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ON CERTAIN SPACE CURVES DUE TO PARALLEL TRANSPORT FRAME IN E^n

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Abstract. In this paper, we study k-type ($k \in \{0, 1, 2, ..., n-1\}$) slant helices due to non-zero parallel transport frame in E^n. We give some characterizations for 0-type, 1-type,..., and in general $(n-1)$-type slant helix due to parallel transport frame in terms of parallel transport curvatures in E^n and with the aid of these characterizations we give an important general theorem which gives the necessary and sufficient condition for any space curve to be k-type slant helices due to parallel transport frame in E^n. We also obtain the characterizations of the curves whose position vectors belong to the normal, rectifying and osculating spaces (called normal, rectifying and osculating curves, respectively) due to parallel transport frame in E^n.
Keywords: parallel transport frame, k-type slant helix, normal curve, rectifying curve, osculating curve.

1. Introduction

In classical differential geometry, the study of space curves has been an area of central importance and attracted many differential geometers due to its importance

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and applications in many interdisciplinary and allied sciences. One of the prominent space curves is a general helix which is defined as a space curve whose tangent vector is inclined at a constant angle with a certain fixed (non-zero) direction. The general helix is characterized by the condition that the ratio of its curvature and torsion is always constant (Lancret theorem; [17]). In 2004, Izumiya and Takeuchi [14] introduced the notion of slant helices in E^3 as a special class of general helices. They called any space curve in E^3, a slant helix, whose principal normal vector observes a constant angle with a fixed (non-zero) direction. They showed that any space curve in E^3 is a slant helix if and only if the geodesic curvature of its principal normal is a constant function. On the other hand, Ali and Turgut [2] extended the study of slant helices to n-dimensional Euclidean space E^n and gave a characterization in the form of curvature integral equations. Also, Gök et al. generalized the idea of slant helices to k-type slant helices (or V_k-slant helices) and characterized them [10]. On the other hand, we refer to ([1, 3, 8, 12, 15, 16, 18, 19, 21, 22, 24], etc.) for more studies about slant helices in 3, 4 or higher dimensional different spaces.

It is well-known that the Frenet frame is constructed for any non-degenerate and 3-time continuously differentiable space curve. This has the possibility that the second derivative of the curve may vanish at some point, i.e. curvature may become zero at some point on it and hence we need an alternative frame to be defined on a space curve which may work in zero-curvature condition. In view of this, Bishop [6] devised a new frame called the parallel transport frame or Bishop frame which is well-defined even if the space curve has a vanishing 2nd derivative. In [23], Ünlütürk, Tozak and Ekici studied k-type slant helices due to a parallel transport frames (Bishop frame) and characterized them in 4-dimensional Euclidean space E^4. Motivated by these developments, our aim is to study k-type slant helices due to parallel transport frame in n-dimensional Euclidean space E^n and characterize them. We also obtain the characterizations of the curves whose position vectors belong to the normal, rectifying and osculating spaces (called normal, rectifying and osculating curves, respectively) in E^n.

2. Preliminaries

Analogous as for a space curve, for an arc-length parametrized curve $x : I \subset R \to E^n$, one can construct a Frenet frame $\{T(s), N_1(s), N_2(s), ...N_i(s), ..., N_{n-1}(s)\}$ that satisfies the equations

$$\left.\begin{array}{l} T'(s) = k_1(s)N_1(s), \\ N_1'(s) = -k_1(s)T(s) + k_2(s)N_2(s), \\ N_2'(s) = -k_2(s)N_1(s) + k_3(s)N_3(s), \\ . \\ N_i'(s) = -k_i(s)N_{i-1}(s) + k_{i+1}(s)N_{i+1}(s), \\ . \\ N_{n-1}'(s) = -k_{n-1}(s)N_{n-2}(s), \end{array}\right\} \quad (2.1)$$

where the functions $k_i(s)$, $i \in \{1, 2, 3, 4, ..., n-1\}$, denote the curvatures of the curve and all $k_i(s)$ are positive. If the curve x is not arc-length parametrized, then the right-hand sides of the equation (2.1) must be multiplied by the speed v of x [9].

Furthermore, the Bishop frame, also known as parallel transport, is an orthonormal frame formed by transporting each component in parallel. In this context, we suppose that x is any unit speed curve in E^n with the tangent vector $T(s)$, then one can choose any convenient arbitrary basis which consists of relatively parallel vector fields $\{M_1(s), M_2(s), ..., M_{n-1}(s)\}$ which are perpendicular to $T(s)$ at each point. The parallel transport frame equations are

$$\begin{bmatrix} T'(s) \\ M_1'(s) \\ M_2'(s) \\ \vdots \\ M_{n-1}'(s) \end{bmatrix} = \begin{bmatrix} 0 & -k_1(s) & -k_2(s) & \dots & -k_{n-1}(s) \\ k_1(s) & 0 & 0 & \dots & \dots \\ k_2(s) & 0 & 0 & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ k_{n-1}(s) & 0 & 0 & \dots & \dots \end{bmatrix} \begin{bmatrix} T(s) \\ M_1(s) \\ M_2(s) \\ \vdots \\ M_{n-1}(s) \end{bmatrix}, \tag{2.2}$$

where $k_i(s)$ are principal curvature functions according to parallel transport frame of the curve x and they are called parallel transport curvature functions (for more details, see [7, 20]).

Definition 2.1. The normal space, rectifying space and osculating space according to parallel transport frame $\{T(s), M_1(s), M_2(s), ..., M_{n-1}(s)\}$ in E^n are defined as

$$\begin{aligned} T^{\perp}(s) &= \{X \in E^n : \langle X, T(s)\rangle = 0\}, \\ M_1^{\perp}(s) &= \{X \in E^n : \langle X, M_1(s)\rangle = 0\}, \\ M_2^{\perp}(s) &= \{X \in E^n : \langle X, M_2(s)\rangle = 0\}, \end{aligned}$$

respectively. Also, normal curve, rectifying curve and osculating curve are defined as curves whose position vectors always lie in their normal space, rectifying space and osculating space, respectively. Hence, position vectors of the normal curve, rectifying curve and osculating curve according to the parallel transport frame in E^n satisfy the equations

$$x(s) = d_1(s)M_1(s) + d_2(s)M_2(s) + d_3(s)M_3(s) + ... + d_{n-1}(s)M_{n-1}(s), \tag{2.3}$$
$$x(s) = d_1(s)T(s) + d_2(s)M_2(s) + d_3(s)M_3(s) + ... + d_{n-1}(s)M_{n-1}(s), \tag{2.4}$$
$$x(s) = d_1(s)T(s) + d_2(s)M_1(s) + d_3(s)M_3(s) + ... + d_{n-1}(s)M_{n-1}(s), \tag{2.5}$$

respectively and here $d_1(s), d_2(s), ..., d_{n-1}(s)$ are differentiable functions (see [4, 5, 8, 11, 13]).

3. Characterization of k-type Slant Helices due to Parallel Transport Frame in E^n

In this section, we give characterizations for 0-type, 1-type,..., $(n-1)$-type slant helix due to parallel transport frame in terms of parallel transport curvatures in E^n

and with the aid of these characterizations we give an important general theorem which gives the necessary and sufficient condition for any space curve to be k-type ($k = 1, 2, ..., n-1$) slant helix due to parallel transport frame in E^n.

Definition 3.1. Let $\{T(s), M_1(s), M_2(s), ..., M_{n-1}(s)\}$ be a parallel transport frame of the curve $x = x(s)$ which is parametrized by arc-length. If there exists a non-zero constant vector field U in E^n such that $\langle M_k(s), U\rangle \neq 0$ is constant for all $s \in I$, where $M_0(s) = T(s)$ and $k \in \{0, 1, 2, ..., n-1\}$, then x is said to be k-type slant helix due to parallel transport frame and U is called the axis of x [23].

Now, let U be any non-zero constant vector in E^n. Then, due to the parallel transport frame, it can be represented as

$$U = c_o(s)T(s) + c_1(s)M_1(s) + c_2(s)M_2(s) + ... + c_{n-1}(s)M_{n-1}(s),$$

where $c_0(s), c_1(s), c_2(s), ..., c_{n-1}(s)$ are differentiable functions. Differentiating U, we get

$$\begin{gathered}(c_o'(s) + c_1(s)k_1(s) + c_2(s)k_2(s) + ... + c_{n-1}(s)k_{n-1}(s))T(s) + (-c_o(s)k_1(s) \\ + c_1'(s))M_1(s) + (-c_o(s)k_2(s) + c_2'(s))M_2(s) + ... + (-c_o(s)k_{n-1}(s) \\ + c_{n-1}'(s))M_{n-1}(s) = 0\end{gathered}$$

which implies that

$$\left.\begin{aligned} c_o'(s) + c_1(s)k_1(s) + c_2(s)k_2(s) + ... + c_{n-1}(s)k_{n-1}(s) = 0, \\ -c_o(s)k_1(s) + c_1'(s) = 0, \\ -c_o(s)k_2(s) + c_2'(s) = 0, \\ . \\ . \\ -c_o(s)k_{n-1}(s) + c_{n-1}'(s) = 0. \end{aligned}\right\} \tag{3.1}$$

Now, let us give a characterization for a 0-type slant helix due to parallel transport frame in terms of parallel transport curvatures in E^n.

Theorem 3.1. *Any smooth curve $x(s)$ in E^n is a 0-type slant helix due to parallel transport frame $\{T(s), M_1(s), M_2(s), ..., M_{n-1}(s)\}$ if and only if the equation*

$$k_1(s)\int k_1(s)ds + k_2(s)\int k_2(s)ds + ... + k_{n-1}(s)\int k_{n-1}(s)ds = 0 \tag{3.2}$$

holds.

Proof. Let $x(s)$ be a 0-type slant helix due to parallel transport frame in E^n. Then, we have

$$< T(s), U >= c_o \neq 0 \ (constant).$$

Combining the above equation with (3.1), we obtain

$$\left.\begin{aligned} c_1(s)k_1(s) + c_2(s)k_2(s) + ... + c_{n-1}(s)k_{n-1}(s) = 0,&\\ c_1(s) = c_o \int k_1(s)ds,&\\ c_2(s) = c_o \int k_2(s)ds,&\\ .&\\ .&\\ c_{n-1}(s) = c_o \int k_{n-1}(s)ds& \end{aligned}\right\}$$

and so

$$c_o \left\{ k_1(s) \int k_1(s)ds + k_2(s) \int k_2(s)ds + ... + k_{n-1}(s) \int k_{n-1}(s)ds \right\} = 0.$$

Since c_o is a non-zero constant, we have our assertion.

Conversely, let us assume that (3.2) holds. If we consider the axis that

$$\begin{aligned} (3.3) \quad U = \Bigg(c_oT(s) + \left(c_o \int k_1(s)ds \right) M_1(s) + \left(c_o \int k_2(s)ds \right) M_2(s) + ...\\ + \left(c_o \int k_{n-1}(s)ds \right) M_{n-1}(s) \Bigg), \end{aligned}$$

where c_0 is a non-zero constant, then by differentiating U and using (2.2) and (3.2), we have $U' = 0$. Thus, the proof is completed. □

The following result can be easily obtained from the Theorem 3.1.

Corollary 3.1. *Let $x(s)$ be a 0-type slant helix with non-zero parallel transport curvatures $k_1(s), k_2(s), \dots, k_{n-1}(s)$ due to parallel transport frame $\{T(s), M_1(s), M_2(s), \dots, M_{n-1}(s)\}$ in E^n. Then, the axis of $x(s)$ is given by (3.3).*

Here, let us characterize a 1-type slant helix due to parallel transport frame in terms of parallel transport curvatures in E^n.

Theorem 3.2. *Any smooth curve $x(s)$ in E^n is a 1-type slant helix due to parallel transport frame $\{T(s), M_1(s), M_2(s), ..., M_{n-1}(s)\}$ if and only if the function*

$$(3.4) \qquad -\frac{c_2k_2(s)}{k_1(s)} - \frac{c_3k_3(s)}{k_1(s)} - .... - \frac{c_{n-1}k_{n-1}(s)}{k_1(s)}$$

is a non-zero constant, where $c_2, c_3, ...c_{n-1}$ are constants.

Proof. Let $x(s)$ be a 1-type slant helix due to parallel transport frame $\{T(s), M_1(s), M_2(s), ..., M_{n-1}(s)\}$ in E^n. Then, we have $\langle M_1(s), U\rangle = c_1 \neq 0$ (constant). Therefore, from the second equation of (3.1), we have $c_o = 0$. Hence, we get the following equations:

$$\left.\begin{aligned} c_1k_1(s) + c_2(s)k_2(s) + ... + c_{n-1}(s)k_{n-1}(s) = 0,\\ c_2(s) = constant,\\ c_3(s) = constant,\\ .\\ .\\ c_{n-1}(s) = constant.\end{aligned}\right\} \tag{3.5}$$

In view of the assumption that c_1 is a non-zero constant, we can say that the function

$$-\frac{c_2k_2(s)}{k_1(s)} - \frac{c_3k_3(s)}{k_1(s)} - .... - \frac{c_{n-1}k_{n-1}(s)}{k_1(s)}$$

is constant, where $c_2, c_3, ...c_{n-1}$ are constants.

Conversely, let us assume that the function (3.4) is a non-zero constant and $c_2, c_3, ...c_{n-1}$ are constants. Then, we can find a fixed non-zero vector U which satisfies $< M_1(s), U >= c_1 \neq 0$ (constant) as follows:

$$U = \left(\left(-\frac{c_2k_2(s)}{k_1(s)} - \frac{c_3k_3(s)}{k_1(s)} - ..... - \frac{c_{n-1}k_{n-1}(s)}{k_1(s)}\right) M_1(s) + c_2M_2(s) + ... + c_{n-1}M_{n-1}(s)\right). \tag{3.6}$$

Differentiating the equation (3.6) and using (2.2), we get $U' = 0$ and this completes the proof. □

The following result can easily be observed from Theorem 3.2.

Corollary 3.2. *Let $x(s)$ be a 1-type slant helix with non-zero parallel transport curvatures $k_1(s), k_2(s), \dots, k_{n-1}(s)$ due to parallel transport frame $\{T(s), M_1(s), M_2(s), \dots, M_{n-1}(s)\}$ in E^n. Then, the axis of $x(s)$ is given by (3.6).*

Similarly, one can characterize 2-type slant helices, 3-type slant helices, etc.

Now, let us characterize in general $(n-1)$-type slant helix due to parallel transport frame $\{T(s), M_1(s), M_2(s), ..., M_{n-1}(s)\}$ in E^n.

Theorem 3.3. *Any smooth curve $x(s)$ in E^n is an $(n-1)$-type slant helix due to parallel transport frame $\{T(s), M_1(s), M_2(s), ..., M_{n-1}(s)\}$ if and only if the function*

$$-\frac{c_1k_1(s)}{k_{n-1}(s)} - \frac{c_2k_2(s)}{k_{n-1}(s)} - .... - \frac{c_{n-2}k_{n-2}(s)}{k_{n-1}(s)} \tag{3.7}$$

is a non-zero constant, where $c_1, c_2, ...c_{n-2}$ *are constants.*

Proof. If $x(s)$ is an $(n-1)$-type slant helix due to parallel transport frame $\{T(s), M_1(s), M_2(s), M_3(s), ..., M_{n-1}(s)\}$ in E^n, then we have $\langle M_{n-1}(s), U\rangle = c_{n-1} \neq 0$ (constant). Using this equality in the last equation of (3.1), we get $c_o = 0$. Hence, we have the following equations:

$$\left.\begin{aligned} c_1(s)k_1(s) + c_2(s)k_2(s) + ... + c_{n-1}k_{n-1}(s) = 0,\\ c_1(s) = constant,\\ c_2(s) = constant,\\ .\\ .\\ c_{n-2}(s) = constant. \end{aligned}\right\} \tag{3.8}$$

In view of the assumption that c_{n-1} is a non-zero constant, the first equation in the above set of equations implies that the function

$$-\frac{c_1k_1(s)}{k_{n-1}(s)} - \frac{c_2k_2(s)}{k_{n-1}(s)} - .... - \frac{c_{n-2}k_{n-2}(s)}{k_{n-1}(s)}$$

is constant, where $c_1, c_2, ..., c_{n-2}$ are constants.

Conversely, let us assume that the function (3.7) is a non-zero constant and $c_1, c_2, ..., c_{n-2}$ are constants. Then, we can find a fixed non-zero vector U which satisfies $< M_{n-1}(s), U >= c_{n-1} \neq 0$ (constant) as follows:

$$U = \Bigg(c_1M_1(s) + c_2M_2(s)...... + c_{n-2}M_{n-2}(s) + \left(-\frac{c_1k_1(s)}{k_{n-1}(s)} - \frac{c_2k_2(s)}{k_{n-1}(s)} - ..... - \frac{c_{n-2}k_{n-2}(s)}{k_{n-1}(s)}\right) M_{n-1}(s)\Bigg). \tag{3.9}$$

It is easy to prove that U is constant and this completes the proof. □

Corollary 3.3. *Let $x(s)$ be an $(n-1)$-type slant helix with non-zero parallel transport curvatures $k_1(s), k_2(s), \ldots, k_{n-1}(s)$ due to parallel transport frame $\{T(s), M_1(s), M_2(s), \ldots, M_{n-1}(s)\}$ in E^n. Then, the axis of $x(s)$ is given by (3.9).*

Hence, from Theorem 3.2 and Theorem 3.3, we can state the following theorem which gives the necessary and sufficient condition for any space curve to be i-type $(i = 1, 2, ..., n-1)$ slant helices due to parallel transport frame in E^n:

Theorem 3.4. *In general, any smooth curve $x(s)$ in E^n is an i-type slant helix due to parallel transport frame $\{T(s), M_1(s), M_2(s), ..., M_{n-1}(s)\}$ if and only if the function*

$$\mathcal{F}(k_1(s), k_2(s), ..., k_{n-1}(s)) = -\frac{1}{k_i(s)}\sum_{j\neq i} c_jk_j(s) \tag{3.10}$$

is non-zero constant, where c_j are constants and $i, j \in \{1, 2, ..., n-1\}$.

4. Normal, Rectifying and Osculating Curves due to Parallel Transport Frame in E^n

In this section, we characterize the normal, rectifying and osculating curves due to parallel transport frame in terms of parallel transport curvatures in E^n.

Theorem 4.1. *Any curve $x(s)$ is a normal curve according to due to parallel transport frame $\{T(s), M_1(s), M_2(s), ..., M_{n-1}(s)\}$ in E^n if and only if the equation*

$$\sum_{i=1}^{n-1} a_i k_i(s) = 1 \tag{4.1}$$

holds for constants a_i, $i \in \{1, 2, ..., n-1\}$.

Proof. Let $x(s)$ be a normal curve due to parallel transport frame $\{T(s), M_1(s), M_2(s), \ldots, M_{n-1}(s)\}$ in E^n. Then, the position vector of the curve $x(s)$ is given by (2.3). Now differentiating (2.3) with respect to 's' and using (2.2), we get

$$\begin{aligned} T(s) \quad &= (d_1(s)k_1(s) + d_2(s)k_2(s) + ... + d_{n-1}(s)k_{n-1}(s))T(s) \\ &+d_1'(s)M_1(s) + d_2'(s)M_2(s) + d_3'(s)M_3(s)... + d_{n-1}'(s)M_{n-1}(s) \end{aligned}$$

which gives

$$\left.\begin{aligned} d_1(s)k_1(s) + d_2(s)k_2(s) + ... + d_{n-1}(s)k_{n-1}(s) = 1, \\ d_1'(s) = 0, \\ d_2'(s) = 0, \\ . \\ . \\ d_{n-1}'(s) = 0. \end{aligned}\right\}$$

Combining the above set of equations gives our result (4.1), where $d_i(s) = a_i$ (constant), $i = 1, 2, ..., n-1$.

Conversely, let us suppose that the curvatures $k_1(s), k_2(s), ...k_{n-1}(s)$ satisfy the equation (4.1) for constants $a_1, a_2, ..., a_{n-1}$ and let us consider the vector $X \in E^n$ given by

$$X(s) = x(s) - (a_1 M_1(s) + a_2 M_2(s) + ... + a_{n-1} M_{n-1}(s)).$$

Differentiating $X(s)$ and using (2.2), we get

$$X'(s) = T(s) - (a_1 k_1(s) + a_2 k_2(s) + ... + a_{n-1} k_{n-1}(s))T(s).$$

Combining the above equation with (4.1), we get $X'(s) = 0$. Hence, $X(s)$ is a constant vector field and so, the curve $x(s)$ is a normal curve. $\square$

Theorem 4.2. *Any curve $x(s)$ is a rectifying curve due to parallel transport frame $\{T(s), M_1(s), M_2(s), ..., M_{n-1}(s)\}$ in E^n if and only if the equation*

$$\sum_{i=1}^{n-2} a_i k_{i+1}(s) = 1 \tag{4.2}$$

holds for constants a_i, $i \in \{1, 2, ..., n-2\}$.

Proof. Let $x(s)$ be a rectifying curve due to parallel transport frame $\{T(s), M_1(s), M_2(s), ..., M_{n-1}(s)\}$ in E^n. Then, the position vector of the curve $x(s)$ is given by (2.4). Now, differentiating (2.4), we get

$$\left.\begin{aligned} d_1'(s) + d_2(s)k_2(s) + d_3(s)k_3(s) + ... + d_{n-1}(s)k_{n-1}(s) = 1,\\ -d_1(s)k_1(s) = 0,\\ -d_1(s)k_2(s) + d_2'(s) = 0,\\ .\\ .\\ -d_1(s)k_{n-1}(s) + d_{n-1}'(s) = 0. \end{aligned}\right\}$$

Combining the above set of equations gives our result (4.2), where $d_i(s) = a_{i-1}$ (constant), $i = 2, 3, ..., n-1$.

Converse can be proved easily by taking the vector $X \in E^n$ as

$$X(s) = x(s) - (a_1 M_2(s) + a_2 M_3(s) + ... + a_{n-2} M_{n-1}(s)),$$

where $a_1, a_2, ..., a_{n-2}$ are constants. $\square$

Using a similar procedure to the proofs of Theorem 4.1 and Theorem 4.2, one can prove the following theorem which characterizes the osculating curve due to parallel transport frame $\{T(s), M_1(s), M_2(s), ..., M_{n-1}(s)\}$ in E^n.

Theorem 4.3. *Any curve $x(s)$ is an osculating curve due to parallel transport frame $\{T(s), M_1(s), M_2(s), ..., M_{n-1}(s)\}$ in E^n if and only if the equation*

$$a_1 k_1(s) + \sum_{i=2}^{n-2} a_i k_{i+1}(s) = 1 \tag{4.3}$$

holds for constants a_i, $i \in \{1, 2, ..., n-2\}$.

5. Conclusion

A general and comprehensive formulation of differentiation theory and parallel transport on manifolds is provided by the n-dimensional parallel transport frame. In particular, this is a crucial step in expanding the basic theoretical framework and comprehending complex differentiation processes in multidimensional spaces.

Additionally, it illustrates the usefulness of this theoretical framework by examining applications of the n-dimensional parallel transport frame in intricate physical theories like general relativity.

The goal of our research is to shed light on how this expanded frame in differential geometry might be applied to better comprehend physical phenomena and construct models. Therefore, we are encouraged to study the n-dimensional parallel transport frame with the goal of exploring useful applications of this extended frame in physical theories, improving the comprehension of parallel transport notions on manifolds, and generalizing differentiation theory.

Hence, in the present study, we study the k-type slant helices due to the parallel transport frame in n-dimensional Euclidean space E^n and obtain some characterizations of the curves whose position vectors belong to the normal, rectifying and osculating spaces due to parallel transport frame in E^n.

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REFERENCES

1. A. T. Ali, R. Lopez and M. Turgut: *k-type partially null and pseudo null slant helices in Minkowski 4-space*. Mathematical Communications **17**(1) (2012), 93–103.
2. A. T. Ali and M. Turgut: *Some characterizations of slant helices in the Euclidean space* E^n. Hacettepe Journal of Mathematics and Statistics **39**(3) (2010), 327–336.
3. F. Ateş, İ. Gök, F. N. Ekmekci and Y. Yayli: *Characterizations of Inclined Curves According to Parallel Transport Frame in* E^4 *and Bishop Frame in* E^3. Konuralp Journal of Mathematics **7**(1) (2019), 16–24.
4. Ö. Bektaş: *Normal curves in n-dimensional Euclidean space*. Advances in Difference Equations **456** (2018).
5. Ö. Bektaş and Z. Bekiryazici: *Generalized Osculating Curves of type* $(n-3)$ *in the n-dimensional Euclidean space*. Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. **71**(1) (2022), 212–225.
6. R. L. Bishop: *There is more than one way to frame a curve*. The American Mathematical Monthly **82**(3) (1975), 246–251.
7. S. Büyükkütük, İ. Kişi and G. Öztürk: *A characterization of curves according to parallel transport frame in Euclidean n-space* E^n. New Trends in Mathematical Sciences **5**(2) (2017), 61–68.
8. S. Cambie, W. Goemans and I. V. D. Bussche: *Rectifying curves in the n-dimensional Euclidean space*. Turkish Journal of Mathematics **40**(1) (2016), 210–223.
9. H. Gluck: *Higher curvatures of curves in Euclidean space*. The American Mathematical Monthly **73**(7) (1966), 699–704.

10. I. Gök, Ç. Camci and H. Hacisalihoğlu: *V_n-slant helices in Euclidean n-space E^n.* Mathematical Communications **14**(2), (2009), 317–329.
11. F. Gökçelik, Z. Bozkurt, İ. Gök, F. N. Ekmekci and Y. Yayli: *Parallel transport frame in 4-dimensional Euclidean space E^4*. Caspian Journal of Mathematical Sciences **3**(1) (2014), 91–103.
12. M. Grbović and E. Nešović: *On generalized Bishop frame of null Cartan curve in Minkowski 3-space.* Kragujevac Journal of Mathematics **43**(4) (2019), 559–573.
13. K. İlarslan, M. Sakaki and A. Uçum: *On Osculating, Normal and Rectifying Bi-Null Curves in R^5_2*. Novi Sad. J. Math. **48**(1) (2018), 9–20.
14. S. Izumiya and N. Takeuchi: *New special curves and developable surfaces.* Turkish Journal of Mathematics **28**(2) (2004), 153–164.
15. S. H. Khan, M. Jamali and M. Aslam: *Characterization of bi-null slant (BNS) helices of (k, m)-type in R^3_1 and R^5_2*. Facta Universitatis, Series: Mathematics and Informatics **38**(4) (2023), 731–740.
16. L. Kula, N. Ekmekçi, Y. Yayli and K. İlarslan: *Characterizations of slant helices in Euclidean 3-space.* Turkish Journal of Mathematics **34**(2), (2010), 261–274.
17. M. A. Lancret: *Mémoire sur les courbesa double courbure.* Memoires presentes alInstitut **1** (1806), 416–454.
18. G. Özturk, B. Bulca, B. Bayram and K. Arslan: *Focal Representation of k-slant Helices in E^{m+1}*. Acta Univ. Sapientiae, Mathematica **7**(2), (2015), 200–209.
19. J. Qian and Y. H. Kim: *Null helix and k-type null slant helices in E^4_1*. Rev. Un. Mat. Argentina **57**(1), (2016), 71–83.
20. C-L. Terng: *Lecture notes on curves and surfaces in R^3*. Preliminary Version and in Progress (2003).
21. M. Turgut and S. Yilmaz: *Some characterizations of type-3 slant helices in Minkowski space-time.* Involve, a Journal of Mathematics **2**(1), (2009), 115–120.
22. S. Uddin, M. S. Stanković, M. Iqbal, S. K. Yadav and M. Aslam: *Slant helices in Minkowski 3-space E^3_1 with Sasai's modified frame fields.* Filomat **36**(1) (2022), 151–164.
23. Y. Ünlütürk, H. Tozak and C. Ekici: *On k-Type Slant Helices Due to Bishop Frame in Euclidean 4-Space E^4*. International J. Math. Combin. **1** (2020), 1–9.
24. D. W. Yoon: *On the inclined curves in Galilean 4-Space.* Applied Mathematical Sciences **7**(44) (2013), 2193–2199.