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LIE IDEALS WITH MULTIPLICATIVE (GENERALIZED)-DERIVATIONS OF SEMIPRIME RINGS

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Abstract. Let R be a $2-$torsion free semiprime ring, U a square-closed Lie ideal of R, f a multiplicative (generalized)-derivation with the additive map d of R. In the present paper, we shall prove that R contains a nonzero central ideal if any one of the following holds: i) $F(x)F(y) \pm [x, y] \in Z$, ii) $F(x)F(y) \pm (x \circ y) \in Z$, iii) $F([x, y]) = \pm(xy \pm\ yx)$, iv) $F(x \circ y) = \pm(xy \pm\ yx)$, v) $F(xy) \pm F(x)F(y) = 0$, vi) $F(xy) \pm F(y)F(x) = 0$ for all $x, y \in U$.
Keywords: semiprime rings, Lie ideals, multiplicative (generalized)-derivations.

1. Introduction

Throughout R will present an associative ring with center Z. For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$ and the symbol $x \circ y$ denotes the anti-commutator $xy+yx$. Recall that a ring R is prime if $xRy = 0$ implies $x = 0$ or $y = 0$, and R is semiprime if for $x \in R$, $xRx = 0$ implies $x = 0$. An additive subgroup U of R is said to be a Lie ideal of R if $[u, r] \in U$, for all $u \in U, r \in R$. U is called a square-closed Lie ideal of R if U is a Lie ideal and $u^2 \in U$ for all $u \in U$. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. For a fixed $a \in R$, the mapping $I_a : R \to R$ given by $I_a(x) = [a, x]$ is a derivation which is said to be an inner derivation. Let S be a nonempty subset

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of R. A mapping F from R to R is called centralizing on S if $[f(x), x] \in Z$, for all $x \in S$ and is called commuting on S if $[f(x), x] = 0$, for all $x \in S$.

The commutativity of prime rings with derivation was initiated by Posner [13]. The history of commuting and centralizing mappings goes back to 1955 when Divinsky [8] proved that a simple Artinian ring is commutative if it has a commuting nontrivial automorphism. In [13], Posner showed that if a prime ring has a nontrivial derivation which is centralizing on the entire ring, then the ring must be commutative. Luh [11] generalized the Divinsky result, we have just mentioned above, to arbitrary prime rings. Mayne [12] proved that in case there exists a nontrivial centralizing automorphism on a prime ring, then the ring is commutative. In [4], Bresar proved that every additive commuting mapping of a prime ring R is of the form $x \to \lambda x + \zeta(x)$ where λ is an element of C and $\zeta : R \to C$ is an additive mapping.

Recently, in [5], Bresar defined the following notation. An additive mapping $F : R \to R$ is called a generalized derivation if there exists a derivation $d : R \to R$ such that

$$F(xy) = F(x)y + xd(y), \text{ for all } x, y \in R.$$

Basic examples are derivations and generalized inner derivations (i.e., maps of type $x \to ax + xb$ for some $a, b \in R$). One may observe that the concept of generalized derivations includes the concept of derivations and of the left multipliers (i.e., $F(xy) = F(x)y$ for all $x, y \in R$). Hence it should be interesting to extend some results concerning these notions to generalized derivations.

In [1], Ashraf, Asma and Shakir have proven to be commutative ring R if provided one of the following conditions: i) $f(x)f(y) \pm xy \in Z$, ii) $f(x)f(y) \pm yx \in Z$ for all $x, y \in R$. This theorem is considered for generalized derivations in [14]. Being inspired by these results, Dhara and Ali have recently discussed the commutativity theorems for prime rings or semiprime rings involving multiplicative generalized derivations in [7]. This work was discussed by Koç and Gölbaşı for the multiplicative generalized derivation of the semiprime ring in [9].

By Daif and Bell, "Let R is semiprime ring, d is nonzero derivation of R and I is nonzero ideal of R. R contains a nonzero central ideal if one of the following conditions is provided; i) $d([x, y]) = [x, y]$ ii) $d([x, y]) = -[x, y]$ for all $x, y \in I$" has been proven in [6]. In [14], Quadri, Khan and Rehman examined the following conditions for generalized derivation: i) $f([x, y]) = [x, y]$, ii) $f([x, y]) = -[x, y]$, iii) $f(x \circ y) = x \circ y$, iv) $f(x \circ y) = -(x \circ y)$. In [3], Bell and Kappe have proved that d is a derivation of R which is either a homomorphism or anti-homomorphism in semiprime ring R or a nonzero right ideal of R then $d = 0$. Koç and Gölbaşı studied multiplicative generalized derivations by generalizing these conditions on semiprime rings in [10]. The conditions discussed above by various authors have been studied by many authors in recent years on different structures and derivation. In the present paper, we shall extend the above results for a nonzero Lie ideal of semiprime rings with multiplicative (generalized)-derivation of R.

Throughout the present paper, we shall make use of the following basic identities without any specific mention:

i) $[x, yz] = y[x, z] + [x, y]z$
ii) $[xy, z] = [x, z]y + x[y, z]$
iii) $xy \circ z = (x \circ z)y + x[y, z] = x(y \circ z) - [x, z]y$
iv) $x \circ yz = y(x \circ z) + [x, y]z = (x \circ y)z + y[z, x]$.

2. Preliminaries

Lemma 2.1. *[2, Theorem 1.3] Let R be a $2-$ torsion free semiprime ring and U a noncentral Lie ideal of R such that $u^2 \in U$ for all $x \in U$. Then there exists a nonzero ideal I of R such that $I \subseteq U$.*

Lemma 2.2. *[6, Lemma 2 (b)] If R is a semiprime ring, then the center of a nonzero ideal of R is contained in the center of R.*

Lemma 2.3. *[15, Theorem 2.1] Let R be a semiprime ring, I a nonzero two-sided ideal of R and $a \in I$ such that $axa = 0$ for all $x \in I$, then $a = 0$.*

Lemma 2.4. *[3, Theorem 3]Let R be a semiprime ring and U a nonzero left ideal. If R admits a derivation D which is nonzero on U and centralizing on U, then R contains a nonzero central ideal.*

Lemma 2.5. *[10, Lemma 2.3]Let R be a semiprime ring. Suppose that R admits a multiplicative generalized derivation F associated with a nonzero additive map d. Then d is a derivation.*

Theorem 2.1. *Let R be a semiprime ring, U a square-closed Lie ideal of R and $F : R \to R$ a multiplicative (generalized)-derivation associated with the additive map $d : R \to R$ such that $F(x)F(y) \pm [x, y] \in Z$ for all $x, y \in U$. Then R contains a nonzero central ideal.*

Proof. By Lemma 2.1, there exists a nonzero ideal I of R such that $I \subseteq U$. By the hypothesis, we have

$$F(x)F(y) + [x, y] \in Z, \text{ for all } x, y \in I. \tag{2.1}$$

Replacing y with yz, $z \in I$ in (2.1), we get

$$(F(x)F(y) + [x, y])z + y[x, z] + F(x)yd(z) \in Z.$$

Commuting this equation with z and using the equation (2.1), we obtain that

$$[F(x)yd(z), z] + [y[x, z], z] = 0. \tag{2.2}$$

Taking x by xz in the last equation, we have

$$[F(x)zyd(z), z] + [xd(z)yd(z), z] + [y[x, z]z, z] = 0. \tag{2.3}$$

Replacing y by zy in (2.2), we have

$$[F(x)zyd(z), z] + [zy[x, z], z] = 0. \tag{2.4}$$

If equation (2.3) subtracts (2.4) from equation, we find that

$$[xd(z)yd(z), z] + [[y[x, z], z], z] = 0. \tag{2.5}$$

Replacing x by xz in the above equation, we see that

$$[xzd(z)yd(z), z] + [[y[x, z], z], z]z = 0. \tag{2.6}$$

Right multiplying by z the equation (2.5) and if this equation is subtracted from equation (2.6), we obtain that

$$[x[d(z)yd(z), z], z] = 0.$$

Replacing x by $d(z)yd(z)x$ in this equation and using this equation, we get

$$[d(z)yd(z), z]I[d(z)yd(z), z] = (0), \text{ for all } y, z \in I.$$

By Lemma 2.3, we find that

$$[d(z)yd(z), z] = 0, \forall y, z \in I,$$

and so

$$d(z)yd(z)z - zd(z)yd(z) = 0.$$

Replacing y by $yd(z)u$, $u \in I$ in the last equation and using this equation, we have

$$d(z)y[d(z), z]ud(z) = 0, \text{ for all } u, y, z \in I. \tag{2.7}$$

Taking y by zy in (2.7), we get

$$d(z)zy[d(z), z]ud(z) = 0, \text{ for all } u, y, z \in I. \tag{2.8}$$

Left multiplying by z this (2.7), we find that

$$zd(z)y[d(z), z]ud(z) = 0, \text{ for all } u, y, z \in I. \tag{2.9}$$

If equations (2.8) and (2.9) are brought to the side, we see that

$$[d(z), z]y[d(z), z]ud(z) = 0, \text{ for all } u, y, z \in I. \tag{2.10}$$

Replacing u by uz in this equation, we find that

$$[d(z), z]y[d(z), z]uzd(z) = 0.$$

Right multiplying by z this (2.10), we find that

$$[d(z), z]y[d(z), z]ud(z)z = 0.$$

If the last two equations hand side of the extraction is done, we see that

$$[d(z), z]y[d(z), z]u[d(z), z] = 0.$$

Right multiplying by $y[d(z), z]$ last equation, we find that

$$[d(z), z]y[d(z), z]I[d(z), z]y[d(z), z] = 0.$$

By Lemma 2.3, we have

$$[d(z), z]y[d(z), z] = 0, \text{ for all } y, z \in I.$$

That is,

$$[d(z), z]I[d(z), z] = 0, \text{ for all } y \in I.$$

Again, using Lemma 2.3, we have $[d(z), z] = 0$, for all $z \in I$. By Lemma 2.4 and Lemma 2.5, we conclude that R contains a nonzero central ideal. "$F(x)F(y) - [x, y] \in Z$" is proved similarly. We complete the proof. □

Theorem 2.2. *Let R be a semiprime ring, U a square-closed Lie ideal of R and $F : R \to R$ a multiplicative (generalized)-derivation associated with the additive map $d : R \to R$ such that $F(x)F(y) \pm (x \circ y) \in Z$ for all $x, y \in U$. Then R contains a nonzero central ideal.*

Proof. By Lemma 2.1, there exists a nonzero ideal I of R such that $I \subseteq U$. By the hypothesis, we have

$$F(x)F(y) + (x \circ y) \in Z, \text{ for all } x, y \in I. \tag{2.11}$$

Replacing y by yz, $z \in I$ in the last equation, we get

$$(F(x)F(y) + (x \circ y))z + y[x, z] + F(x)yd(z) \in Z.$$

Commuting this equation with z and using the equation (2.11), we obtain that

$$[F(x)yd(z), z] + [y[x, z], z] = 0, \text{ for all } x, y, z \in I.$$

This expression is the same as equation (2.2) in Theorem 2.1. Therefore, using the methods in Theorem 2.1, we get R contains a nonzero central ideal."$F(x)F(y) \pm (x \circ y) \in Z$" is proved similarly. We complete the proof. □

Theorem 2.3. *Let R be a semiprime ring, U a square-closed Lie ideal of R and $F : R \to R$ a multiplicative (generalized)-derivation associated with the additive map $d : R \to R$ such that $F([x, y]) = \pm(xy \pm yx)$ for all $x, y \in U$. Then R contains a nonzero central ideal.*

Proof. By Lemma 2.1, there exists a nonzero ideal I of R such that $I \subseteq U$. By the hypothesis, we get

$$F([x,y]) = \pm(xy \pm \ yx), \text{ for all } x, y \in I.$$

Replacing y by yx in this equation, we obtain that

$$F([x,y])x + [x,y]d(x) = \pm(xy \pm \ yx)x.$$

Using the hypothesis, we get

$$[x,y]d(x) = 0, \text{ for all } x, y \in I.$$

Replacing y by $d(x)y$ in this equation and using this equation, we get

$$[x, d(x)]yd(x) = 0. \tag{2.12}$$

Replacing y by yx in (2.12), we find that

$$[x, d(x)]yxd(x) = 0 \text{ for all } x, y \in I. \tag{2.13}$$

Multiplying (2.12) on the right by x, we have

$$[x, d(x)]yd(x)x = 0 \text{ for all } x, y \in I. \tag{2.14}$$

Subtracting (2.13) from (2.14), we arrive at

$$[x, d(x)]y[x, d(x)] = 0 \text{ for all } x, y \in I.$$

Using Lemma 2.3, we conclude that $[x, d(x)] = 0$ for all $x \in I$. By Lemma 2.4 and Lemma 2.5, we conclude that R contains a nonzero central ideal. □

Theorem 2.4. *Let R be a semiprime ring, U a square-closed Lie ideal of R and $F : R \to R$ a multiplicative (generalized)-derivation associated with the additive map $d : R \to R$ such that $F(x \circ y) = \pm(xy \pm \ yx)$ for all $x, y \in U$. Then R contains a nonzero central ideal.*

Proof. By Lemma 2.1, there exists a nonzero ideal I of R such that $I \subseteq U$. By the hypothesis, we get

$$F(x \circ y) = \pm(xy \pm \ yx) \text{ for all } x, y \in I.$$

Replacing y by yx in this equation, we see that

$$F(x \circ y)x + (x \circ y)d(x) = \pm(xy \pm \ yx)x.$$

Using the hypothesis, we have

$$(x \circ y)d(x) = 0, \text{ for all } x, y \in I.$$

Replacing y by yz in the above equation and using this equation, we obtain that

$$[x,y]zd(x), \text{ for all } x,y \in I.$$

Replacing y by $d(x)$ in this equation, we get

$$[x,d(x)]zd(x) = 0 \text{ for all } x,z \in I. \tag{2.15}$$

Taking z by zx in (2.15), we see that

$$[x,d(x)]zxd(x) = 0 \text{ for all } x,z \in I. \tag{2.16}$$

Multiplying (2.15) on the right by x, we have

$$[x,d(x)]zd(x)x = 0 \text{ for all } x,z \in I. \tag{2.17}$$

Subtracting (2.16) from (2.17), we get

$$[x,d(x)]z[x,d(x)] = 0 \text{ for all } x,z \in I.$$

Using Lemma 2.3, we conclude that $[x,d(x)] = 0$ for all $x \in I$. By Lemma 2.4 and Lemma 2.5, we conclude that R contains a nonzero central ideal. □

Theorem 2.5. *Let R be a semiprime ring, U a square-closed Lie ideal of R and $F : R \to R$ a multiplicative (generalized)-derivation associated with the additive map $d : R \to R$ such that $F(xy) \pm F(x)F(y) = 0$ for all $x,y \in U$. Then R contains a nonzero central ideal.*

Proof. By Lemma 2.1, there exists a nonzero ideal I of R such that $I \subseteq U$. By the hypothesis, we get

$$F(xy) \pm F(x)F(y) = 0 \text{ for all } x,y \in I.$$

Replacing y by yz, $z \in I$ in this equation, we get

$$F(xy)z + xyd(z) \pm F(x)\{F(y)z + yd(z)\} = 0.$$

Using the hypothesis, we have

$$(F(x) \pm x)yd(z) = 0.$$

Since I is an ideal and left multiplying this equation by $F(y)$, we obtain that

$$F(y)(F(x) \pm x)Ryd(z) = (0), \text{ for all } x,y,z \in I. \tag{2.18}$$

Using the hypothesis, we get

$$F(x)y + xd(y) \pm F(x)F(y) = 0.$$

and so,

$$F(x)(F(y) \pm y) = xd(y).$$

If this equation is used in the equation (2.18), we have

$$yd(x)Ryd(z) = (0).$$

Replacing z by x in this equation, we find that

$$yd(x)Ryd(x) = (0) \text{ for all } x, y \in I.$$

Since R is semiprime ring, we see that

$$yd(x) = 0 \text{ for all } x, y \in I.$$

We conclude that $[x, d(x)]y[x, d(x)] = 0$ for all $x, y \in I$. By Lemma 2.3, we get $[x, d(x)] = 0$ for all $x \in I$. By Lemma 2.4 and Lemma 2.5, we conclude that R contains a nonzero central ideal. $\square$

Theorem 2.6. *Let R be a semiprime ring, U a square-closed Lie ideal of R and $F : R \to R$ a multiplicative (generalized)-derivation associated with the additive map $d : R \to R$ such that $F(xy) \pm F(y)F(x) = 0$ for all $x, y \in U$. Then R contains a nonzero central ideal.*

Proof. By Lemma 2.1, there exists a nonzero ideal I of R such that $I \subseteq U$. By the hypothesis, we get

$$F(xy) \pm F(y)F(x) = 0 \text{ for all } x, y \in I.$$

Replacing x with xy in this equation, we get

$$F(xy)y + xyd(y) \pm F(y)\{F(x)y + xd(y)\} = 0.$$

Using the hypothesis, we have

$$xyd(y) \pm F(y)xd(y) = 0 \text{ for all } x, y \in I. \tag{2.19}$$

That is

$$(xy \pm F(y)x)d(y) = 0.$$

Taking x with rx, $r \in R$ in the above equation, we obtain that

$$(rxy \pm F(y)rx)d(y) = 0 \text{ for all } x, y \in I, r \in R.$$

Replacing r by $F(z), z \in I$ in the last equation, we get

$$(F(z)xy \pm F(y)F(z)x)d(y) = 0 \text{ for all } x, y \in I, r \in R.$$

Left multiplying (2.19) by $F(z)$, we have

$$F(z)xyd(y) \pm F(z)F(y)xd(y) = 0 \text{ for all } x, y \in I.$$

If these two equations are used, we see that

$$\{F(y)F(z) - F(z)F(y)\}xd(y) = 0.$$

Using the hypothesis, we get

$$\{F(zy) - F(yz)\}xd(y) = 0, \text{ for all } x, y, z \in I. \tag{2.20}$$

Replacing z by zy in this equation, we get

$$0 = \{F(zy)y + zyd(y) - F(yz)y - yzd(y)\}xd(y) = (F(zy) - F(yz))yxd(y) + [z, y]d(y)xd(y)$$

Using (2.20), we obtain that

$$[z, y]d(y)xd(y) = 0 \text{ for all } x, y, z \in I.$$

Replacing x by $x[z, y]$, we have

$$[z, y]d(y)x[z, y]d(y) = 0 \text{ for all } x, y, z \in I.$$

By Lemma 2.3, we have

$$[z, y]d(y) = 0 \text{ for all } y, z \in I.$$

The rest of the proof is the same as an equation in Theorem 2.3. This completes the proof. □

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