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ON DOUBLE WIJSMAN STRONG DEFERRED CESÀRO SUMMABLE SET SEQUENCES

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Abstract. By this study, for double sequences of sets, we first presented strongly r-deferred Cesàro summability ($0 < r < \infty$) concept (in Wijsman sense), and we examined relations between this concept and deferred statistical convergence concept. Then, we gave theorems associated with Wijsman strongly r-deferred Cesàro summability concept.
Keywords: Wijsman strongly r-deferred Cesàro summability, deferred statistical convergence.

1. Introduction

This study is based on Wijsman convergence concept, which is one of convergence concepts for sequences of sets. For double sequences, Wijsman convergence concept was developed by Nuray et al. [9]. Also, Wijsman Cesàro summability and Wijsman statistical convergence concepts for double sequences were presented by Nuray et al. in [9] and [8], respectively.

For sequences, the deferred Cesàro mean, acquainted by Agnew [1], was developed and presented as deferred statistical convergence concept by Küçükaslan and Yılmaztürk [7]. For double sequences, deferred Cesàro summability and statistical

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convergence concepts were given by Dağadur-Sezgek [4, 10]. Also, for sequences of sets, Altınok et al. [2] presented Wijsman deferred Cesàro summability and statistical convergence concepts. Recently, Ulusu-Gülle [11] studied on Wijsman deferred Cesàro summability and Wijsman deferred statistical convergence concepts for double sequences. See [5, 6] for more details.

2. Basic Concepts

In this part of the study, we remind some fundamental definitions and notations (See, [3, 4, 8–11]).

In a metric space (X,ρ), the distance function $\mu=\mu(x,C):=\mu_x(C)$ is defined by

$$\mu_x(C)=\inf_{y\in C}\rho(x,y)$$

for any non-empty $C\subseteq X$ and any $x\in X$.

On a non-empty set X, for a function $g:\mathbb{N}\to 2^X$, $g(j)=C_j\in 2^X$, the sequence $\{C_j\}=\{C_1,C_2,\ldots\}$ is said to be sequences of sets.

All through this work, (X,ρ) is conceived as a metric space and C, C_{jk} $(j,k\in\mathbb{N})$ are conceived any non-empty closed subsets of X.

A double sequence $\{C_{jk}\}$ is (in Wijsman sense);

i. convergent to set C provided that

$$\lim_{j,k\to\infty}\mu_x(C_{jk})=\mu_x(C),$$

ii. strongly r-Cesàro summable $(0<r<\infty)$ to set C provided that

$$\lim_{u,v\to\infty}\frac{1}{uv}\sum_{j,k=1,1}^{u,v}|\mu_x(C_{jk})-\mu_x(C)|^r=0,$$

iii. statistically convergent to set C provided that for every $\varepsilon>0$

$$\lim_{u,v\to\infty}\frac{1}{uv}\Big|\Big\{(j,k):j\le u,\,k\le v:|\mu_x(C_{jk})-\mu_x(C)|\ge\varepsilon\Big\}\Big|=0,$$

for each $x\in X$. The notations $C_{jk}\xrightarrow{W_2}C$, $C_{jk}\xrightarrow{W_2[\sigma_r]}C$ and $C_{jk}\xrightarrow{W_2(S)}C$ are used respectively.

For a double sequence $x=(x_{jk})$, the deferred Cesàro mean $D_{\varphi,\psi}$ is defined by

$$(D_{\varphi,\psi}x)_{uv}=\frac{1}{\varphi_u\psi_v}\sum_{j=p_u+1}^{q_u}\sum_{k=s_v+1}^{t_v}x_{jk}:=\frac{1}{\varphi_u\psi_v}\sum_{\substack{j=p_u+1\\k=s_v+1}}^{q_u,t_v}x_{jk},$$

where (p_u), (q_u), (s_v), (t_v) are non-negative integer sequences satisfying the following conditions:

$$p_u < q_u, \lim_{u\to\infty} q_u = \infty; \quad s_v < t_v, \lim_{v\to\infty} t_v = \infty \tag{2.1}$$

and

$$q_u - p_u = \varphi_u; \quad t_v - s_v = \psi_v. \tag{2.2}$$

Note here that the method $D_{\varphi,\psi}$ is openly regular for any selection of the sequences (p_u), (q_u), (s_v), (t_v).

All through this study, except where otherwise stated, (p_u), (q_u), (s_v), (t_v) are conceived non-negative integer sequences satisfying (2.1) and (2.2).

A double sequence $\{C_{jk}\}$ is (in Wijsman sense);

i. deferred Cesàro summable to set C provided that

$$\lim_{u,v\to\infty} \frac{1}{\varphi_u\psi_v} \sum_{\substack{j=p_u+1\\k=s_v+1}}^{q_u,t_v} \mu_x(C_{jk}) = \mu_x(C),$$

ii. deferred statistical convergent to set C provided that for every $\varepsilon > 0$

$$\lim_{u,v\to\infty} \frac{1}{\varphi_u\psi_v} \left|\left\{(j,k) : p_u < j \le q_u,\, s_v < k \le t_v,\, |\mu_x(C_{jk}) - \mu_x(C)| \ge \varepsilon\right\}\right| = 0,$$

for each $x \in X$. The notations $C_{jk} \xrightarrow{W_2D} C$ and $C_{jk} \xrightarrow{W_2DS} C$ are used respectively. Also, $\{W_2DS\}$ denotes the class of W_2DS-convergent double sequences of sets.

3. Main Definition and Theorems

In this part of the study, for double sequences of sets, we first presented strongly r-deferred Cesàro summability $(0 < r < \infty)$ concept (in Wijsman sense), and we examined the relations between this concept and the deferred statistical convergence concept. Then, we proved theorems associated with Wijsman strongly r-deferred Cesàro summability concept.

Definition 3.1. A double sequence $\{C_{jk}\}$ is Wijsman strongly r-deferred Cesàro summable $(0 < r < \infty)$ to set C provided that

$$\lim_{u,v\to\infty} \frac{1}{\varphi_u\psi_v} \sum_{\substack{j=p_u+1\\k=s_v+1}}^{q_u,t_v} |\mu_x(C_{jk}) - \mu_x(C)|^r = 0,$$

for each $x \in X$ and the notation $C_{jk} \xrightarrow{W_2[D]^r} C$ is used.

The double sequence is called Wijsman strongly deferred Cesàro summable to set C if $r = 1$ and the notation $C_{jk} \overset{W_2[D]}{\longrightarrow} C$ is used.

The class of Wijsman strongly r-deferred Cesàro summable double sequences of sets will be denoted by $\{W_2[D]^r\}$.

Remark 3.1.

(i) For $p_u = 0$, $q_u = u\,; s_v = 0$, $t_v = v$, this newly defined concept matches with Wijsman strongly r-Cesàro summability concept in [9].

(ii) For $p_u = j_{u-1}$, $q_u = j_u\,; s_v = k_{v-1}$, $t_v = k_v$ ((j_u, k_v) states double lacunary sequence), this newly defined concept matches with Wijsman strongly r-lacunary summability concept in [9].

Theorem 3.1. *If* $C_{jk} \overset{W_2[D]^r}{\longrightarrow} C$ *where* $0 < r < \infty$*, then* $C_{jk} \overset{W_2DS}{\longrightarrow} C$*.*

Proof. Assume that $C_{jk} \overset{W_2[D]^r}{\longrightarrow} C$ and $0 < r < \infty$. For every $\varepsilon > 0$, we can write the following inequality

$$\sum_{\substack{j=p_u+1\\k=s_v+1}}^{q_u,t_v} |\mu_x(C_{jk}) - \mu_x(C)|^r$$

$$\geq \sum_{\substack{j=p_u+1\\k=s_v+1\\|\mu_x(C_{jk})-\mu_x(C)|\geq\varepsilon}}^{q_u,t_v} |\mu_x(C_{jk}) - \mu_x(C)|^r$$

$$\geq \varepsilon^r \left|\left\{(j,k) : p_u < j \leq q_u,\ s_v < k \leq t_v,\ |\mu_x(C_{jk}) - \mu_x(C)| \geq \varepsilon\right\}\right|,$$

for each $x \in X$ and so

$$\frac{1}{\varepsilon^r}\frac{1}{\varphi_u\psi_v} \sum_{\substack{j=p_u+1\\k=s_v+1}}^{q_u,s_v} |\mu_x(C_{jk}) - \mu_x(C)|^r$$

$$\geq \frac{1}{\varphi_u\psi_v}\left|\left\{(j,k) : p_u < j \leq q_u,\ s_v < k \leq t_v,\ |\mu_x(C_{jk}) - \mu_x(C)| \geq \varepsilon\right\}\right|.$$

For $u, v \to \infty$, considering our acceptance, we obtain that $C_{jk} \overset{W_2DS}{\longrightarrow} C$. □

The converse of Theorem (3.1) is provided when the sequence $\{C_{jk}\}$ is bounded. Otherwise, it is not provided (see, Example 3.3 in [11]).

A double sequence $\{C_{jk}\}$ is bounded provided that $\sup \mu_x(C_{jk}) < \infty$, for each $x \in X$. Also, all bounded double sequences of sets are denoted by L^2_∞.

Theorem 3.2. *If* $\{C_{jk}\} \in L^2_\infty$ *and* $C_{jk} \overset{W_2DS}{\longrightarrow} C$*, then* $C_{jk} \overset{W_2[D]^r}{\longrightarrow} C$ *where* $0 < r < \infty$.

Proof. Assume that $\{C_{jk}\} \in L^2_\infty$ and $C_{jk} \overset{W_2DS}{\longrightarrow} C$. Since $\{C_{jk}\} \in L^2_\infty$, there is an $\mathcal{M} > 0$ such that

$$|\mu_x(C_{jk}) - \mu_x(C)| \leq \mathcal{M}$$

for all $j, k \in \mathbb{N}$ and each $x \in X$.
Thus, for every $\varepsilon > 0$, we have

$$\begin{aligned}
&\frac{1}{\varphi_u \psi_v} \sum_{\substack{j=p_u+1 \\ k=s_v+1}}^{q_u, t_v} |\mu_x(C_{jk}) - \mu_x(C)|^r \\
&= \frac{1}{\varphi_u \psi_v} \sum_{\substack{j=p_u+1 \\ k=s_v+1 \\ |\mu_x(C_{jk})-\mu_x(C)| \geq \varepsilon}}^{q_u, t_v} |\mu_x(C_{jk}) - \mu_x(C)|^r \\
&\quad + \frac{1}{\varphi_u \psi_v} \sum_{\substack{j=p_u+1 \\ k=s_v+1 \\ |\mu_x(C_{jk})-\mu_x(C)| < \varepsilon}}^{q_u, t_v} |\mu_x(C_{jk}) - \mu_x(C)|^r \\
&\leq \frac{\mathcal{M}^r}{\varphi_u \psi_v} \left| \left\{ (j,k) : p_u < j \leq q_u,\ s_v < k \leq t_v,\ |\mu_x(C_{jk}) - \mu_x(C)| \geq \varepsilon \right\} \right| + \varepsilon^r
\end{aligned}$$

for each $x \in X$. For $u, v \to \infty$, considering our acceptance, we obtain that $C_{jk} \overset{W_2[D]^r}{\longrightarrow} C$. □

Corollary 3.1. $L^2_\infty \cap \{W_2[D]^r\} = L^2_\infty \cap \{W_2DS\}$.

Theorem 3.3. *If* $\{A_{jk}\}$, $\{B_{jk}\}$ *and* $\{C_{jk}\}$ *are double sequences of sets such that* $A_{jk} \subset B_{jk} \subset C_{jk}$ *for all* $j, k \in \mathbb{N}$*, then*

$$A_{jk} \overset{W_2[D]^r}{\longrightarrow} C \text{ and } C_{jk} \overset{W_2[D]^r}{\longrightarrow} C \Rightarrow B_{jk} \overset{W_2[D]^r}{\longrightarrow} C.$$

Proof. Assume that $A_{jk} \overset{W_2[D]^r}{\longrightarrow} C$, $C_{jk} \overset{W_2[D]^r}{\longrightarrow} C$ and $A_{jk} \subset B_{jk} \subset C_{jk}$. For all $j, k \in \mathbb{N}$,

$$\begin{aligned}
&A_{jk} \subset B_{jk} \subset C_{jk} \\
&\qquad \Rightarrow \mu_x(C_{jk}) \leq \mu_x(B_{jk}) \leq \mu_x(A_{jk}) \\
&\qquad \Rightarrow |\mu_x(C_{jk}) - \mu_x(C)| \leq |\mu_x(B_{jk}) - \mu_x(C)| \leq |\mu_x(A_{jk}) - \mu_x(C)| \\
&\qquad\quad (\text{ or } |\mu_x(A_{jk}) - \mu_x(C)| \leq |\mu_x(B_{jk}) - \mu_x(C)| \leq |\mu_x(C_{jk}) - \mu_x(C)|\,)
\end{aligned}$$

is held for each $x \in X$.

Thus, we have

$$\frac{1}{\varphi_u \psi_v} \sum_{\substack{j=p_u+1 \\ k=s_v+1}}^{q_u, t_v} |\mu_x(C_{jk}) - \mu_x(C)|^r$$

$$\leq \frac{1}{\varphi_u \psi_v} \sum_{\substack{j=p_u+1 \\ k=s_v+1}}^{q_u, t_v} |\mu_x(B_{jk}) - \mu_x(C)|^r$$

$$\leq \frac{1}{\varphi_u \psi_v} \sum_{\substack{j=p_u+1 \\ k=s_v+1}}^{q_u, t_v} |\mu_x(A_{jk}) - \mu_x(C)|^r.$$

Both two cases, for $u, v \to \infty$, considering our acceptance, we obtain that $B_{jk} \overset{W_2[D]^r}{\longrightarrow} C$. □

Theorem 3.4. *If* $\left(\frac{p_u}{\varphi_u}\right)$ *and* $\left(\frac{s_v}{\psi_v}\right)$ *are bounded, then* $C_{jk} \overset{W_2[\sigma_r]}{\longrightarrow} C$ *implies that* $C_{jk} \overset{W_2[D]^r}{\longrightarrow} C$.

Proof. Assume that $\left(\frac{p_u}{\varphi_u}\right)$ and $\left(\frac{s_v}{\psi_v}\right)$ are bounded. Here, we can write the following equality

$$\frac{1}{\varphi_u \psi_v} \sum_{\substack{j=p_u+1 \\ k=s_v+1}}^{q_u, t_v} |\mu_x(C_{jk}) - \mu_x(C)|^r$$

$$= \frac{1}{\varphi_u \psi_v} \left[\sum_{\substack{j=1 \\ k=1}}^{q_u, t_v} - \sum_{\substack{j=1 \\ k=1}}^{p_u, t_v} - \sum_{\substack{j=1 \\ k=1}}^{q_u, s_v} + \sum_{\substack{j=1 \\ k=1}}^{p_u, s_v} \right] |\mu_x(C_{jk}) - \mu_x(C)|^r$$

$$= \frac{q_u t_v}{\varphi_u \psi_v} \left(\frac{1}{q_u t_v} \sum_{\substack{j=1 \\ k=1}}^{q_u, t_v} |\mu_x(C_{jk}) - \mu_x(C)|^r \right) - \frac{p_u t_v}{\varphi_u \psi_v} \left(\frac{1}{p_u t_v} \sum_{\substack{j=1 \\ k=1}}^{p_u, t_v} |\mu_x(C_{jk}) - \mu_x(C)|^r \right)$$

$$- \frac{q_u s_v}{\varphi_u \psi_v} \left(\frac{1}{q_u s_v} \sum_{\substack{j=1 \\ k=1}}^{q_u, s_v} |\mu_x(C_{jk}) - \mu_x(C)|^r \right) + \frac{p_u s_v}{\varphi_u \psi_v} \left(\frac{1}{p_u s_v} \sum_{\substack{j=1 \\ k=1}}^{p_u, s_v} |\mu_x(C_{jk}) - \mu_x(C)|^r \right)$$

for each $x \in X$. Clearly, Wijsman strongly r-deferred Cesàro summability of the sequence $\{C_{jk}\}$ is equal to the linear combination of Wijsman strongly r-Cesàro summability of same sequence. Here, this linear combination can be considered as a matrix transformation. For this matrix transformation to be regular (which is desirable), the sequence

$$\left\{ \frac{(p_u + q_u)(s_v + t_v)}{\varphi_u \psi_v} \right\}$$

must be bounded that similar situation was shown in [10]. Thus, the proof is completed. □

The last two theorems were considered under the following prerequisites:

$$p_u \leq p'_u < q'_u \leq q_u \quad \text{and} \quad s_v \leq s'_v < t'_v \leq t_v$$

for all $u, v \in \mathbb{N}$, where these are non-negative integer sequences.

Theorem 3.5. *If* $\left(\frac{\varphi'_u \, \psi'_v}{\varphi_u \, \psi_v}\right) \to L \in \mathbb{R}$, *then* $\{W_2[D]^r\}_{[\varphi,\psi]} \subseteq \{W_2[D]^r\}_{[\varphi',\psi']}$.

Proof. Assume that $\{C_{jk}\} \in \{W_2[D]^r\}_{[\varphi,\psi]}$ and $C_{jk} \overset{W_2[D]^r_{[\varphi,\psi]}}{\longrightarrow} C$. Here, we can write the following inequality

$$\begin{aligned} \frac{1}{\varphi_u \psi_v} \sum_{\substack{j=p_u+1 \\ k=s_v+1}}^{q_u, t_v} |\mu_x(C_{jk}) - \mu_x(C)|^r \quad &\geq \quad \frac{1}{\varphi_u \psi_v} \sum_{\substack{j=p'_u+1 \\ k=s'_v+1}}^{q'_u, t'_v} |\mu_x(C_{jk}) - \mu_x(C)|^r \\ &\geq \quad \frac{\varphi'_u \psi'_v}{\varphi_u \psi_v} \left(\frac{1}{\varphi'_u \psi'_v} \sum_{\substack{j=p'_u+1 \\ k=s'_v+1}}^{q'_u, t'_v} |\mu_x(C_{jk}) - \mu_x(C)|^r \right) \end{aligned}$$

for each $x \in X$. Then, for $u, v \to \infty$, considering our acceptance, we obtain that $C_{jk} \overset{W_2[D]^r_{[\varphi',\psi']}}{\longrightarrow} C$ and $\{C_{jk}\} \in \{W_2[D]^r\}_{[\varphi',\psi']}$. Consequently, $\{W_2[D]^r\}_{[\varphi,\psi]} \subseteq \{W_2[D]^r\}_{[\varphi',\psi']}$. □

Theorem 3.6. *If the sets* $\{j : p_u < j \leq p'_u\}$, $\{j : q'_u < j \leq q_u\}$, $\{k : s_v < k \leq s'_v\}$, $\{k : t'_v < k \leq t_v\}$ *are finite for all* $u, v \in \mathbb{N}$, *then*

$$L^2_\infty \cap \{W_2[D]^r\}_{[\varphi',\psi']} \subseteq L^2_\infty \cap \{W_2[D]^r\}_{[\varphi,\psi]}.$$

Proof. Assume that $\{C_{jk}\} \in L^2_\infty \cap \{W_2[D]^r\}_{[\varphi',\psi']}$ and $C_{jk} \overset{W_2[D]^r_{[\varphi',\psi']}}{\longrightarrow} C$. Since $\{C_{jk}\} \in L^2_\infty$, there is an $\mathcal{M} > 0$ such that

$$|\mu_x(C_{jk}) - \mu_x(C)| \leq \mathcal{M}$$

for all $j, k \in \mathbb{N}$ and each $x \in X$.

Thus, we have

$$
\begin{aligned}
\frac{1}{\varphi_u\psi_v} & \sum_{\substack{j=p_u+1\\k=s_v+1}}^{q_u,t_v} |\mu_x(C_{jk})-\mu_x(C)|^r \\
&= \frac{1}{\varphi_u\psi_v}\left(\sum_{\substack{j=p_u+1\\k=s_v+1}}^{p'_u,s'_v} + \sum_{\substack{j=p_u+1\\k=s'_v+1}}^{p'_u,t'_v} + \sum_{\substack{j=p_u+1\\k=t'_v+1}}^{p'_u,t_v}\right)|\mu_x(C_{jk})-\mu_x(C)|^r \\
&\quad + \frac{1}{\varphi_u\psi_v}\left(\sum_{\substack{j=p'_u+1\\k=s_v+1}}^{q'_u,s'_v} + \sum_{\substack{j=p'_u+1\\k=s'_v+1}}^{q'_u,t'_v} + \sum_{\substack{j=p'_u+1\\k=t'_v+1}}^{q'_u,t_v}\right)|\mu_x(C_{jk})-\mu_x(C)|^r \\
&\quad + \frac{1}{\varphi_u\psi_v}\left(\sum_{\substack{j=q'_u+1\\k=s_v+1}}^{q_u,s'_v} + \sum_{\substack{j=q'_u+1\\k=s'_v+1}}^{q_u,t'_v} + \sum_{\substack{j=q'_u+1\\k=t'_v+1}}^{q_u,t_v}\right)|\mu_x(C_{jk})-\mu_x(C)|^r \\
&\leq \frac{1}{\varphi'_u\psi'_v}\sum_{\substack{j=p'_u+1\\k=s'_v+1}}^{q'_u,t'_v}|\mu_x(C_{jk})-\mu_x(C)|^r + 8\,\frac{\mathcal{M}^r}{\varphi'_u\psi'_v}
\end{aligned}
$$

for each $x \in X$. Then, for $u, v \to \infty$, considering our acceptance, we obtain that $C_{jk} \overset{W_2[D]^r_{[\varphi,\psi]}}{\longrightarrow} C$ and $\{C_{jk}\} \in L^2_\infty \cap \{W_2[D]^r\}_{[\varphi,\psi]}$. Consequently, $L^2_\infty \cap \{W_2[D]^r\}_{[\varphi',\psi']} \subseteq L^2_\infty \cap \{W_2[D]^r\}_{[\varphi,\psi]}$. $\square$

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