y_1 - y_2\|\right). \tag{3.2}$$

Using the same method as before, we obtain

$$\|\mathcal{A}_2(x_1,y_1) - \mathcal{A}_2(x_2,y_2)\| \leq \left(n_1 + n_2\right)\Sigma_2\left(\|x_1 - x_2\| + \|y_1 - y_2\|\right) \tag{3.3}$$

The inequalities in (3.2) and (3.3) allow us to write:

$$\|\mathcal{A}(x_1,y_1) - \mathcal{A}(x_2,y_2)\| \leq \left[\left(m_1 + m_2\right)\Sigma_1 + \left(n_1 + n_2\right)\Sigma_2\right]\left(\|x_1 - x_2\| + \|y_1 - y_2\|\right)$$

Using (3.1), we deduce that the operator $\mathcal{A}$ is a contraction. Then the system (1.1) has a unique solution in $[a, b]$.

The second existence result relies on the following lemma.

Lemma 3.1. *[9] (Leray-Schauder alternative)*
Let $\mathcal{P} : \mathcal{E} \to \mathcal{E}$ be a completely continuous operator. Let also

$$\zeta(\mathcal{P}) = \{\tau \in \mathcal{E} : \tau = \lambda \mathcal{P}(\tau) \text{ for some } 0 < \lambda < 1\}.$$

Then, either set $\zeta(\mathcal{P})$ is unbounded, or $\mathcal{P}$ has at least one fixed point.

Theorem 3.2. *Assume that (H_2) and (H_3) are satisfied. Then, the system (1.1) has at least one solution on $[a, b]$.*

Proof. in the first step, we show that the operator $\mathcal{A}$ is completely continuous. Since the functions F, G, η, ϑ, then the operator $\mathcal{A}$ is continuous. We have

$$\begin{aligned}
|\mathcal{A}_1(x,y)(t)| &= \left|\eta_{x,y}(t)\left[\theta_1 -^H I^{\beta_1}_{a^+}\left({}^R I^{\alpha_1}_{a^+} f_{x,y}\right)(b) +^H I^{\beta_1}_{a^+}\left({}^R I^{\alpha_1}_{a^+} f_{x,y}\right)(t)\right]\right| \\
&\leq M_\eta \left[|\theta_1| + \left| {}^H I^{\beta_1}_{a^+}\left({}^R I^{\alpha_1}_{a^+} f_{x,y}\right)(b)\right| + \left| {}^H I^{\beta_1}_{a^+}\left({}^R I^{\alpha_1}_{a^+} f_{x,y}\right)(t)\right| \right] \\
&\leq M_\eta \left[|\theta_1| + 2 \ {}^H I^{\beta_1 R} I^{\alpha_1} \left(f_{x,y}\right)(b)\right]
\end{aligned}$$

With the aid of (H_2), we obtain

$$|\mathcal{A}_1(x,y)(t)| \leq M_\eta \left[|\theta_1| + 2\rho_1 \ {}^H I^{\beta_1 R} I^{\alpha_1} (1)(b)\right]. \tag{3.4}$$

Similarly,

$$|\mathcal{A}_2(x,y)(t)| \leq M_\vartheta \left[|\theta_2| + 2\rho_2 \ {}^R I^{\alpha_2 H} I^{\beta_2} (1)(b)\right]. \tag{3.5}$$

The inequalities (3.4) and (3.5) allow us to obtain

$$|\mathcal{A}(x,y)(t)| \leq M_\eta \left[|\theta_1| + 2\rho_1 \ {}^H I^{\beta_1 R} I^{\alpha_1} (1)(b)\right] + M_\vartheta \left[|\theta_2| + 2\rho_2 \ {}^R I^{\alpha_2 H} I^{\beta_2} (1)(b)\right].$$

Hence, the operator $\mathcal{A}$ is uniformly bounded.
In the next step, we show the equi-continuity of $\mathcal{A}$. For this, we consider $\xi_1, \xi_2 \in$

$[a, b]$ with $\xi_1 < \xi_2$. Then $\forall(\tau, \varsigma) \in \Omega$, we have

$$\begin{aligned}
&\left|\mathcal{A}_1(x,y)(t_2) - \mathcal{A}_1(x,y)(t_1)\right| \\
&\quad \leq M_\eta \left| \left(\theta_1 - {}^H I^{\beta_1 R} I^{\alpha_1}(f_{x,y})(b) + {}^H I^{\beta_1 R} I^{\alpha_1}(f_{x,y})(t_2)\right) \right. \\
&\qquad \left. - \left(\theta_1 - {}^H I^{\beta_1 R} I^{\alpha_1}(f_{x,y})(b) + {}^H I^{\beta_1 R} I^{\alpha_1}(f_{x,y})(t_1)\right) \right| \\
&\quad \leq M_\eta \left| {}^H I^{\beta_1 R} I^{\alpha_1} |f_{x,y}|(t_2) - {}^H I^{\beta_1 R} I^{\alpha_1} |f_{x,y}|(t_1) \right| \\
&\quad \leq \frac{\rho_1 M_\eta}{\Gamma(\beta_1)\Gamma(\alpha_1+1)} \left| \int_a^{t_1} \left[\left(\log\frac{t_2}{s}\right)^{\beta_1-1} - \left(\log\frac{t_1}{s}\right)^{\beta_1-1} \right] \right. \\
&\qquad \left. \times (s-a)^{\alpha_1}\frac{ds}{s} + \int_{t_1}^{t_2} \left(\log\frac{t_2}{s}\right)^{\beta_1-1} (s-a)^{\alpha_1}\frac{ds}{s} \right|
\end{aligned}$$

and then,

$$\begin{aligned}
&|\mathcal{A}_1(x,y)(t_2) - \mathcal{A}_1(x,y)(t_1)| \\
&\quad \leq \frac{\rho_1 M_\eta (b-a)^{\alpha_1}}{\Gamma(\beta_1+1)\Gamma(\alpha_1+1)} \left[2\left(\log\frac{t_2}{t_1}\right)^{\beta_1} + \left| \left(\log\frac{t_2}{a}\right)^{\beta_1} - \left(\log\frac{t_1}{a}\right)^{\beta_1} \right| \right].
\end{aligned} \tag{3.6}$$

While $\xi_1 \to \xi_2$, the right side of the inequality (3.6) tends to zero.
Following the same steps as before, we get

$$|\mathcal{A}_2(\tau,\varsigma)(\xi_2) - \mathcal{A}_2(\tau,\varsigma)(\xi_1)| \to 0 \text{ as } \xi_1 \to \xi_2.$$

So, $\mathcal{A}\Omega$ is equicontinuous.
Using Arzela-Ascoli theorem, we conclude that $\mathcal{A}$ is completely continuous.
In the last step we show the boundedness of

$$\mathcal{F} = \{(\tau,\varsigma) \in X \times Y : (\tau,\varsigma) = \lambda\mathcal{A}(\tau,\varsigma),\, 0 \leq \lambda \leq 1\}.$$

Let $(\tau,\varsigma) \in \mathcal{F}$, then $\forall \xi \in [a,b]$, the equation $(\tau,\varsigma) = \lambda\mathcal{A}(\tau,\varsigma)$ gives us

$$\tau(\xi) = \lambda\mathcal{A}_1(\tau,\varsigma)(\xi), \quad \varsigma(\xi) = \lambda\mathcal{A}_2(\tau,\varsigma)(\xi).$$

With the aid of (H_2), we find

$$\|x\| \leq M_\eta|\theta_1| + \rho_1\Sigma_1$$

$$\|y\| \leq M_\vartheta|\theta_2| + \rho_2\Sigma_2.$$

Consequently, we have

$$\|(\tau,\varsigma)\| \leq M_\eta|\theta_1| + M_\vartheta|\theta_2| + \rho_1\Sigma_1 + \rho_2\Sigma_2. \tag{3.7}$$

The above inequality (3.7) shows that $\mathcal{F}$ is bounded.
Using the above lemma, we deduce that the problem (1.1) has at least one solution on $[a, b]$.

4. Example

We consider the following example:

$$
(4.1)\quad \begin{cases}
\left({}^{C}D^{\frac{1}{5}}\,{}^{H}D^{\frac{4}{9}}\right)\left(\dfrac{\tau(\xi)}{\eta(\xi,\tau(\xi),\varsigma(\xi))}\right) = F(\xi,\tau(\xi),\varsigma(\xi)), \quad \xi\in[2,5] \\[2ex]
\left({}^{H}D^{\frac{2}{3}}\,{}^{C}D^{\frac{3}{4}}\right)\left(\dfrac{\varsigma(\xi)}{\vartheta(\xi,\tau(\xi),\varsigma(\xi))}\right) = G(\xi,\tau(\xi),\varsigma(\xi)), \quad \xi\in[2,5] \\[2ex]
{}^{H}D^{\frac{4}{9}}\left(\frac{\tau(\xi)}{\eta(\xi,\tau(\xi),\varsigma(\xi))}\right)\Big|_{\xi=2} = 0, \quad {}^{C}D^{\frac{3}{4}}\left(\frac{\varsigma(\xi)}{\vartheta(\xi,\tau(\xi),\varsigma(\xi))}\right)\Big|_{\xi=2} = 0, \\[2ex]
\tau(5) = \dfrac{17}{25}\eta\left(b,\tau(b),\varsigma(b)\right), \quad \varsigma(5) = \dfrac{31}{7}\vartheta\left(5,\tau(5),\varsigma(5)\right).
\end{cases}
$$

We have

$$
\begin{aligned}
&\alpha_1=\frac{1}{5},\ \alpha_2=\frac{3}{4},\ \beta_1=\frac{4}{9},\ \beta_2=\frac{2}{3},\ \theta_1=\frac{17}{25},\ \theta_2=\frac{31}{7},\\
&\eta(t,x(t),y(t))=\frac{1}{100}\left|\sin x(t)\right|+\left|\cos y(t)\right|, \quad \vartheta(t,x(t),y(t))=\frac{1}{50}\left|\sin y(t)\right|+3,
\end{aligned}
$$

1. In order to apply Theorem 3.1, we consider two nonlinear functions F,G : $[2,5]\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ such that

$$
\begin{aligned}
f(t,x(t),y(t)) &= e^{2t}+\frac{5}{20\sqrt{t^2+12}}\cos x(t)+\frac{1}{15}\frac{y(t)}{t+2},\\
g(t,u(t),v(t)) &= \frac{\sin x(t)}{t^2+3}+\frac{tan^{-1}y(t)}{t^3}+\frac{8}{11}.
\end{aligned}
$$

Since

$$
\begin{aligned}
&\left|f\left(t,x_1,y_1\right)-f\left(t,x_2,y_2\right)\right| \le \frac{5}{80}\left|x_1-x_2\right|+\frac{1}{60}\left|y_1-y_2\right|,\\
&\left|g\left(t,x_1,y_1\right)-g\left(t,x_2,y_2\right)\right| \le \frac{1}{7}\left|x_1-x_2\right|+\frac{1}{8}\left|y_1-y_2\right|,\\
&|\eta(t,x(t),y(t))| \le 0,105\ , \quad |\vartheta(t,x(t),y(t))| \le 0,06,\\
&\left(m_1+m_2\right)\Sigma_1+\left(n_1+n_2\right)\Sigma_2 = 0,1387 < 1.
\end{aligned}
$$

Consequently, F,G satisfy (H_1) and η,ϑ satisfy condition (H_3). Thus, the problem (4.1) satisfies the conditions of Theorem 3.1.

2. Now, we consider: $F,G:[2,5]\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ defined as

$$
\begin{aligned}
f(t,x,y) &= e^{-t}+\frac{x(t)}{1+x(t)}+\frac{1}{30}\tan^{-1}y(t),\\
g(t,x,y) &= \cos t+\frac{1}{7}\tan^{-1}x(t)+\frac{|y(t)|\,e^{-t}}{4(1+|y(t)|)}.
\end{aligned}
$$

Then, $\forall \xi \in [2,5]$, we have

$$|f(t,x(t),y(t))| \leq 2 + \frac{\pi}{15} \quad \text{and} \quad |g(t,x(t),y(t))| \leq \frac{5}{4} + \frac{2\pi}{7}.$$

Thus, all conditions of Theorem 3.2 hold. Thus, the problem (4.1) has at least one solution defined on $[2,5]$.

5. Conclusion

In this paper, we have investigated a hybrid system of sequential type that integrates both Caputo and Hadamard fractional derivatives. By utilizing the fixed point principle, we have derived new and significant results regarding the existence and uniqueness of solutions to this complex system. Furthermore, we extended our analysis by applying Schaefer's fixed point theorem, which has broadened the scope and applicability of our findings.

Theoretical results are often abstract, but their true value lies in their applicability to real-world problems. To demonstrate this, we provided a detailed example that illustrates how our results can be applied to practical scenarios. This example not only validates our theoretical work but also offers insights into how such hybrid systems can be modelled and analysed using the methodologies we have developed.

The results presented in this paper contribute to the growing body of knowledge on fractional calculus and hybrid systems. They offer a robust framework for further exploration in both theoretical and applied contexts. Future research could build on our findings by exploring the effects of different fractional derivatives or by applying these techniques to more complex systems in various scientific disciplines. We hope that our work will serve as a foundation for future studies and inspire further advancements in this evolving field.

REFERENCES

1. B. Ahmad, K. Ntouyas and A. Alsaedi: *Existence theory for nonlocal boundary value problems involving mixed fractional derivatives.* Nonlinear analysis: modelling and control **24**(6) (2019), 937–957.
2. M. Ahmad, A. Zada, M. Ghaderi, R. George and S. Rezapour: *On the existence and stability of a neutral stochastic fractional differential system.* Fractal and Fractional **6**(4) (2022), 203.
3. M. Arab, M. Awadalla, M. Manigandan, K. Abuasbeh, N. I. Mahmudov and T. Nandha Gopal: *On the existence results for a mixed hybrid fractional differential equations of sequential type.* Fractal and Fractional **7**(3) (2023), 229.
4. M. Awadalla and N. I. Mahmudov: *On system of mixed fractional hybrid differential equations.* Journal of Function Spaces (2022).
5. R. L. Bagley and P. Torvik: *A theoretical basis for the application of fractional calculus to viscoelasticity.* Journal of Rheology **27**(3) (1983), 201–210.

6. D. Baleanu, A. Jajarmi, H. Mohammadi and S. Rezapour: *A new study on the mathematical modelling of human liver with Caputo–Fabrizio fractional derivative.* Chaos, Solitons & Fractals **134** (2020), 109705.

7. S. Chávez-Vázquez, J. E. Lavín-Delgado, J. F. Gómez-Aguilar, J. R. Razo-Hernández, S. Etemad, and S. Rezapour: *Trajectory tracking of Stanford robot manipulator by fractional-order sliding mode control.* Applied Mathematical Modelling **120** (2023), 436–462.

8. R. Gorenflo, F. Mainardi and I. Podlubny: *Fractional differential equations.* Academic Press **8** (1999), 683–699.

9. A. Granas and J. Dugundji: *Fixed point theory.* vol. 14. Springer (2003).

10. K. Hilal and A. Kajouni: *Boundary value problems for hybrid differential equations with fractional order.* Advances in Difference Equations **2015**(1) (2015), 1–19.

11. R. Hilfer: *Applications of fractional calculus in physics.* World scientific (2000).

12. F. Jarad, T. Abdeljawad and D. Baleanu: *Caputo-type modification of the Hadamard fractional derivatives.* Advances in Difference Equations **2012** (2012), 1–8.

13. C. Kiataramkul, W. Yukunthorn, S. K. Ntouyas and J. Tariboon: *Sequential riemann-liouville and hadamard-caputo fractional differential systems with nonlocal coupled fractional integral boundary conditions.* Axioms **10**(3) (2021), 174.

14. M. M. Matar: *Solution of sequential hadamard fractional differential equations by variation of parameter technique.* In Abstract and Applied Analysis, vol. 2018. Hindawi (2018).

15. M. M. Matar, J. Alzabut and J. M. Jonnalagadda: *A coupled system of nonlinear caputo-hadamard langevin equations associated with nonperiodic boundary conditions.* Mathematical Methods in the Applied Sciences **44**(3) (2021), 2650–2670.

16. H. Mohammadi, S. Kumar, S. Rezapour and S. Etemad: *A theoretical study of the Caputo–Fabrizio fractional modeling for hearing loss due to Mumps virus with optimal control.* Chaos, Solitons & Fractals **144** (2021), 110668.

17. B. Ross: *Fractional calculus and its applications.* Proceedings of the international conference held at the University of New Haven, June 1974, vol. 457. Springer (2006).

18. A. Samadi, S. K. Ntouyas and J. Tariboon: *Nonlocal coupled hybrid fractional system of mixed fractional derivatives via an extension of Darbo's theorem.* AIMS Mathematics **6**(4) (2021), 3915–3926.

19. Y. Zhao, S. Sun, Z. Han and Q. Li: *Theory of fractional hybrid differential equations.* Computers and Mathematics with Applications **62**(3) (2011), 1312–1324.