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ON RICCI SOLITONS AND SUBMANIFOLDS WITH A SEMI-SYMMETRIC NON-METRIC CONNECTION

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Abstract. We consider Ricci solitons with a semi-symmetric non-metric connection. We find some properties, when the potential vector field is torse-forming. Applications to submanifolds are also given.
Keywords: Ricci soliton, semi-symmetric non-metric connection, torse-forming vector field, quasi-Einstein manifold, hyper-generalized quasi-Einstein manifold.

1. Introduction

A *semi-symmetric connection is a* linear connection on a Riemannian manifold (M, g) whose torsion tensor T is of the form

$$T(X_1, X_2) = \phi(X_2)X_1 - \phi(X_1)X_2,$$

where ϕ is a 1-form defined by $\phi(X_1) = g(X_1, U)$, and U is a vector field on M [12].

Let ∇ be the Levi-Civita connection of a Riemannian manifold (M, g). The semi-symmetric non-metric connection $\widetilde{\nabla}$ (briefly $SSNMC$) is defined by

$$\widetilde{\nabla}_{X_1} X_2 = \nabla_{X_1} X_2 + \phi(X_2)X_1, \tag{1.1}$$

where X_1, X_2 are vector fields on M [1]. Let $\widetilde{R}$ and R denote Riemannian curvature tensor fields of $\widetilde{\nabla}$ and ∇, respectively. Then from (1.1), it is easy to see that

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$$\widetilde{R}(X_1, X_2)X_3 = R(X_1, X_2)X_3 - \alpha(X_2, X_3)X_1 + \alpha(X_1, X_3)X_2, \tag{1.2}$$

where α is a tensor of type $(0, 2)$ of M given by

$$\alpha(X_1, X_2) = (\nabla_{X_1}\phi)X_2 - \phi(X_1)\phi(X_2). \tag{1.3}$$

Let $\widetilde{Ric}$ and Ric state the Ricci tensor fields of the connections $\widetilde{\nabla}$ and ∇, respectively. Then from (1.2), it is easy to see that

$$\widetilde{Ric} = Ric - (n-1)\alpha, \tag{1.4}$$

(see [1]).

Let (M, g) be a Riemannian manifold. R. S. Hamilton [14] presented the *Ricci flow* for the first time as

$$\frac{\partial}{\partial t}g(t) = -2Ric(g(t)).$$

The Ricci flow is an evolution equation for Riemannian metrics. There is a correspondence between Ricci solitons and self-similar solutions of Ricci flow. A smooth vector field υ on a Riemannian manifold (M, g) is considered to define a *Ricci soliton* [13], if there exists a real constant λ such that

$$\frac{1}{2}\pounds_{\upsilon}g + Ric = \lambda g, \tag{1.5}$$

where $\pounds_{\upsilon}$ denotes the Lie derivative operator in the direction of the vector field υ, Ric denotes the Ricci tensor field of (M, g).

We denote Ricci soliton by (υ, λ). It is obvious that Ricci solitons are natural generalizations of Einstein metrics, any Einstein metric gives a trivial Ricci soliton. A Ricci soliton (υ, λ) on a (semi)-Riemannian manifold (M, g) is considered to be *shrinking*, *steady* or *expanding* according to λ is positive, zero or negative, respectively [13].

Quite a few geometers have lately studied the geometry of Ricci solitons. Refer to, for instance, [8,17,18] and the references therein. Ricci solitons on submanifolds have also become a quite popular study subject. For such studies refer to, for example, [3,5,8] and the references therein.

In the present study, we consider some properties of Ricci solitons on Riemannian manifolds equipped with an $SSNMC$ when the potential vector field is torse-forming with respect to an $SSNMC$. As recent studies on torse-forming vector fields see [**?**, 5, 9, 16].

The paper is organized as follows: In Section 2, we discover the geometric properties of Ricci solitons on Riemannian manifolds when the potential vector field is torse-forming. In Section 3, we get some applications for submanifolds.

2. Ricci solitons on Riemannian manifolds with an $SSNMC$

In this section, we consider Ricci solitons on Riemannian manifolds with an $SSNMC$.

The Euclidean 3-space, hyperbolic 3-space and Minkowski motion group are included in the following 3-parameter family of Riemannian homogeneous spaces $(\mathbb{R}^3, g[\mu_1,\mu_2,\mu_3])$ with left-invariant metric

$$g[\mu_1,\mu_2,\mu_3] = e^{-2\mu_1 t}dx^2 + e^{-2\mu_2 t}dy^2 + \mu_3^2 dt^2.$$

Here μ_1,μ_2 are real constants and μ_3 is a positive constant.

The Lie group $G(\mu_1,\mu_2,\mu_3)$ can be realised as a closed subgroup of affine transformation group $GL_3\mathbb{R} \ltimes \mathbb{R}^3$ of $\mathbb{R}^3$.

The Levi-Civita connection ∇ of $G(\mu_1,\mu_2,\mu_3)$ is given by the following formula:

$$\nabla_{E_1}E_1 = \frac{\mu_1}{\mu_3}E_3,\ \nabla_{E_1}E_2 = 0,\ \nabla_{E_1}E_3 = -\frac{\mu_1}{\mu_3}E_1 \tag{2.1}$$

$$\nabla_{E_2}E_1 = 0,\ \nabla_{E_2}E_2 = \frac{\mu_2}{\mu_3}E_3,\ \nabla_{E_2}E_3 = -\frac{\mu_2}{\mu_3}E_2,$$

$$\nabla_{E_3}E_1 = \ \nabla_{E_3}E_2 = \ \nabla_{E_3}E_3 = 0.$$

The Ricci tensor field Ric of G is given by

$$R_{11} = -\frac{\mu_1(\mu_1+\mu_2)}{\mu_3^2},\ R_{22} = -\frac{\mu_2(\mu_1+\mu_2)}{\mu_3^2}, R_{33} = -\frac{\mu_1^2+\mu_2^2}{\mu_3^2}$$

and the scalar curvature τ of G is given by

$$\tau = -\frac{2}{\mu_3^2}\left(\mu_1^2+\mu_2^2+\mu_1\mu_2\right).$$

(see [15]).

Using (2.1), the Levi-Civita connection ∇ of $G(1,1,1)$ is given by the following formula:

$$\nabla_{E_1}E_1 = E_3,\ \nabla_{E_1}E_2 = 0,\ \nabla_{E_1}E_3 = -E_1,$$

$$\nabla_{E_2}E_1 = 0,\ \nabla_{E_2}E_2 = E_3,\ \nabla_{E_2}E_3 = -E_2,$$

$$\nabla_{E_3}E_1 = \ \nabla_{E_3}E_2 = \ \nabla_{E_3}E_3 = 0.$$

Then we can state the following example:

Example 2.1. Assume that $\upsilon = 2E_3$ is the potential vector field. Then $G(1,1,1)$ is an *Ricci soliton* with respect to an $SSNMC$.

Using (1.1), we reach the Lie derivative as follows

$$(\widetilde{\mathcal{L}}_{\upsilon} g)(X_1, X_2) = g(\widetilde{\nabla}_{X_1} \upsilon, X_2) + g(X_1, \widetilde{\nabla}_{X_2} \upsilon) - 2\phi(\upsilon) g(X_1, X_2), \tag{2.2}$$

Therefore, using equation (2.2), the soliton equation (1.5) with respect to an $SSNMC$ could be written as

$$\frac{1}{2}\left(g(\widetilde{\nabla}_{X_1}\upsilon, X_2) + g(X_1, \widetilde{\nabla}_{X_2}\upsilon)\right) + \widetilde{Ric}(X_1, X_2) = (\lambda + \phi(\upsilon))\, g(X_1, X_2). \tag{2.3}$$

A vector field υ on a Riemannian manifold (M, g) is called *torse-forming* [20], if

$$\nabla_{X_1}\upsilon = cX_1 + \omega(X_1)\upsilon,$$

where c is a smooth function, ω is a 1-form and ∇ is the Levi-Civita connection of g.

Specifically, if $\omega = 0$, then υ is called a concircular vector field [11] and if $c = 0$, then υ is called a *recurrent vector field* [18].

Presume that U is a parallel unit vector field with respect to the Levi-Civita connection ∇. Using (1.1), we reach

$$\widetilde{\nabla}_{X_1} U = X_1.$$

Consequently, we obtain the proposition below:

Proposition 2.1. *Let (M, g) be a Riemannian manifold endowed with an $SSNMC$. If U is a parallel unit vector field with respect to the Levi-Civita connection ∇ then, U is a torse-forming potential vector field (briefly $TF - PVF$) with respect to an $SSNMC$ of the form $\widetilde{\nabla}_{X_1} U = X_1$.*

A non-flat Riemannian manifold (M, g) $(n \geq 3)$ is called *a hyper-generalized quasi-Einstein manifold* [19], if its Ricci tensor field is not likewise zero and provides

$$Ric = a_1 g + a_2 A \otimes A + a_3 (A \otimes B + B \otimes A) + a_4 (A \otimes D + D \otimes A),$$

where a_1, a_2, a_3 and a_4 are scalars and A, B and D are non-zero 1-forms. If $a_4 = 0$, then M is called a *generalized quasi-Einstein manifold* in the sense of Chaki [6]. If $a_3 = a_4 = 0$, then M is called a *quasi-Einstein manifold* [7]. If $a_2 = a_3 = a_4 = 0$, then (M, g) is *an Einstein manifold* [4]. *The functions* a_1, a_2, a_3 and a_4 are called *associated functions.*

Now let (M, g) be a Riemannian manifold equipped with an $SSNMC$ and υ a $TF - PVF$ with respect to an $SSNMC$ on M. Then $\widetilde{\nabla}_{X_1}\upsilon = cX_1 + \omega(X_1)\upsilon$. So by (2.3), we can write

$$\begin{aligned} Ric(X_1, X_2) &= (\lambda - c + \phi(\upsilon))\, g(X_1, X_2) \\ &-\frac{1}{2}\left\{g(X_1, \upsilon)\omega(X_2) + g(X_2, \upsilon)\omega(X_1)\right\}. \end{aligned} \tag{2.4}$$

Hence we have:

Corollary 2.1. *Let (M, g) be a Riemannian manifold endowed with an $SSNMC$ and υ a $TF-PVF$ with respect to an $SSNMC$ on M. Assume that a 1-form η is the g-dual of υ. Then (M, g) is a Ricci soliton (υ, λ) if and only if*

$$Ric(X_1, X_2) = (\lambda - c + \phi(\upsilon))\, g(X_1, X_2) - \frac{1}{2}\{\eta(X_1)\omega(X_2) + \eta(X_2)\omega(X_1)\}.$$

Thus, we can state the following theorems:

Theorem 2.1. *Let (M, g) be a Riemannian manifold endowed with an $SSNMC$ and υ a $TF-PVF$ with respect to an $SSNMC$ on M. Assume that a 1-form η is the g-dual of υ. Then (M, g) is a Ricci soliton (υ, λ) if and only if M is a hyper-generalized quasi-Einstein manifold with associated functions $(\lambda - c + \phi(\upsilon)), 0, 0$ and $-\frac{1}{2}$.*

Theorem 2.2. *Let (M, g) be a Riemannian manifold endowed with an $SSNMC$ and υ a $TF-PVF$ with respect to an $SSNMC$ on M. Assume that a 1-form η is the g-dual of υ and $\omega = \eta$. Then (M, g) is a Ricci soliton (υ, λ) if and only if M is a quasi-Einstein manifold with associated functions $(\lambda - c + \phi(\upsilon)), -1$.*

If υ is a concircular potential vector field (briefly $C - PVF$) with respect to an $SSNMC$, then the following theorem can be expressed:

Theorem 2.3. *Let (M, g) be a Riemannian manifold endowed with an $SSNMC$ and υ a $C - PVF$ with respect to an $SSNMC$ on M. Assume that a 1-form η is the g-dual of υ. Then (M, g) is a Ricci soliton. (υ, λ) if and only if M is an Einstein manifold with an associated function $(\lambda - c + \phi(\upsilon))$.*

Using (1.4), the equation (2.4) can be written as

$$\widetilde{Ric}(X_1, X_2) = (\lambda - c + \phi(\upsilon))\, g(X_1, X_2) - (n-1)\alpha(X_1, X_2) - \frac{1}{2}\{g(X_1, \upsilon)\omega(X_2) + g(X_2, \upsilon)\omega(X_1)\}. \tag{2.5}$$

Thus, the following corollary can be expressed:

Corollary 2.2. *Let (M, g) be a Riemannian manifold endowed with an $SSNMC$ and υ a $TF - PVF$ with respect to an $SSNMC$ on M. Then (M, g) is a Ricci soliton if and only if the Ricci tensor field of an $SSNMC$ is of the form (2.5).*

Now assume that U is a parallel unit vector field with respect to the Levi-Civita connection, i.e., $\nabla U = 0$ and $\|U\| = 1$. Then

$$(\nabla_{X_1}\phi)X_2 = \nabla_{X_1}\phi(X_2) - \phi(\nabla_{X_1}X_2) = 0.$$

So from (1.3), $\alpha(X_1, X_2) = -\phi(X_1)\phi(X_2)$. Thus by (2.5), we have

$$\widetilde{Ric}(X_1, X_2) = (\lambda - c + \phi(\upsilon))g(X_1, X_2) + (n-1)\,\phi(X_1)\phi(X_2) - \frac{1}{2}\left\{g(X_1, \upsilon)\omega(X_2) + g(X_2, \upsilon)\omega(X_1)\right\}. \tag{2.6}$$

Hence we have:

Corollary 2.3. *Let (M, g) be a Riemannian manifold endowed with an $SSNMC$, U a parallel unit vector field with respect to the Levi-Civita connection ∇ and υ a $TF-PVF$ with respect to an $SSNMC$ on M. Then (M, g) is a Ricci soliton if and only if the Ricci tensor field of an $SSNMC$ is of the form (2.6).*

Thus, the following theorems can be stated:

Theorem 2.4. *Let (M, g) be a Riemannian manifold endowed with an $SSNMC$, U a parallel unit vector field with respect to the Levi-Civita connection ∇ and υ a $TF-PVF$ with respect to an $SSNMC$ on M. Assume that a 1-form ϕ is the g-dual of υ. Then (M, g) is a Ricci soliton (υ, λ) if and only if M is a generalized quasi-Einstein manifold with respect to an $SSNMC$ with associated functions $\left(\lambda - c + \|\upsilon\|^2\right)$, $(n-1)$ and $-\frac{1}{2}$.*

Theorem 2.5. *Let (M, g) be a Riemannian manifold endowed with an $SSNMC$, U a parallel unit vector field with respect to the Levi-Civita connection ∇ and υ a $C-PVF$ with respect to an $SSNMC$ on M. Assume that a 1-form ϕ is the g-dual of υ. Then (M, g) is a Ricci soliton (υ, λ) if and only if M is a quasi-Einstein manifold with respect to an $SSNMC$ with associated functions $\left(\lambda - c + \|\upsilon\|^2\right)$ and $(n-1)$.*

3. Submanifolds

Let $\left(\widetilde{M}, \widetilde{g}\right)$ be an $(n+d)$-dimensional Riemannian manifold endowed with an $SSNMC$ $\widetilde{\nabla}$ and the Levi-Civita connection ∇. Decomposing the vector field U on M uniquely into its tangential and normal components U^T and $U^\perp$, respectively, we have

$$U = U^T + U^\perp.$$

Let M be an n-dimensional submanifold of $\left(\widetilde{M}, \widetilde{g}\right)$. On the submanifold M, let us denote the induced $SSNMC$ by $\overset{\circ}{\widetilde{\nabla}}$ and the induced Levi-Civita connection by $\overset{\circ}{\nabla}$.

The Gauss formulas and Weingarten formulas with respect to ∇ and $\widetilde{\nabla}$ could be written as follows:

$$\nabla_{X_1} X_2 = \overset{\circ}{\nabla}_{X_1} X_2 + h(X_1, X_2),$$

$$\widetilde{\nabla}_{X_1} X_2 = \overset{\circ}{\widetilde{\nabla}}_{X_1} X_2 + \overset{\circ}{h}(X_1, X_2), \quad X_1, X_2 \in \chi(M),$$

and

$$\nabla_{X_1} N = -A_N X_1 + \overset{\circ}{\nabla}{}^{\perp}_{X_1} N,$$

$$\widetilde{\nabla}_{X_1} N = -\overset{\circ}{A}_N X_1 + \overset{\circ}{\widetilde{\nabla}}{}^{\perp}_{X_1} N,$$

respectively, where X_1, $X_2 \in \chi(M)$, h is the second fundamental form, N is a unit normal vector field and A_N is the shape operator of M in $\left(\widetilde{M}, \widetilde{g}\right)$ and $\overset{\circ}{h}$ is a normal valued $(0,2)$-tensor field and $\overset{\circ}{A}$ is a $(1,1)$-tensor field on M [2]. The tangential and normal parts of U are denote by U^T and $U^{\perp}$, respectively. Then from [2], we get

$$\overset{\circ}{h}(X_1, X_2) = h(X_1, X_2) \tag{3.1}$$

and

$$\overset{\circ}{A}_N X_1 = A_N X_1 - \phi(N) X_1. \tag{3.2}$$

It is known from [2] that the induced connection $\overset{\circ}{\widetilde{\nabla}}$ on a submanifold of a Riemannian manifold endowed with an $SSNMC$ is also an $SSNMC$.

Now assume that $(\widetilde{M}, \widetilde{g})$ is a Riemannian manifold admitting an $SSNMC$ and υ is a $TF-PVF$ with respect to an $SSNMC$ on $\widetilde{M}$. Let (M, g) be a submanifold of $(\widetilde{M}, \widetilde{g})$. Denote by υ^T and $\upsilon^{\perp}$, the tangential and normal parts of υ, respectively. Then using (1.1), it can be written as follows

$$\begin{aligned}
\widetilde{\nabla}_{X_1} \upsilon = \widetilde{\nabla}_{X_1} \left(\upsilon^T + \upsilon^{\perp}\right) &= \widetilde{\nabla}_{X_1} \upsilon^T + \widetilde{\nabla}_{X_1} \upsilon^{\perp} \\
= \nabla_{X_1} \upsilon^T + \phi(\upsilon^T) X_1 &+ \nabla_{X_1} \upsilon^{\perp} + \phi(\upsilon^{\perp}) X_1 \\
&= c X_1 + \omega(X_1) \upsilon^T + \omega(X_1) \upsilon^{\perp}.
\end{aligned}$$

So by using Gauss and Weingarten formulas and by the equality of the tangential and normal parts, we have

$$\overset{\circ}{\nabla}_{X_1} \upsilon^T = \left(c - \phi(\upsilon)\right) X_1 + A_{\upsilon^{\perp}} X_1 + \omega(X_1) \upsilon^T \tag{3.3}$$

and

$$h\left(X_1, \upsilon^T\right) + \overset{\circ}{\nabla}{}^{\perp}_{X_1} \upsilon^{\perp} = \omega(X_1) \upsilon^{\perp}.$$

Then in view of (3.3), we get

$$\begin{aligned}(\pounds_{\upsilon^T} g)(X_1, X_2) &= g(\overset{\circ}{\nabla}_{X_1}\upsilon^T, X_2) + g(X_1, \overset{\circ}{\nabla}_{X_2}\upsilon^T)\\ &= 2\,(c-\phi(\upsilon))\,g(X_1,X_2) + 2\widetilde{g}(h(X_1,X_2),\upsilon^\perp)\\ &\quad +\omega(X_1)g(X_2,\upsilon^T) + \omega(X_2)g(X_1,\upsilon^T).\end{aligned}$$

Thus, equation (1.5) gives us

$$\begin{aligned}\overset{\circ}{Ric}(X_1,X_2) &= (\lambda - c + \phi(\upsilon))\,g(X_1,X_2) - \widetilde{g}(h(X_1,X_2),\upsilon^\perp)\\ &\quad -\frac{1}{2}\left\{\omega(X_1)g(X_2,\upsilon^T) + \omega(X_2)g(X_1,\upsilon^T)\right\}.\end{aligned} \tag{3.4}$$

So we reach the following corollary:

Corollary 3.1. *Let M be an n-dimensional submanifold isometrically immersed into a Riemannian manifold $\left(\widetilde{M},\widetilde{g}\right)$ endowed with an $SSNMC$ and υ a $TF-PVF$ with respect to an $SSNMC$ on $\widetilde{M}$. Then (M,g) is a Ricci soliton (υ^T,λ) if and only if the condition (3.4) holds on M.*

If M is $\upsilon^\perp$-umbilical, then $A_{\upsilon^\perp} = kI$, where k is a function on M and I is the identity map [8]. Therefore, we obtain $\widetilde{g}(h(X_1,X_2),\upsilon^\perp) = g(A_{\upsilon^\perp}X_1,X_2) = kg(X_1,X_2)$. Then from (3.4), we have

$$\begin{aligned}\overset{\circ}{Ric}(X_1,X_2) &= (\lambda - c + \phi(\upsilon) - k)\,g(X_1,X_2)\\ &\quad -\frac{1}{2}\left\{\omega(X_1)g(X_2,\upsilon^T) + \omega(X_2)g(X_1,\upsilon^T)\right\}.\end{aligned} \tag{3.5}$$

Hence we can state the following corollary:

Corollary 3.2. *Let M be an n-dimensional $\upsilon^\perp$-umbilical submanifold isometrically immersed into a Riemannian manifold $\left(\widetilde{M},\widetilde{g}\right)$ endowed with an $SSNMC$ and υ a $TF-PVF$ with respect to an $SSNMC$ on $\widetilde{M}$. Then (M,g) is a Ricci soliton (υ^T,λ) if and only if the condition (3.5) holds on M.*

Thus, we can state the following theorems:

Theorem 3.1. *Let M be an n-dimensional $\upsilon^\perp$-umbilical submanifold isometrically immersed into a Riemannian manifold $\left(\widetilde{M},\widetilde{g}\right)$ endowed with an $SSNMC$ and υ a $TF-PVF$ with respect to an $SSNMC$ on $\widetilde{M}$. Assume that a 1-form η is the g dual of υ^T. Then (M,g) is a Ricci soliton (υ^T,λ) if and only if it is a hyper-generalized quasi-Einstein manifold with related functions $(\lambda - c + \phi(\upsilon) - k)$, $0, 0$ and $-\frac{1}{2}$.*

Theorem 3.2. *Let M be an n-dimensional $\upsilon^{\perp}$-umbilical submanifold isometrically immersed into a Riemannian manifold $\left(\widetilde{M}, \widetilde{g}\right)$ endowed with an $SSNMC$ and υ a $TF-PVF$ with respect to an $SSNMC$ on $\widetilde{M}$. Accept that a 1-form ω is the g dual of υ^{T}. Then (M,g) is a Ricci soliton (υ^{T},λ) if and only if it is a quasi-Einstein manifold with associated functions $(\lambda - c + \phi(\upsilon) - k)$, -1.*

Since the induced connection $\overset{\circ}{\widetilde{\nabla}}$ on a submanifold of a Riemannian manifold endowed with an $SSNMC$ is also an $SSNMC$, from (1.4), (3.1) and (3.4), we also have

$$\overset{\circ}{\widetilde{Ric}}(X_1,X_2) = (\lambda - c + \phi(\upsilon))\, g(X_1,X_2) - (n-1)\alpha(X_1,X_2) \tag{3.6}$$
$$-\widetilde{g}(\overset{\circ}{h}(X_1,X_2),\upsilon^{\perp}) - \frac{1}{2}\left\{\omega(X_1)g(X_2,\upsilon^{T}) + \omega(X_2)g(X_1,\upsilon^{T})\right\}$$

where $\overset{\circ}{\widetilde{Ric}}$ denotes the Ricci tensor of the induced $SSNMC$.

So we get the following corollary:

Corollary 3.3. *Let M be an n-dimensional submanifold isometrically immersed into a Riemannian manifold $\left(\widetilde{M}, \widetilde{g}\right)$ endowed with an $SSNMC$ and υ a $TF-PVF$ with respect to an $SSNMC$ on $\widetilde{M}$. Then (M,g) is a Ricci soliton (υ^{T},λ) with respect to the $SSNMC$ if and only if the condition (3.6) holds on M.*

If U is a parallel unit vector field with respect to the Levi-Civita connection ∇, then we get

$$\overset{\circ}{\widetilde{Ric}}(X_1,X_2) = (\lambda - c + \phi(\upsilon))\, g(X_1,X_2) + (n-1)\phi(X_1)\phi(X_2) \tag{3.7}$$
$$-\widetilde{g}(\overset{\circ}{h}(X_1,X_2),\upsilon^{\perp}) - \frac{1}{2}\left\{\omega(X_1)g(X_2,\upsilon^{T}) + \omega(X_2)g(X_1,\upsilon^{T})\right\}.$$

If M is $\upsilon^{\perp}$-umbilical, then by (3.2), we have

$$\overset{\circ}{A}_{\upsilon^{\perp}} X_1 = \left(k - \phi(\upsilon^{\perp})\right) X_1,$$

which brings us

$$\left(k - \phi(\upsilon^{\perp})\right) g\left(X_1,X_2\right) = g\left(\overset{\circ}{A}_{\upsilon^{\perp}} X_1, X_2\right) = \widetilde{g}(\overset{\circ}{h}(X_1,X_2),\upsilon^{\perp}).$$

Hence from (3.7), we have

$$\overset{\circ}{\widetilde{Ric}}(X_1,X_2) = (\lambda - c - k + \phi(\upsilon) + \phi(\upsilon^{\perp}))\ g(X_1,X_2) + (n-1)\phi(X_1)\phi(X_2)$$
$$-\frac{1}{2}\left\{\omega(X_1)g(X_2,\upsilon^{T}) + \omega(X_2)g(X_1,\upsilon^{T})\right\}.$$

So we get the following theorem:

Theorem 3.3. *Let M be a $\upsilon^{\perp}$-umbilical submanifold isometrically immersed into a Riemannian manifold $\left(\widetilde{M}, \widetilde{g}\right)$ endowed with an $SSNMC$ and υ a $TF-PVF$ with respect to an $SSNMC$ on $\widetilde{M}$. Assume that a 1-form ϕ is the g dual of υ^T. Then (M, g) is a Ricci soliton (υ^T, λ) if and only if it is a hyper-generalized quasi-Einstein manifold with respect to an $SSNMC$ with associate functions $\big(\lambda - c - k + \|\upsilon\|^2 + \|\upsilon^{\perp}\|^2\big)$, $(n-1)$, 0 and $-\frac{1}{2}$.*

When υ is a $C-PVF$ with respect to an $SSNMC$ on $\widetilde{M}$, we have the following theorem:

Theorem 3.4. *Let M be a $\upsilon^{\perp}$-umbilical submanifold isometrically immersed into a Riemannian manifold $\left(\widetilde{M}, \widetilde{g}\right)$ endowed with an $SSNMC$ and υ a $C-PVF$ with respect to an $SSNMC$ on $\widetilde{M}$. Assume that a 1-form ϕ is the g dual of υ^T. Then (M, g) is a Ricci soliton (υ^T, λ) if and only if it is a quasi-Einstein manifold with respect to an $SSNMC$ with associate functions $\left(\lambda - c - k + \|\upsilon\|^2 + \|\upsilon^{\perp}\|^2\right)$ and $(n-1)$.*

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