FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 40, No 2 (2025), 365–373
https://doi.org/10.22190/FUMI240326027A
Original Scientific Paper

HARMONIC MAPS ON COTANGENT AND UNIT COTANGENT BUNDLES

Murat Altunbaş

Department of Mathematics, Faculty of Sciences and Arts
Erzincan Binali Yildirim University, 24100 Erzincan, Turkey

ORCID ID: Murat Altunbaş https://orcid.org/0000-0002-0371-9913

Abstract. Let (M, g) be an $n-$dimensional Riemannian manifold and $(T^*M, \tilde{g})$ be its cotangent bundle with a metric $\tilde{g}$ that generalizes Sasaki and Cheeger-Gromoll metrics. In this paper, we investigate the harmonicity of the canonical projection $\pi : (T^*M, \tilde{g}) \to (M, g)$, the harmonicity of 1-forms regarded as maps $\sigma : (M, g) \to (T^*M, \tilde{g})$ and the harmonicity of the identity maps $I_1 : (T^*M, \tilde{g}) \to (T^*M, {}^S g)$ and $I_2 : (T^*M, {}^S g) \to (T^*M, \tilde{g})$, where ${}^S g$ is the Sasaki metric. Moreover, we consider the same problems on the unit cotangent bundle T_1^*M.
Keywords: Riemannian manifold, cotangent bundle, Sasaki metrics, Cheeger-Gromoll metrics, harmonic maps.

1. Introduction

Sasaki was the first who constructed a Riemannian metric on the tangent bundle of a Riemannian manifold [18]. About 30 years later, inspired by the paper of Cheeger and Gromoll [3], Musso and Tricceri introduced another natural Riemannian metric on the tangent bundle [10]. Today these metrics are known as Sasaki and Cheeger-Gromoll metrics, respectively. In further studies, natural metrics on tangent bundles are generally obtained by deformations of the horizontal and vertical parts of the Sasaki and Cheeger-Gromoll metrics (for a history of tangent bundles see [9]).

Received March 26, 2024, accepted: August 24, 2024
Communicated by Ljubica Velimirović
Corresponding Author: Murat Altunbaş. E-mail addresses: maltunbas@erzincan.edu.tr
2020 *Mathematics Subject Classification.* Primary 53C43; Secondary 53C25, 53B21

Due to the duality of tangent and cotangent spaces, most geometric objects in tangent bundles can be considered in cotangent bundles. Therefore, Sasaki and Cheeger-Gromoll type metrics have been discussed on cotangent bundles by many authors (for example see [1, 6, 7, 16, 17]).

Let (M, g) be a Riemannian manifold and TM be the tangent bundle of M. When a vector field X on M is given, it may be thought of as a map $X : M \to TM$. In the cases of TM is endowed with Sasaki, Cheeger-Gromoll, or complete lift type metrics, some conditions were found under which X is an an isometric immersion, a totally geodesic or a harmonic map (see [8, 12, 13]). In addition, some conditions were examined for the canonical projection $\pi : TM \to M$ to be a totally geodesic or harmonic map. Also, the methods used to investigate these properties were considered for identity maps $I : TM \to TM$ when the domain bundle and the target bundle have different metrics (see [13]).

Similar results were obtained by replacing the tangent bundle TM by cotangent bundle T^*M; Sasaki, Cheeger-Gromoll or complete lift type metrics by Riemannian extension type metrics and the vector field X by a 1-form σ (see [14, 15]).

This paper has two parts. In the first part, we discuss the harmonicity of the canonical projection $\pi : T^*M \to M$, the harmonicity of a 1-form which defines a map $\sigma : M \to T^*M$ and the harmonicity of the identity maps $I : T^*M \to T^*M$. In the second part, the same problems are considered on the unit cotangent bundle T_1^*M. Here, we assume a general metric $\tilde{g}$. This metric can be considered as the correspondence of the metric in [2] on the cotangent bundle.

We assume in the sequel that the manifolds, functions, tensor fields and connections under consideration are all smooth, i.e. of differentiable of class C^∞. The Einstein summation convention is used, the range of the indices i, j, s being always $\{1, 2, ..., n\}$. We shall denote by $\chi(M)$ the module of vector fields on M and by $\Lambda^1(M)$ the module of 1-forms on M.

2. The cotangent bundle T^*M and the metric $\tilde{g}$

Let (M, g) be an $n-$dimensional Riemannian manifold and denote by $\pi : T^*M \to M$ its cotangent bundle with fibres the cotangent spaces to M. Then T^*M may be endowed with a structure of a $2n-$dimensional manifold, induced by the structure on the base manifold. If $(U, x^i); i = 1, ..., n$ is a system of local coordinates, then the system of local coordinates $\left(\pi^{-1}(U), x^i, x^{\bar{i}} = p_i\right), \bar{i} = n + i = n + 1, ..., 2n$ is defined on T^*M, where $x^{\bar{i}} = p_i$ are the components of covectors p in each cotangent space T_x^*M, $x \in U$ with respect to the natural coframe $\{dx^i\}$.

Let $X = X^i \frac{\partial}{\partial x^i}$ and $\omega = \omega_i dx^i$ be the local expressions in U of a vector field X and a 1-form ω, respectively. Then the vertical lift ${}^V\omega$ of ω and the horizontal lift ${}^HX$ of X are given, with respect to the induced coordinates, by

$$ {}^V\omega = \omega_i \frac{\partial}{\partial p_i} \tag{2.1} $$

and

$$ {}^{H}X = X^i \frac{\partial}{\partial x^i} + p_h \Gamma^h_{ij} X^j \frac{\partial}{\partial p_i}, \tag{2.2} $$

where Γ^h_{ij} are the coefficients of the Levi-Civita connection ∇ of g. Using the equations (2.1) and (2.2), we obtain

$$ E_j = \frac{\partial}{\partial x^j} + p_s \Gamma^s_{hj} \frac{\partial}{\partial p_h}, \tag{2.3} $$

$$ E_{\bar{j}} = \frac{\partial}{\partial p_j}. \tag{2.4} $$

These $2n$ vector fields are linearly independent and generate the horizontal distribution of ∇ and the vertical distribution of T^*M. Indeed, we have ${}^{H}X = X^j E_j$ and ${}^{V}\omega = \omega_j E_{\bar{j}}$. The set $\{E_\beta = E_j, E_{\bar{j}}\}$ is called the frame adapted to the affine connection ∇ on $\pi^{-1}(U) \subset T^*M$ (for details concerning T^*M see [19]).

For each $x \in M$, the scalar product $g^{-1} = (g^{ij})$ is defined on the cotangent space $\pi^{-1}(x) = T^*_x M$ by

$$ g^{-1}(\omega, \theta) = g^{ij} \omega_i \theta_j \tag{2.5} $$

for all $\omega, \theta \in \Lambda^1(M)$.

Now, we consider the Riemannian metric $\tilde{g}$ on T^*M as follows:

$$ \begin{aligned} \tilde{g}({}^{H}X, {}^{H}Y) &= {}^{V}(g(X,Y)) = g(X,Y) \circ \pi, \\ \tilde{g}({}^{H}X, {}^{V}\omega) &= 0, \\ \tilde{g}({}^{V}\omega, {}^{V}\theta) &= a g^{-1}(\omega, \theta) + b g^{-1}(\omega, p) g^{-1}(\theta, p), \end{aligned} \tag{2.6} $$

for any $X, Y \in \chi(M)$ and $\omega, \theta \in \Lambda^1(M)$, where a and b are smooth functions of $t = \frac{1}{2} g^{-1}(p,p)$ such that $a > 0$ and $a + 2tb > 0$ [11]. The matrix representations of $\tilde{g}$ and its inverse $\tilde{g}^{-1}$ are given by respectively

$$ \tilde{g} = \begin{pmatrix} g_{ij} & 0 \\ 0 & a g^{ij} + b g^{i0} g^{j0} \end{pmatrix} \tag{2.7} $$

and

$$ g^{-1} = \begin{pmatrix} g^{ij} & 0 \\ 0 & \frac{1}{a} g_{ij} - \frac{b}{a(a + b g^{00})} p_i p_j \end{pmatrix}, \tag{2.8} $$

where $g^{i0} = g^{ik} p_k$ and $g^{00} = g^{i0} p_i$.

For the Levi-Civita connection $\tilde{\nabla}$ of the metric $\tilde{g}$, we get the following proposition by a straightforward computation.

Proposition 2.1. *Let (M,g) be a Riemannian manifold and $(T^*M, \tilde{g})$ be its cotangent bundle. The Levi-Civita connection $\tilde{\nabla}$ of the metric $\tilde{g}$ satisfies the relations*

$$ \begin{cases} \tilde{\nabla}_{E_i} E_j = \Gamma^h_{ij} E_h + \frac{1}{2} R_{ijh}{}^{0} E_{\bar{h}}, \\ \tilde{\nabla}_{E_{\bar{i}}} E_j = \frac{a}{2} R^{hi0}_{.j.} E_h, \\ \tilde{\nabla}_{E_i} E_{\bar{j}} = \frac{a}{2} R^{hj0}_{.i.} E_h - \Gamma^j_{ih} E_{\bar{h}}, \\ \tilde{\nabla}_{E_{\bar{i}}} E_{\bar{j}} = F_1 (g^{i0} E_{\bar{j}} + g^{j0} E_{\bar{i}}) + (F_2 g^{ij} + F_3 g^{i0} g^{j0}) C = A^{ij}_l E_{\bar{l}} \end{cases} \tag{2.9} $$

with respect to the adapted frame, where $F_1 = \frac{a'}{2a}, F_2 = \frac{2b-a'}{2(a+2tb)}, F_3 = \frac{ab'-2a'b}{2a(a+2tb)}$ *,* $C = p_h E_{\bar{h}}$ *and* $A_l^{ij} = \{(2F_1 + F_2)g^{ij} + F_3 g^{i0} g^{j0}\}C$*. Also,* Γ_{ih}^j *are the Christoffel symbols of* ∇*,* $R_{ijk}{}^h$ *are the local coordinate components of the curvature tensor field of* ∇*,* $R_{ijh}{}^0 = p_m R_{ijh}{}^m$ *and* $R_{.j.}^{hi\;0} = g^{sh} g^{it} p_m R_{sjt}{}^m$*.*

3. Harmonic maps on the cotangent bundle T^*M

Let (M, g_1) and (N, g_2) be two Riemannian manifolds of dimensions m and n, respectively and $f : M \to N$ be a smooth map. Let $U \subset M$ be a domain with coordinates $(x^1, ..., x^m)$ and $V \subset N$ be a domain with coordinates $(y^1, ..., y^n)$ such that $f(U) \subset V$, and suppose that f is locally represented by $y^\alpha = f^\alpha(x^1, ..., x^m), \alpha = 1, ..., n$. Then the second fundamental form of f, denoted by $\beta(f)$, is locally given by

$$\beta(f)(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})^\gamma = \{\frac{\partial^2 f^\gamma}{\partial x^i \partial x^j} -^M \Gamma_{ij}^k \frac{\partial f^\gamma}{\partial x^k} +^N \Gamma_{\alpha\beta}^\gamma \frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial x^j}\}\frac{\partial}{\partial y^\gamma} \tag{3.1}$$

and that of the tension field $\tau(f)$ of f is

$$\tau(f) = tr\beta(f) = g^{ij}\{\frac{\partial^2 f^\gamma}{\partial x^i \partial x^j} -^M \Gamma_{ij}^k \frac{\partial f^\gamma}{\partial x^k} +^N \Gamma_{\alpha\beta}^\gamma \frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial x^j}\}\frac{\partial}{\partial y^\gamma}, \tag{3.2}$$

where ${}^M\Gamma_{ij}^k$ and ${}^N\Gamma_{\alpha\beta}^\gamma$ are the Christoffel symbols of the Levi-Civita connections of the metrics g_1 and g_2 respectively.

The map f is a totally geodesic map if and only if $\beta(f) = 0$, and the map f is said to be harmonic if $\tau(f) = 0$ [5].

First, we shall study the harmonicity of the canonical projection $\pi : T^*M \to M$, which is a Riemannian submersion. From (2.9) and (3.1), we obtain

$$\begin{aligned} \beta(\pi)(E_i, E_j) &= \beta(\pi)(E_{\bar{i}}, E_{\bar{j}}) = 0, \\ \beta(\pi)_p(E_{\bar{i}}, E_j) &= -\frac{a}{2} R_{.j.}^{hi0}(\pi(p))\frac{\partial}{\partial x^h}. \end{aligned} \tag{3.3}$$

So, we write

Theorem 3.1. *Let* (M, g) *be a Riemannian manifold and* $(T^*M, \tilde{g})$ *be its cotangent bundle. The Riemannian submersion* $\pi : (T^*M, \tilde{g}) \to (M, g)$ *is totally geodesic if and only if* M *is locally flat. Moreover,* π *is a harmonic map.*

Let d be another Riemannian metric on M. For the projection $\pi : (T^*M, \tilde{g}) \to (M, d)$, we get the relations

$$\begin{aligned} \beta(\pi)(E_{\bar{i}}, E_{\bar{j}}) &= 0, \\ \beta(\pi)_u(E_{\bar{i}}, E_j) &= -\frac{a}{2} R_{.j.}^{hi0}(\pi(p))\frac{\partial}{\partial x^h}, \\ \beta(\pi)(E_i, E_j) &= \{{}^d\Gamma_{ij}^h - \Gamma_{ij}^h\}\frac{\partial}{\partial x^h}, \end{aligned} \tag{3.4}$$

where ${}^d\Gamma^h_{ij}$ are the Christoffel symbols of the metric d. Hence we have the proposition below.

Proposition 3.1. *Let (M,g) be a Riemannian manifold and $(T^*M,\tilde{g})$ be its cotangent bundle. The map $\pi : (T^*M,\tilde{g}) \to (M,d)$ is totally geodesic if and only if (M,g) is flat and the identity map $I : (M,g) \to (M,d)$ is totally geodesic.*

Note that if $\pi : (T^*M,\tilde{g}) \to (M,d)$ is totally geodesic then (M,g) and (M,d) are locally flat.

On the other hand, we know that d is harmonic with respect to g if $g^{ij}\{{}^d\Gamma^h_{ij} - \Gamma^h_{ij}\} = 0$ [4]. From (3.4), we obtain

Proposition 3.2. *Let (M,g) be a Riemannian manifold and $(T^*M,\tilde{g})$ be its cotangent bundle. The map $\pi : (T^*M,\tilde{g}) \to (M,d)$ is harmonic if and only if d is harmonic with respect to g.*

Now, let σ be a 1-form on M. We consider σ as a smooth map from M to T^*M. For investigation of the harmonicity of σ, we should prove the following proposition.

Proposition 3.3. *Let (M,g) be a Riemannian manifold and $(T^*M,\tilde{g})$ be its cotangent bundle. The map $\sigma : M \to T^*M$ is an isometric immersion if and only if $\nabla\sigma = 0$.*

Proof. We have

$$\sigma_{*,p}X = ({}^HX + {}^V(\nabla_X\sigma))_{\sigma(p)}, \ \forall p \in M. \tag{3.5}$$

If we assume

$$\begin{aligned} g'_p(X,Y) &= \tilde{g}_{\sigma(p)}(\sigma_{*,p}X, \sigma_{*,p}Y) = g_p(X,Y) + a g_p^{-1}(\nabla_X\sigma, \nabla_Y\sigma) \\ &\quad + b g_p^{-1}(\nabla_X\sigma,\sigma) g_p^{-1}(\nabla_Y\sigma,\sigma), \end{aligned}$$

then σ is an isometric immersion if and only if $g' = g$. It is obvious that $g' = g$ if and only if $\nabla_X\sigma = 0$ for every vector field X on M, i.e., $\nabla\sigma = 0$. □

Since $\sigma : (M,g) \to (T^*M,\tilde{g})$ is an immersion, we can use the formula below:

$$\beta(\sigma)(X,Y) = \tilde{\nabla}_{\sigma_*X}\sigma_*Y - \sigma_*(\nabla_XY), \ \ \forall X,Y \in \chi(M) \tag{3.6}$$

(see [14]). We have

$$\sigma_*(\frac{\partial}{\partial x^i}) = E_i + \nabla_i\sigma_j E_{\bar{j}}. \tag{3.7}$$

Using the equations (2.9), (3.5) and the formula (3.6), we obtain

Proposition 3.4. *Let (M,g) be a Riemannian manifold and $(T^*M,\tilde{g})$ be its cotangent bundle. The second fundamental form $\beta(\sigma)$ and the tension field $\tau(\sigma)$ of the map $\sigma:(M,g)\to(T^*M,\tilde{g})$ are given by respectively*

(3.8)
$$\begin{aligned}\beta(\sigma)(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}) &= \frac{a}{2}\{(\nabla_i\sigma_s)R^{hsm}_{.j.}\sigma_m+(\nabla_j\sigma_k)R^{hkm}_{.i.}\sigma_m\}E_h\\ &+\{-\frac{1}{2}R_{ijh}{}^{m}\sigma_m+\nabla_i\nabla_j\sigma_h+(\nabla_i\sigma_m)(\nabla_j\sigma_n)A^{mn}_h\}E_{\overline{h}},\end{aligned}$$

(3.9)
$$\tau(\sigma)=\{ag^{ij}(\nabla_j\sigma_s)R^{hsm}_{.i.}\sigma_m\}E_h+\{g^{ij}(\nabla_i\nabla_j\sigma_h)+g^{ij}(\nabla_i\sigma_m)(\nabla_j\sigma_n)A^{mn}_h\}E_{\overline{h}}.$$

The equation (3.9) gives the following results.

Theorem 3.2. *Let (M,g) be a Riemannian manifold and $(T^*M,\tilde{g})$ be its cotangent bundle. The map $\sigma:(M,g)\to(T^*M,\tilde{g})$ is harmonic if and only if it satisfies the system:*

$$g^{ij}(\nabla_j\sigma_s)R^{hsm}_{.i.}\sigma_m=0,\quad g^{ij}(\nabla_i\nabla_j\sigma_h)+g^{ij}(\nabla_i\sigma_m)(\nabla_j\sigma_n)A^{mn}_h=0. \tag{3.10}$$

Corollary 3.1. *The tension field $\tau(\sigma)$ is collinear with $C=p_i\frac{\partial}{\partial p_i}$ if and only if $g^{ij}(\nabla_i\nabla_j\sigma_h)=f\sigma_h$, $g^{ij}(\nabla_j\sigma_s)R^{hsm}_{.i.}\sigma_m=0$, where f is a smooth function.*

At the end of the first part, we investigate the harmonicity of the identity maps $I_{\tilde{g}}:(T^*M,\tilde{g})\to(T^*M,{}^Sg)$ and $I_{{}^Sg}:(T^*M,{}^Sg)\to(T^*M,\tilde{g})$, where ${}^Sg$ denotes the Sasaki metric on T^*M. Therefore, before we proceed, we need to recall the Levi–Civita connection ${}^S\nabla$ of the metric ${}^Sg$. From [1], the Levi-Civita connection components of ${}^Sg$ are given as follows.

$$\begin{cases}{}^S\nabla_{E_i}E_j=\Gamma^h_{ij}E_h+\frac{1}{2}R_{ijh}{}^{0}E_{\bar{h}},\\ {}^S\nabla_{E_{\bar{i}}}E_j=\frac{1}{2}R^{hi0}_{.j.}E_h,\\ {}^S\nabla_{E_i}E_{\bar{j}}=\frac{1}{2}R^{hj0}_{.i.}E_h-\Gamma^j_{ih}E_{\bar{h}},\\ {}^S\nabla_{E_{\bar{i}}}E_{\bar{j}}=0,\end{cases} \tag{3.11}$$

where Γ and R are given as in Proposition 2.1. Using (2.9), (3.1) and (3.11), we get

$$\begin{aligned}\beta(I_{\tilde{g}})(E_i,E_j) &= 0,\ \ \beta(I_{\tilde{g}})(E_{\bar{i}},E_j)=\frac{1-a}{2}R^{hi0}_{.j.}E_h,\\ \beta(I_{\tilde{g}})(E_{\bar{i}},E_{\bar{j}}) &= -A^{ij}_lE_{\bar{l}}\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}\beta(I_{{}^Sg})(E_i,E_j) &= 0,\ \ \beta(I_{{}^Sg})(E_{\bar{i}},E_j)=\frac{a-1}{2}R^{hi0}_{.j.}E_h,\\ \beta(I_{{}^Sg})(E_{\bar{i}},E_{\bar{j}}) &= A^{ij}_lE_{\bar{l}}.\end{aligned} \tag{3.13}$$

Thus, we write

Proposition 3.5. *(i) The map* $I_{\tilde{g}} : (T^*M, \tilde{g}) \to (T^*M, {}^S g)$ *cannot be totally geodesic.*

(ii) The tension field $\tau_{I_{\tilde{g}}}$ *of* $I_{\tilde{g}}$ *is* $\tau_{I_{\tilde{g}}} = -\frac{(na+g^{00}(nb-1))(2F_1+F_2)+ag^{00}F_3}{a(a+bg^{00})}$. *So, the map* $I_{\tilde{g}} : (T^*M, \tilde{g}) \to (T^*M, {}^S g)$ *cannot be harmonic, i.e.,* ${}^S g$ *cannot be harmonic with respect to* $\tilde{g}$.

Proposition 3.6. *(i) The map* $I_{{}^S g} : (T^*M, {}^S g) \to (T^*M, \tilde{g})$ *cannot be totally geodesic.*

(ii) The tension field $\tau_{I_{{}^S g}}$ *of* $I_{\tilde{g}}$ *is* $\tau_{I_{{}^S g}} = n(2F_1 + F_2) + g^{00}F_3$. *So,* $I_{{}^S g} : (T^*M, {}^S g) \to (T^*M, \tilde{g})$ *cannot be harmonic, i.e.,* $\tilde{g}$ *cannot be harmonic with respect to* ${}^S g$.

4. Harmonic maps on the unit cotangent bundle T_1^*M

The unit cotangent bundle T_1^*M of a Riemannian manifold (M, g) is the $(2n-1)-$dimensional hypersurface given by $T_1^*M = \{\omega \in T^*M : g^{-1}(\omega, \omega) = 1\}$. If we denote by (x^i, p_i) local coordinates on T^*M, then T_1^*M can be expressed as $g^{00} = 1$, where $g^{i0} = g^{ij}p_j$. The local vector fields $\{E_i, Y^i\}$ generate a system for T_1^*M, where

$$Y^i = \frac{\partial}{\partial p_i} - g^{i0}C, \quad C = p_j \frac{\partial}{\partial p_j}. \tag{4.1}$$

The induced metric from $(T^*M, \tilde{g})$ on T_1^*M is given as follows:

$$g_a(E_i, E_j) = g_{ij}, g_a(E_i, Y^j) = 0, g_a(Y^i, Y^j) = a(g^{ij} - g^{i0}g^{j0}), \tag{4.2}$$

where a is a constant that satisfy $a > 0$. It is easy to check that $g_a(E_i, C) = g_a(Y^i, C) = 0$, i.e. C is orthogonal on T_1^*M with respect to g_a.

We have

Proposition 4.1. *Let* (M, g) *be a Riemannian manifold and* (T_1^*M, g_a) *be its unit cotangent bundle. The Levi-Civita connection* ${}^a\nabla$ *of the metric* g_a *satisfies the relations*

$$\begin{cases} {}^a\nabla_{E_i} E_j = \Gamma^h_{ij} E_h + \frac{1}{2} R_{ijh}{}^0 Y^h, \\ {}^a\nabla_{Y^i} E_j = \frac{a}{2} R^{hi0}_{.j.} E_h, \\ {}^a\nabla_{E_i} Y^j = \frac{a}{2} R^{hj0}_{.i.} E_h - \Gamma^j_{ih} Y^h, \\ {}^a\nabla_{Y^i} Y^j = -g^{j0} Y^i, \end{cases} \tag{4.3}$$

where Γ *and* R *are given as in Proposition 2.1.*

Now, we examine the harmonicity of the projection $\tilde{\pi} = \pi \mid_{T_1^*M}$. From (3.1) and (4.3), we get

$$\begin{aligned} \beta(\widetilde{\pi})(E_i, E_j) &= \beta(\widetilde{\pi})(Y^i, Y^j) = 0, \\ \beta(\widetilde{\pi})_p(Y^i, E_j) &= -\frac{a}{2} R^{hi0}_{.j.}(\pi(p)) \frac{\partial}{\partial x^h}. \end{aligned} \tag{4.4}$$

The equations in (4.4) give us the following theorem.

Theorem 4.1. *Let (M,g) be a Riemannian manifold and (T_1^*M, g_a) be its unit cotangent bundle. The map $\tilde{\pi} : (T_1^*M, g_a) \to (M,g)$ is totally geodesic if and only if M is locally flat. Moreover $\tilde{\pi}$ is a harmonic map.*

Denote by ${}^S\overline{g}$ the induced Sasaki metric on T_1^*M. For investigation of the harmonicity of the identity map from (T_1^*M, g_a) to $(T_1^*M, {}^S\overline{g})$ (resp. from $(T_1^*M, {}^S\overline{g})$ to (T_1^*M, g_a)), we should express the following proposition.

Proposition 4.2. *Let (M,g) be a Riemannian manifold and $(T_1^*M, {}^S\overline{g})$ be its unit cotangent bundle. The Levi-Civita connection ${}^S\bar{\nabla}$ of the metric ${}^S\overline{g}$ satisfies the relations*

$$\begin{cases} {}^S\bar{\nabla}_{E_i} E_j = \Gamma_{ij}^h E_h + \frac{1}{2} R_{ijh}{}^0 Y^h, \\ {}^S\bar{\nabla}_{Y^i} E_j = \frac{1}{2} R_{.j.}^{hi0} E_h, \\ {}^S\bar{\nabla}_{E_i} Y^j = \frac{1}{2} R_{.i.}^{hj0} E_h - \Gamma_{ih}^j Y^h, \\ {}^S\bar{\nabla}_{Y^i} Y^j = -g^{j0} Y^i, \end{cases} \tag{4.5}$$

where Γ and R are given as in Proposition 2.1.

Using (3.1), (4.3) and (4.5), we obtain the following expressions for the identity maps $I_{g_a} : (T_1^*M, g_a) \to (T_1^*M, {}^S\overline{g})$ and $I_{{}^S\overline{g}} : (T_1^*M, {}^S\overline{g}) \to (T_1^*M, g_a)$, respectively:

$$\beta(I_{g_a})(E_i, E_j) = \beta(I_{g_a})(Y^i, Y^j) = 0, \quad \beta(I_{g_a})(Y^i, E_j) = \frac{1-a}{2} R_{.j.}^{hi0} E_h \tag{4.6}$$

and

$$\beta(I_{{}^S\overline{g}})(E_i, E_j) = \beta(I_{{}^S\overline{g}})(Y^i, Y^j) = 0, \quad \beta(I_{{}^S\overline{g}})(Y^i, E_j) = \frac{a-1}{2} R_{.j.}^{hi0} E_h. \tag{4.7}$$

Consequently, we state

Proposition 4.3. *(i) The map $I_{g_a} : (T_1^*M, g_a) \to (T_1^*M, {}^S\overline{g})$ (resp. the map $I_{{}^S\overline{g}} : (T_1^*M, {}^S\overline{g}) \to (T_1^*M, g_a)$) is totally geodesic if and only if (M,g) is locally flat.*

(ii) The maps $I_{g_a} : (T_1^*M, g_a) \to (T_1^*M, {}^S\overline{g})$ and $I_{{}^S\overline{g}} : (T_1^*M, {}^S\overline{g}) \to (T_1^*M, g_a)$ are harmonic.

Let σ be a 1-form with $g^{-1}(\sigma, \sigma) = 1$. Denote by $\sigma_{\tilde{g}} : (M,g) \to (T^*M, \tilde{g}), \sigma_{\tilde{g}}(q) = \sigma(q)$ and $\bar{\sigma}_{g_a} : (M,g) \to (T_1^*M, g_a)$, $\bar{\sigma}_{g_a}(q) = \sigma(q)$, $\forall q \in M$. We have

$$\bar{\sigma}_*(\frac{\partial}{\partial x^i}) = E_i + (\nabla_i \sigma_j) Y^j. \tag{4.8}$$

So from (4.3) and (4.8), we get the expression for $\tau(\bar{\sigma}_{g_a})$:

$$\tau(\bar{\sigma}_{g_a}) = \{g^{ij} (\nabla_i \nabla_j \sigma_h)\} Y^h + a\{g^{ij} (\nabla_j \sigma_k) R_{.i.}^{hkm} \sigma_m\} E_h. \tag{4.9}$$

Having in mind Corollary 3.1 and the expression (4.9), we give the final theorem of the paper.

Theorem 4.2. *Let (M,g) be a Riemannian manifold and (T_1^*M, g_a) be its unit cotangent bundle. The map $\bar{\sigma}_{g_a} : (M,g) \to (T_1^*M, g_a)$ is harmonic if and only if $\tau(\sigma_{\tilde{g}})$ is collinear with C.*

REFERENCES

1. S. Akbulut, M. Ozdemir and A. Salimov: *Diagonal lift in the cotangent bundle and its applications.* Turk. J. Math. **25** (2001), 491–502.
2. M. Anastasie: *Locally conformal Kaehler structures on tangent bundle of a space form.* Libertas Math. **19** (1999), 71–76.
3. J. Cheeger and D. Gromoll: *On the structure of complete manifolds of nonnegative curvature.* Ann. of Math. **96** (1972), 413–443.
4. B. Y. Chen and T. Nagano: *Harmonic metrics, harmonic tensors and Gauss maps.* J. Math. Soc. Japan **36**(2) (1984), 295–313.
5. J. Eells and L. Lemaire: *Selected topics in harmonic maps.* Conf. Board of the Math. Sci. A.M.S. **50**, Providince (1983).
6. A. Gezer and M. Altunbaş: *Notes on the rescaled Sasaki type metric on the cotangent bundle.* Acta Math. Sci. **34B**(1) (2014), 162–174.
7. A. Gezer and M. Altunbaş: *On the rescaled Riemannian metric of Cheeger-Gromoll type on the cotangent bundle.*Hacettepe J. Math. Stat. **45**(2) (2016), 355–365.
8. S. Ishihara: *Harmonic sections of tangent bundles.* J. Math. Tokushima Univ. **13** (1979), 23–27.
9. M. I. Munteanu: *Some aspects on the geometry of the tangent bundles and tangent sphere bundles of a Riemannian manifold.* Mediterr. J. Math. **5**(1) (2008), 43–59.
10. E. Musso and F. Tricerri: *Riemannian metrics on tangent bundles.* Ann. Mat. Pur. Appl. **150**(4), (1988), 1–19.
11. F. Ocak: *g-natural metrics on the cotangent bundle.* Int. Elec. J. Geo. **6**(1) (2013), 129–146.
12. C. Oniciuc: *On the harmonic sections of tangent bundles.* An. Univ. Bucuresti **47**(1) (1998), 67–72.
13. C. Oniciuc: *Tangent bundles and harmonicity.* Ann. Sti. Ale. Uni. "Al. I. Cuza" Iasi **XLIII**(1) (1997), 151–172.
14. V. Opriou: *On harmonic sections of cotangent bundles.* Rendiconti Sem. Fac. Sci. Uni. Cagliari **59**(2) (1989), 177–184.
15. V. Opriou and N. Papaghiuc: *Some results on harmonic sections of cotangent bundles.* Ann. Stiint. Univ. "Al. I. Cuza" Iasi **45** (1999), 275–290.
16. A. A. Salimov and F. Agca: *Some properties of Sasakian metrics in cotangent bundles.* Mediterr. J. Math. **8** (2010), 243–255.
17. A. A. Salimov and S. Kazimova: *Geodesics of the Cheeger-Gromoll Metric.* Turk. J. Math. **33**(1) (2009), 99–105.
18. S. Sasaki: *On the differential geometry of tangent bundles of Riemannian manifolds.* Tohoku Math. J. **10** (1958), 338–358.
19. K. Yano and S. Ishihara: *Tangent and cotangent bundles*, Marcel Dekker Inc., New York (1973).