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DEFORMATION OF SASAKIAN METRIC AS A YAMABE SOLITON

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Abstract. In this paper, we investigate Yamabe solitons on deformed Sasakian manifolds. We proved that the Yamabe soliton constant is invariant under new deformation of contact manifolds that deforms metric and structure tensor simultaneously. Further, we show that the scalar curvature is equal to the soliton constant and the potential vector field of Yamabe soliton reduces to an affine vector field.
Keywords: deformation, Yamabe soliton, Sasakian metric, projective vector field, affine vector field.

1. Introduction

The notions of Yamabe and Ricci flow were introduced by Hamilton [11] and Yamabe soliton introduced as a tool to produce the Riemannian metric of constant scalar curvature. The evolution of a time-dependent Riemannian or semi-Riemannian metric g defines Yamabe flow by the equation

$$\frac{\partial g(t)}{\partial t} = -rg(t), \quad g(0) = g_0, \tag{1.1}$$

where r is the scalar curvature that corresponds to g. A self similar solution to the Yamabe flow known as Yamabe soliton is defined on a pseudo-Riemannian or Riemannian manifold (M, g) by the equation [1]

$$(L_V g)(X, Y) = 2(r - \lambda)g(X, Y), \tag{1.2}$$

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for all vector fields X and Y on M, where L_V denotes the Lie derivative with respect to V and λ is a soliton constant. If λ in (1.2) is a smooth function on M then g is referred to as an almost Yamabe soliton. According as $\lambda < 0$, $\lambda = 0$, $\lambda > 0$, Yamabe soliton is shrinking, steady and expanding respectively. Kundu [12] proved that 3-dimensional α-Sasakian manifold (M, g) with g as a Yamabe soliton is of constant curvature and gave an example of such a manifold. Sasakian metric on a 3-dimensional manifold as a Yamabe soliton was studied by Sharma [17]. The notion of Yamabe soliton generalized to quasi Yamabe soliton by Deshmukh and Chen [6]. Suh and De [18] extended the study to almost co-Kahler manifolds and De and co-authors [7–10] extensively studied Yamabe solitons in different contexts such as space time manifolds and para contact manifolds. Several authors [5, 15, 16] studied Yamabe solitons on manifolds with different contact structures. The action of projective vector fields on Riemannian manifolds with contact structures reveals intrinsic properties of the manifolds. Romero and Sanchez [14] studied a projective vector field on the Lorentzian manifold and established some relations between the casual character of projective vector field and curvature in a Lorentzian manifold.
Nagaraja and Sharma [13] while studying almost Ricci soliton on D-homothetically deformed K-contact metric established that η-Einstein K-contact metric as an almost gradient Ricci soliton is fixed D-homothetically and the potential vector field preserves the structure tensor ϕ. Bouzir and Beldjilali [4] introduced a new deformation of almost contact metric structure, where the deformation is for metric g and the structure tensor ϕ simultaneously. Beldjilai and Akyol [2] further investigated the above deformation and established relations among different classes of almost contact metrics.
In this paper, we study new deformation introduced by Bouzir and Beldjilali on Sasakian manifold (M, g) with g as a Yamabe soliton and establish conditions for g to have constant scalar curvature. Further, we show that on (M, g), the projective vector field becomes affine.

2. Preliminaries

A $(2n + 1)$ dimensional C^∞ manifold M together with a 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ is called a contact manifold. There is a global vector field ξ called Reeb vector field or characteristic vector field on M satisfying $d\eta(\xi, X) = 0$ for any vector field X on M and $\eta(\xi) = 1$.
An odd-dimensional manifold M has an almost contact structure (ϕ, ξ, η), if it admits a tensor field ϕ of type (1, 1), a vector field ξ and a 1-form η satisfying

$$\eta(\xi) = 1, \quad \phi^2 = -I + \eta \otimes \xi. \tag{2.1}$$

From (2.1), it is easy to deduce that

$$\eta \circ \phi = 0, \quad \phi\xi = 0. \tag{2.2}$$

If there is a Riemannian metric g on manifold M with a (ϕ, ξ, η)-structure such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\,\eta(Y), \tag{2.3}$$

for any vector fields X and Y on M, then we say that M has almost contact metric structure and $M(\phi,\, \xi,\, \eta,\, g)$ is called an almost contact metric manifold.
It follows from (2.3) that

$$\eta(X) = g(X, \xi), \tag{2.4}$$

for any vector field X on M. A Sasakian manifold is a normal contact metric manifold and an almost contact metric structure ($\phi,\, \xi,\, \eta,\, g$) on a manifold M is Sasakian if and only if

$$(\nabla_X \phi)(Y) = -\eta(Y)X + g(X,\, Y)\xi, \tag{2.5}$$

for any vector fields X and Y on M. The formula (2.5) implies

$$\nabla_X \xi = -\phi X \tag{2.6}$$

and

$$(\nabla_X \eta)Y = -g(\phi X, Y). \tag{2.7}$$

The following equations hold in a Sasakian manifold :

$$R(X,\, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.8}$$

$$S(X,\ \xi)\ = 2n\,\eta(X), \tag{2.9}$$

where R and S are respectively the Riemannian curvature tensor and Ricci tensor of g.

Definition 1. *On a Riemannian manifold, a vector field is called a solenoidal vector field if and only if its divergence is zero.*

Definition 2. *A vector field V on a Riemannian manifold M with connection ∇ is called a projective vector field if there is a scalar 1-form P-satisfying*

$$(L_V \nabla)(X,\, Y) = P(X)Y + P(Y)X. \tag{2.10}$$

It is to be noted that if $L_V \nabla = 0$, V is called an affine vector field and V is Killing if $L_V g = 0$.
On a $(2n+1)$-dimensional manifold M with almost contact metric structure $(\phi,\, \xi,\, \eta,\, g)$, a new structure $(\overline{\phi},\, \overline{\xi},\, \overline{\eta},\, \overline{g})$ is defined on M in the form [4]

$$\overline{\xi} = \xi, \quad \overline{\phi}X = \phi X + \theta(\phi X)\xi, \quad \overline{\eta} = \eta - \theta, \tag{2.11}$$

and

$$\overline{g}(X,\, Y) = fg(X,\, Y) + \overline{\eta}(X)\,\overline{\eta}(Y) - f\eta(X)\,\eta(Y), \tag{2.12}$$

for all vector fields X and Y on M, smooth function f and a closed 1-form θ orthogonal to η on M.
The relation between the connections ∇ and $\overline{\nabla}$ corresponding to the Riemannian metrics g and $\overline{g}$ is given by [3]

$$\overline{\nabla}_X Y = \nabla_X Y + \theta(X)\,\overline{\phi}Y + \theta(Y)\,\overline{\phi}X - (\nabla_X\theta)(Y)\,\xi. \tag{2.13}$$

We give the following result due to Benaoumeur and Gherici [3] for later use.

Remark 2.1. *[Lemma 1 of [3]]: Let r and $\overline{r}$ denote the scalar curvatures respectively for the metric g and deformed metric $\overline{g}$, the metric obtained from g under* (2.11)-(2.12). *Then $\overline{r} = r$.*

Throughout this paper, we use $\{e_i\}$ as an orthonormal basis for tangent space T_pM, $p \in M$ and its deformed orthonormal basis as $\{\overline{e}_i = e_i + \theta(e_i)\xi\}_{1\le i\le 2n+1}$.

3. Sasakian manifold as a Yamabe soliton under new deformation

Let the metric g in a Sasakian manifold (M, g) be a Yamabe soliton. Then we have

$$g(\nabla_X V, Y) + g(\nabla_Y V, X) = 2(-\lambda + r)\, g(X, Y). \tag{3.1}$$

We write

$$\begin{aligned} g(\nabla_X V, Y) = &\frac{1}{2}[g(\nabla_X V, Y) + g(\nabla_Y V, X)] \\ &+\frac{1}{2}\,[g(\nabla_X V, Y) - g(\nabla_Y V, X)]. \end{aligned} \tag{3.2}$$

Using $(L_V g)(X, Y) = g(\nabla_X V, Y) + g(\nabla_Y V, X)$ in (3.2), we get

$$g(\nabla_X V, Y) = \frac{1}{2}(L_V g)(X, Y) + du(X, Y), \tag{3.3}$$

where

$$2du(X, Y) = g(\nabla_X V, Y) - g(\nabla_Y V, X). \tag{3.4}$$

Using (1.2) in (3.3), we have

$$g(\nabla_X V, Y) = (-\lambda + r)\, g(X, Y) + du(X, Y). \tag{3.5}$$

Interchanging X and Y in (3.5), the symmetry of g gives

$$g(\nabla_Y V, X) = (-\lambda + r) g(X, Y) + du(Y, X). \tag{3.6}$$

Adding (3.5)-(3.6) and using (1.2), we obtain

$$du(X, Y) = -du(Y, X). \tag{3.7}$$

As du is skew-symmetric, we write using tensor field ϕ of type (1, 1) as

$$du(X,\,Y)=g(\phi X,\,Y). \tag{3.8}$$

From (3.5) and (3.8), we get

$$g(\nabla_X V,\,Y)=(-\lambda+r)\,g(X,\,Y)+g(\phi X,\,Y)$$

or

$$\nabla_X V=(r-\lambda)X+\phi X$$

and so

$$\nabla_\xi V=(r-\lambda)\xi. \tag{3.9}$$

The following lemma proves the invariance of soliton constant under deformation (2.11)-(2.12).

Lemma 1. *Let the Yamabe Soliton $(M,\overline{g},V,\overline{\lambda})$ be obtained from the Yamabe Soliton (M,g,V,λ) by deformation (2.11)-(2.12). Then $\lambda=\overline{\lambda}$. i.e., the Yamabe soliton consatnt is invariant under the deformaton of Sasakian Manifolds.*

Proof. Let $(M,\,g)$ be a Sasakian manifold and $(M,\,\overline{g})$ be obtained by deformation (2.11)-(2.12) of $(M,\,g)$. Suppose that $(M,\,g,\,V,\,\lambda)$ and $(\overline{M},\,\overline{g},\,V,\,\overline{\lambda})$ are Yamabe solitons. Then we have

$$(\overline{L}_V\overline{g})(X,\,Y)=\;2(\overline{r}-\overline{\lambda})\overline{g}(X,\,Y). \tag{3.10}$$

Applying the Lie derivative of (2.12) with respect to V, following the expression for the Yamabe soliton under new deformation, we obtain

$$\overline{L}_V\overline{g}(X,\,Y)=\overline{L}_Vfg(X,\,Y)-\overline{L}_Vf\eta(X)\eta(Y)+\overline{L}_V\overline{\eta}(X)\overline{\eta}(Y). \tag{3.11}$$

Using (2.3) in (3.11), we get

$$\begin{aligned}(\overline{L}_V\overline{g})(X,\,Y)\quad &=Vfg(\phi X,\phi Y)+f\left[g(\overline{L}_VX,\,Y)+g(X,\,\overline{L}_VY)+(\overline{L}_Vg)(X,\,Y)\right]\\ &-f\left[\eta(Y)\left(\eta(\overline{L}_VX)+(\overline{L}_V\eta)(X)\right)+\eta(X)\left((\eta(\overline{L}_VY)+\overline{L}_V\eta)(Y)\right)\right]\\ &+\overline{\eta}(Y)\left((\overline{L}_V\overline{\eta})(X)+\overline{\eta}(\overline{L}_VX)\right)+\overline{\eta}(Y)\left((\overline{L}_V\overline{\eta})(Y)+\overline{\eta}(\overline{L}_VY)\right)\\ &-g(\overline{L}_VX,\,Y)-g(X,\overline{L}_VY).\end{aligned} \tag{3.12}$$

For the unit vector field V corresponding to 1-form θ, we have

$$\theta(X)=g(X,\,V),\;\;\theta(V)=1.$$

$$Also\;\;\theta(\xi)=0,\;\;\eta(V)=0,\;\;\overline{\eta}(V)=-1. \tag{3.13}$$

Making use of (2.11) and (2.13) in $(\overline{L}_V g)(X,\, Y) = g(\overline{\nabla}_X V,\, Y) + g(X,\, \overline{\nabla}_Y V)$, we obtain

$$\begin{aligned}(\overline{L}_V g)(X,\, Y) &= (L_V g)(X,\, Y) + \theta(X)\, g(\phi V,\, Y) + \theta(\phi X)\, \eta(X) + \theta(Y) g(X, \phi V)\\ &\quad +\theta(\phi Y)\eta(X) - (\nabla_X \theta)(V).\end{aligned} \tag{3.14}$$

Differentiation of $g(V, V) = 1$ with respect to X gives $\theta(\nabla_X V) = 0$, which then imply that

$$(\nabla_X \theta)(V) = 0. \tag{3.15}$$

Using equation (3.15) in (3.14), we get

$$\begin{aligned}(\overline{L}_V g)(X,\, Y) &= (L_V g)(X,\, Y) + g(\phi V, Y)\, \theta(X) + \eta(Y)\, \theta(\phi X)\\ &\quad +g(X, \phi V)\, \theta(Y) + \theta(\phi Y)\eta(X).\end{aligned} \tag{3.16}$$

Let us take $f = 1$. Using (3.16) in (3.12) and replacing ξ for X and Y in the resulting equation, we can get easily

$$\begin{aligned}(\overline{L}_V \overline{g})(\xi,\, \xi) &= (L_V g)(\xi,\, \xi) + 2[g(\overline{\nabla}_V \xi,\, \xi) - g(\overline{\nabla}_\xi V,\, \xi) - \overline{g}(\overline{\nabla}_V \xi,\, \xi)]\\ &\quad -2(\overline{L}_V g)(\xi,\, \xi) + 4g(\nabla_\xi V,\, \xi) - 2(\nabla_\xi \theta)(V) + 2(\overline{L}_V \overline{g})(\xi, \xi).\end{aligned} \tag{3.17}$$

Setting $X = Y = \xi$ in (3.16), we obtain

$$(\overline{L}_V g)(\xi,\, \xi) = (L_V g)(\xi,\, \xi). \tag{3.18}$$

Now differentiating $g(V,\, \xi) = \theta(\xi) = 0$ with respect to ξ and using (3.13), we get

$$g(\nabla_\xi V,\, \xi) = 0. \tag{3.19}$$

Differentiating $g(V, V) = 1$ with respect to ξ, we obtain $g(\nabla_\xi V, V) = 0$. i.e., $\theta(\nabla_\xi V) = 0$. This with $\nabla_\xi \theta(V) = 0$ gives

$$(\nabla_\xi \theta)(V) = 0. \tag{3.20}$$

Using (3.18), (3.19) and (3.20) in (3.17), we obtain

$$(\overline{L}_V \overline{g})(\xi,\, \xi) = (L_V g)(\xi,\, \xi) + 2[g(\overline{\nabla}_V \xi, \xi) - g(\overline{\nabla}_\xi V, \xi) - \overline{g}(\overline{\nabla}_V \xi, \xi)]. \tag{3.21}$$

Using (2.11) and (2.12) in (3.21), we get

$$(\overline{L}_V \overline{g})(\xi,\, \xi) = (L_V g)(\xi,\, \xi) + 2\left(g(\overline{\nabla}_V \xi,\, \xi) - g(\overline{\nabla}_\xi V, V)\right). \tag{3.22}$$

Expanding $\overline{\nabla}_V \xi$ using (2.13), (3.13) and noting that $\theta(\phi V) = 0$, after simplification from (3.22), we get

$$(\overline{L}_V \overline{g})(\xi,\, \xi) = (L_V g)(\xi,\, \xi). \tag{3.23}$$

Using (1.2) and (3.10) in (3.23) and from Remark 2.1, we have

$$\overline{\lambda} = \lambda. \tag{3.24}$$

Proof is completed. □

Theorem 1. *Under the deformation* (2.11)-(2.12) *with* $f = 1$ *of Sasakian metric as a Yamabe Soliton, the Yamabe Soliton constant* λ *is equal to the scalar curvature* r.

Proof. Let the Sasakian manifold (M, g) be Yamabe soliton with V as potential vector field and λ as a soliton constant. We write

$$(\overline{L}_V\overline{g})(X, Y) = \overline{g}(\overline{\nabla}_X V, Y) + \overline{g}(X, \overline{\nabla}_Y V). \tag{3.25}$$

Using (2.1), (2.6), (2.11), (2.12) and (2.13) in (3.25), we obtain

$$\begin{aligned}(\overline{L}_V\overline{g})(X,Y) \;&= (L_V g)(X,Y) + \theta(X)\,(g(\phi V, Y) - \eta(\nabla_Y V)) + \theta(Y)(g(\phi V, X)\\ &\quad -\eta(\nabla_X V)) - \nabla_X\theta(V)\overline{\eta}(Y) - \nabla_Y\theta(V)\overline{\eta}(X).\end{aligned} \tag{3.26}$$

Using (2.12) and (3.26) in (3.10), we get

$$\begin{aligned}(L_V g)(X,Y) &= 2(\overline{r} - \overline{\lambda})\,[g(X, Y) - \eta(X)\,\eta(Y) + \overline{\eta}(X)\,\overline{\eta}(Y)]\\ &\quad -\theta(Y)\;(\,g(\phi V,\, X) \,-\eta(\nabla_X V)) - \theta(X)(\,g(\phi V,\, X)\\ &\quad -\eta(\nabla_X V)) + \nabla_X\theta(V)\overline{\eta}(Y) + \nabla_Y\theta(V)\overline{\eta}(X).\end{aligned} \tag{3.27}$$

Substituting e_i for X and Y in (3.27) and using Remark **2.1**, we get

$$\begin{aligned}(L_V g)(e_i, e_i) \;&= 2(r - \overline{\lambda})\,[g(e_i,\, e_i) - \,\eta(e_i)\,\eta(e_i) \,+\, \overline{\eta}(e_i)\,\overline{\eta}(e_i)]\,.\\ &\quad -2\theta(e_i)\,g(\phi V, e_i) - 2\theta(e_i)\,\eta(\nabla_{e_i} V) + 2\nabla_{e_i}\theta(V)\overline{\eta}(e_i).\end{aligned} \tag{3.28}$$

We see that

$$\theta(e_i)\,\eta(\nabla_{e_i} V) = g(\nabla_V V, \xi) = g(\nabla_{e_i} V, \xi)g(e_i, V). \tag{3.29}$$

Differentiating $g(\xi, V) = 0$ along V, we obtain

$$0 = g(\nabla_V \xi,\, V) + g(\xi,\, \nabla_V V). \tag{3.30}$$

Use of $0 = \theta(\phi V) = g(\nabla_V \xi, V)$ in (3.30) gives $g(\nabla_V V, \xi) = 0$. This with (3.29) gives

$$\theta(e_i)\eta(\nabla_{e_i} V) = 0. \tag{3.31}$$

Using (1.2), (3.10), (3.31) and noting that $\theta(\phi V) = 0$ and $\theta(V) = 1$ in (3.28), we get

$$(\overline{\lambda} - \lambda)(2n + 1) = (r - \overline{\lambda}). \tag{3.32}$$

From (3.32) and Lemma 1, it follows that

$$r = \lambda. \tag{3.33}$$

This proves Theorem 1. □

Corollary 1. *Under the deformation* (2.11)-(2.12) *with* $f = 1$, *Sasakian metric as a Yamabe soliton has constant scalar curvature.*

From equation (1.2) and Theorem 1, we have

Corollary 2. *Under the deformation of Sasakian metric as a Yamabe Soliton, the potential vector field* V *is Killing.*

In the following part, we show that the potential vector field is solenoidal.

Theorem 2. *Under the deformation of the Sasakian metric as a Yamabe Soliton, the potential vector field is solenoidal.*

Proof. We now write

$$(\overline{L}_V\overline{g})(X, Y) = \overline{g}(\overline{\nabla}_X V, Y) + \overline{g}(X, \overline{\nabla}_Y V). \tag{3.34}$$

Substituting from (2.13) to (3.34), we get

$$\begin{aligned}(\overline{L}_V\overline{g})(X,Y) &= \overline{g}(\nabla_X V, Y) + \theta(X)\,\overline{g}(\overline{\phi}V, Y) - (\nabla_X\theta)(V)\overline{g}(\xi, Y) + \overline{g}(X, \nabla_Y V)\\ &+\theta(Y)\,\overline{g}(\overline{\phi}V, X) - (\nabla_Y\theta)(V)\,\overline{g}(\xi, X).\end{aligned} \tag{3.35}$$

Taking $X = Y = \overline{e}_i = \{e_i + \theta(e_i)\xi\}_{0\leq i\leq 2n}$ in (3.35), we get

$$\begin{aligned}\sum_{i=0}^{2n}(\overline{L}_V\overline{g})(e_i + \theta(e_i)\xi,\, e_i + \theta(e_i)\xi) &= 2\textstyle\sum_{i=0}^{2n}\{\overline{g}\left(\nabla_{e_i+\theta(e_i)\xi}V,\, e_i + \theta(e_i)\xi\right)\\ &+\theta(e_i + \theta(e_i)\xi)\overline{g}(\overline{\phi}V,\, e_i + \theta(e_i)\xi)\\ &-(\nabla_{e_i+\theta(e_i)\xi}\theta)V)\,\overline{g}(\xi, e_i + \theta(e_i)\xi)\},\end{aligned} \tag{3.36}$$

where $\theta(e_i) = -\overline{g}(e_i, \xi)$, $\overline{g}(\overline{e}_0, \overline{e}_0) = \overline{g}(\xi, \xi) = 1$ and for $i \in \{1, \ldots 2n\}$.
Using (2.12), (3.10) and (3.13) in (3.36), we obtain

$$\begin{aligned}2(\overline{r} - \overline{\lambda})\sum_{i=1}^{2n} g(e_i, e_i) &= 2\mathbf{div}\,V - 2\sum_{i=1}^{2n}\overline{g}(e_i, \xi)\overline{g}(\nabla_{e_i}V, \xi)\\ &- 2(\nabla_\xi\theta)(V).\end{aligned} \tag{3.37}$$

By Remark 2.1, we get

$$(r - \overline{\lambda})(2n + 1) = \mathbf{div}\,V - \overline{g}(\nabla_\xi V, \xi) - (\nabla_\xi\theta)(V). \tag{3.38}$$

Using (3.9), (3.24) and (3.33) in (3.38), we get

$$\mathbf{div}\,V = (\nabla_\xi\theta)(V). \tag{3.39}$$

From (3.20) and (3.39), it follows that

$$\mathbf{div}\,V = 0. \tag{3.40}$$

This leads to the Theorem 2. □

4. Projective vector field on Sasakian manifold (M, g) with g as a Yamabe soliton

Theorem 3. *Let (M, g, V, λ) be a Sasakian manifold with the metric g as a Yamabe soliton and the vector field V as a projective vector field. If scalar curvature r is constant along ξ, then M has constant scalar curvature and the vector field V reduces to an affine vector field.*

Proof. Let the metric g be a Yamabe soliton. Then

$$\frac{1}{2}(L_V g)(Z, X) = (r - \lambda) g(Z, X). \tag{4.1}$$

Differentiating (4.1) on both sides covariantly with respect to Y, we have

$$\begin{aligned}\nabla_Y\,(L_V g)(Z,\,X) &= 2\nabla_Y((r-\lambda)g(Z,\,X))\\ &= 2(Yr)g(Z,X) + 2(r-\,\lambda)\,(g(\nabla_Y Z, X)\,+g(Z,\nabla_Y X))\\ &= 2(Yr)g(Z,X) + 2(r-\lambda)g(\nabla_Y Z,X) + 2(r-\lambda)g(Z,\nabla_Y X)\\ &\quad -(L_V g)g(\nabla_Y Z,X) - (L_V g)g(Z,\nabla_Y X).\end{aligned} \tag{4.2}$$

Using (1.2) in (4.2), we get

$$(\nabla_Y(L_V g))(Z,\,X) = 2(Yr)g(Z,\,X). \tag{4.3}$$

We make use of $\nabla g = 0$ in the commutation formula [19],

$$(L_V\nabla_Y g - \nabla_Y L_V g - \nabla_{[V,Y]}g)(Z,\,X) \;\;= -g\left((L_V\nabla)(Y,\,Z),X\right) - g\left((L_V\nabla)(Y,\,X),Z\right),$$

to obtain

$$(\nabla_Y(L_V g))(Z,\,X)\;= g\left((L_V\nabla)(Y,\,Z),X\right) + g\left((L_V\nabla)(Y,\,X),\,Z\right). \tag{4.4}$$

Using (4.3) in (4.4), we obtain

$$2(Yr)g(Z,\,X) = g((L_V\nabla)(Y,Z),X) + g((L_V\nabla)(Y,X),Z).$$

As V is a projective vector field, the above equation with (2.10) gives

$$2(Yr)g(Z,\,X) = g(P(Y)\,Z + P(Z)\,Y,\,X) + g(P(Y)\,X + P(X)\,Y,\,Z). \tag{4.5}$$

Substitution of e_i for X and Z in (4.5) gives

$$P(Y) = \frac{(2n+1)(Yr)}{2(n+1)}. \tag{4.6}$$

Substituting ξ for X and Y in (4.5), we get

$$P(Z) = (2(\xi r) - 3P(\xi))\,\eta(Z). \tag{4.7}$$

Replace Z by ξ in (4.7) to get

$$P(\xi) = \frac{1}{2}(\xi r). \tag{4.8}$$

If r is constant along ξ, then

$$P(\xi) = 0.$$

Now (4.7) gives

$$P(Z) = 0. \tag{4.9}$$

Thus from (4.9) and (4.6), it follows that $Xr = 0$, i.e., r is a constant.
Also $P(X) = 0$ in (2.10) gives $(L_V \Delta)(X, Y) = 0$, which makes V an affine vector field. $\square$

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