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A STUDY OF THE HASIMOTO SURFACES CONSTRUCTED BY THE TANGENT SPHERICAL INDICATRICES IN EUCLIDEAN 3-SPACE

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Abstract. In this study, we investigate the vortex filament equation formed by the tangent spherical indicatrix of a unit-speed moving curve in Euclidean 3-space. For this purpose, the relations between the Frenet frames of the space curve and its tangent spherical indicatrix are considered. Then we introduce how the Hasimoto surfaces are constructed by the tangent spherical indicatrices and put forth some characterizations via related new findings. In the meantime, we present the first and second fundamental forms, Gaussian, and mean curvatures of this type of Hasimoto surface. Finally, we express some properties of the parameter curves of these Hasimoto surfaces with respect to the characters of both the tangent indicatrix and the original curve.
Keywords: Hasimoto surfaces, Frenet frames.

1. Introduction

In recent years, the theory of surfaces with the connection of the motion of space curves and differential equations has been a subject of research attention. At the same time, the applications of this subject in differential geometry and physics have attracted attention. Hasimoto examined the movement of a vortex filament in

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1971 and then showed that the vortex filament (smoke ring) equation is equivalent to the non-linear Schrödinger equation in 1972 [11, 12]. The relationship between the integrable equations and the theory of surfaces produces new types of surfaces. The Hasimoto surfaces are one of these surfaces. If the position vector of the vortex filament that can be viewed as a dynamical system on the space of curves is $\delta = \delta(s,t)$ which is a position vector of a unit speed moving curve δ for all t parameters in Euclidean 3-space E^3, then the curve velocity relation

$$\delta_t = \delta_s \wedge \delta_{ss} = \kappa B \tag{1.1}$$

holds, where the subscripts indicate the partial derivatives with respect to time parameter t and arc-length parameter s. Here κ and B denote the curvature and binormal vector of the curve δ, respectively, for each fixed time parameter t.

The geometric properties of the Hasimoto surfaces swept out by the binormal motion were investigated in detail by [1,19]. Furthermore, these surfaces were recreated using the Gauss-Weingarten equation from the fundamental forms. Numerous studies on the Hasimoto surfaces have been conducted using the various frames found in the Euclidean and Minkowski spaces [3–10, 14, 16]. The spherical indicatrix of a curve was redefined, and the $k-$slant curve and its' characterizations were investigated with the help of the spherical indicatrix [2]. In 3-dimensional Euclidean space, if the tangent lines of a curve make a fixed angle with a fixed line, this curve is called a cylindrical helix or general helix. It was expressed by Lancret in 1802, and it was proved in 1845 that "a necessary and sufficient condition to be a cylindrical helix of a curve is that the function

$$f = \frac{\tau}{\kappa}$$

is constant along the curve, where κ and τ are the curvature and torsion of the space curve" [20]. In 2004, Izumiya and Takeuchi defined the curves as slant helix if the normal lines of a curve make a fixed angle with a fixed line and also they gave the characterization of these slant curves. According to the characterization given by Izumiya and Takeuchi, a necessary and sufficient condition for a curve to be a slant helix with non-zero curvature is that the function

$$\sigma = \frac{\kappa^2}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\left(\frac{\tau}{\kappa}\right)'$$

is a constant [15]. Many researchers have been working on helices and spherical indicatrices in different spaces using different frames [13, 17, 18, 21, 22].
The aim of this study is to investigate the vortex filament equation formed by the tangent spherical indicatrix of a unit-speed moving curve. The connection between the binormal motion of the tangent spherical indicatrix and the Frenet element of the original curve is considered. With the help of the general equation forming the Hasimoto surfaces constructed by the tangent spherical indicatrices, the characterizations of their curvatures are obtained. This allows us to find the conditions for the parameter curves of these surfaces to be geodesics, asymptotic

curves, and lines of curvature with respect to the character of both the tangent indicatrix and the original curve. Finally, an example is presented by plotting Hasimoto surfaces for a given unit-speed curve and its tangent spherical indicatrix.

2. Preliminaries

Let δ be a unit speed space curve with arc-length s in Euclidean 3-space E^3. T, N, and B are tangent, principal normal, and binormal unit vectors at any point $\delta(s)$ of the curve δ, respectively. Therefore, the Frenet formulas for this curve δ are given as follows:

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix}_s = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

where κ and τ are the curvature and torsion of the curve δ, respectively. Now, let's define the tangent spherical indicatrices of the space curve δ in the Euclidean 3-space.

Definition 2.1. Let δ be a unit speed regular curve in the Euclidean 3-space with Frenet vectors T, N, and B. The unit tangent vectors emanating from the origin form a curve $T(s)$ on the unit sphere, and this curve is called the tangent spherical indicatrix of the curve δ or more commonly, the tangent indicatrix of the curve δ [17, 18, 20].

If $\delta = \delta(s)$ is a natural representation of the curve δ, then $\delta_T = T(s)$ is a representation of δ_T.

Let $\delta_T = T(s_T)$ be an arclength reparameterization of the tangent indicatrix of the unit speed regular curve δ, then Frenet frame $\{T_T, N_T, B_T\}$ satisfies the Frenet formulas:

$$\begin{bmatrix} T_T \\ N_T \\ B_T \end{bmatrix}_{s_T} = \begin{bmatrix} 0 & \kappa_T & 0 \\ -\kappa_T & 0 & \tau_T \\ 0 & -\tau_T & 0 \end{bmatrix} \begin{bmatrix} T_T \\ N_T \\ B_T \end{bmatrix}$$

such that

$$T_T = N, \quad N_T = \frac{-T+fB}{\sqrt{1+f^2}}, \quad B_T = \frac{fT+B}{\sqrt{1+f^2}},$$

and

$$s_T = \int \kappa(s)ds, \quad \kappa_T = \sqrt{1+f^2}, \quad \tau_T = \sigma\sqrt{1+f^2}$$

where

$$f=\frac{\tau(s)}{\kappa(s)} \quad \text{and} \quad \sigma=\frac{f'(s)}{\kappa(s)\left(1+f^2(s)\right)^{\frac{3}{2}}}.$$

Here, κ_T and τ_T are the curvature and torsion of the tangent indicatrix curve $\delta_T=T$ with the arc-length parameter s_T. Consequently, it is seen that $\sigma=\frac{\tau_T}{\kappa_T}$ [2].

3. Hasimoto surfaces constructed via the tangent spherical indicatrices

In this section, the Hasimoto surfaces obtained from tangent spherical indicator curves that provide the equation of a special velocity relation are investigated.

Definition 3.1. Let $\delta_T=T(s,t)$ be the tangent spherical indicatrix of a unit speed moving curve $\delta(s,t)$ for each fixed time parameter t. Then $\delta_T=T(s_T,t)$ generates a Hasimoto surface for all t by the motion satisfying the vortex filament equation

$$(\delta_T)_t=(\delta_T)_{s_T}\wedge(\delta_T)_{s_Ts_T}=\kappa_T(s_T)\,B_T(s_T)=\frac{\tau(s)\,T(s)+\kappa(s)\,B(s)}{\kappa(s)}$$

where $\{T,N,B,\kappa,\tau\}$ and $\{T_T,N_T,B_T,\kappa_T,\tau_T\}$ are the Frenet apparatus of $\delta(s,t)$ and $\delta_T(s_T,t)$ such that $s_T=\int\kappa(s)ds$.

Theorem 3.1. *Suppose $\delta_T=\delta_T(s_T,t)$ is a Hasimoto surface with tangent indicatrix such that $\delta_T=\delta_T(s_T,t)$ is a unit speed vector for each fixed t parameter in the Euclidean 3-space. The time evolution of Frenet frame $\{T_T,N_T,B_T\}$ of the tangent indicatrix curve δ_T satisfies*

$$\begin{bmatrix} T_T \\ N_T \\ B_T \end{bmatrix}_t=\begin{bmatrix} 0 & -\kappa_T\tau_T & (\kappa_T)_{s_T} \\ \kappa_T\tau_T & 0 & \frac{(\kappa_T)_{s_Ts_T}-\kappa_T{\tau_T}^2}{\kappa^T} \\ -(\kappa_T)_{s_T} & \frac{\kappa_T{\tau_T}^2-(\kappa_T)_{s_Ts_T}}{\kappa_T} & 0 \end{bmatrix}\begin{bmatrix} T_T \\ N_T \\ B_T \end{bmatrix}$$

where κ_T and τ_T are the curvature and the torsion of tangent indicatrix curve $\delta_T=T$ with the arc-length parameter s_T.

Proof. The time evolution of the Frenet frame $\{T_T,N_T,B_T\}$ of the tangent indicatrix curve δ_T can be written in matrix form as follows:

$$\begin{bmatrix} T_T \\ N_T \\ B_T \end{bmatrix}_t=\begin{bmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{bmatrix}\begin{bmatrix} T_T \\ N_T \\ B_T \end{bmatrix}$$

where α, β and γ are smooth functions. Under the compatibility conditions $(T_T)_{s_T t} = (T_T)_{t s_T}$, $(N_T)_{s_T t} = (N_T)_{t s_T}$ and $(B_T)_{s_T t} = (B_T)_{t s_T}$, we obtain

$$\begin{aligned} \alpha_{s_T} &= (\kappa_T)_t \alpha + \tau_T \beta, \\ \beta_{s_T} &= \kappa_T \gamma - \tau_T \alpha, \\ \gamma_{s_T} &= (\tau_T)_t - \kappa_T \beta. \end{aligned} \tag{3.1}$$

We suppose that the velocity of tangent indicatrix curve $\delta_T = T$ is given by

$$(\delta_T)_t = \frac{d\delta_T}{dt} = aT_T + bN_T + cB_T. \tag{3.2}$$

From the imposition of condition $(\delta_T)_{s_T t} = (\delta_T)_{t s_T}$, we find the following equations

$$\begin{aligned} 0 &= a_{s_T} - b\kappa_T, \\ \alpha &= a\kappa_T + b_{s_T} - c\tau_T, \\ \beta &= b\tau_T + c_{s_T} \end{aligned} \tag{3.3}$$

where a, b, and c are functions of the parameters s_T and t correspond to the coefficients of the tangent, principal normal, and binormal vectors of the velocity of tangent indicatrix curve $\delta_T = T$, respectively. Substituting the equations (3.3) into the second equation of (3.1), we get

$$\gamma = \frac{1}{\kappa_T} \left((b\tau_T + c_{s_T})_{s_T} + \tau_T (a\kappa_T + b_{s_T} - c\tau_T) \right). \tag{3.4}$$

For a solution of the equation of the vortex filament, the velocity vector of tangent indicatrix curve $\delta_T = T$ is given by

$$(\delta_T)_t = (\delta_T)_{s_T} \wedge (\delta_T)_{s_T s_T} = \kappa_T B_T. \tag{3.5}$$

Thus, from the equations (3.2) and (3.5), we get

$$a = 0, b = 0, c = \kappa_T. \tag{3.6}$$

Substituting the equations (3.6) into the equations (3.3) and (3.4), we get

$$\begin{aligned} \alpha &= -\kappa_T \tau_T, \\ \beta &= (\kappa_T)_{s_T}, \\ \gamma &= \frac{(\kappa_T)_{s_T s_T} - \kappa_T {\tau_T}^2}{\kappa_T}. \end{aligned}$$

Thus, the evolution equation of the curvature and the torsion of tangent indicatrix δ_T are

$$\begin{aligned} (\kappa_T)_t &= -2(\kappa_T)_{s_T} - \kappa_T (\tau_T)_{s_T}, \\ (\tau_T)_t &= \kappa_T (\kappa_T)_{s_T} - 2\tau_T (\tau_T)_{s_T} + \left(\frac{(\kappa_T)_{s_T s_T}}{\kappa_T} \right)_{s_T}. \end{aligned}$$

□

Theorem 3.2. *Let $\delta_T = \delta_T(s_T, t)$ be a Hasimoto surface such that $\delta_T = \delta_T(s_T, t)$ is a unit speed vector for all t parameter in the Euclidean 3-space. The Gaussian and the mean curvatures of the Hasimoto surface $\delta_T = \delta_T(s_T, t)$ are*

$$K_T = \frac{-(\kappa_T)_{s_T s_T}}{\kappa_T},$$
$$H_T = \frac{1}{2\kappa_T}\left(\frac{(\kappa_T)_{s_T s_T}}{\kappa_T} - {\kappa_T}^2 - {\tau_T}^2\right),$$

respectively, where $\kappa_T \neq 0$.

Proof. The tangent vectors of the Hasimoto surface $\delta_T = \delta_T(s_T, t)$ are $(\delta_T)_{s_T} = T_T$ and $(\delta_T)_t = \kappa_T B_T$. The normal vector of the Hasimoto surface $\delta_T = \delta_T(s_T, t)$ is

$$n_T = \frac{(\delta_T)_{s_T} \wedge (\delta_T)_t}{\left\|(\delta_T)_{s_T} \wedge (\delta_T)_t\right\|} = \frac{T_T \wedge B_T}{\|T_T \wedge B_T\|} = -N_T. \tag{3.7}$$

The coefficients of the first fundamental form of the Hasimoto surface $\delta_T = \delta_T(s_T, t)$ are

$$\begin{aligned} E &= \left\langle (\delta_T)_{s_T}, (\delta_T)_{s_T} \right\rangle = 1, \\ F &= \left\langle (\delta_T)_{s_T}, (\delta_T)_t \right\rangle = 0, \\ G &= \langle (\delta_T)_t, (\delta_T)_t \rangle = {\kappa_T}^2. \end{aligned} \tag{3.8}$$

The second-order partial derivatives of $\delta_T(s_T, t)$ according to the parameters s_T and t are

$$\begin{aligned} (\delta_T)_{s_T s_T} &= \kappa_T N_T, \\ (\delta_T)_{s_T t} &= -\kappa_T \tau_T N_T + (\kappa_T)_{s_T} B_T, \\ (\delta_T)_{tt} &= -\kappa_T (\kappa_T)_{s_T} T_T \\ &+ \left(\kappa_T {\tau_T}^2 - (\kappa_T)_{s_T s_T}\right) N_T + (\kappa_T)_t B_T. \end{aligned} \tag{3.9}$$

The coefficients of the second fundamental form of the Hasimoto surface $\delta_T = \delta_T(s_T, t)$ are

$$\begin{aligned} K &= \left\langle (\delta_T)_{s_T s_T}, n_T \right\rangle = -{\kappa_T}^2, \\ L &= \left\langle (\delta_T)_{s_T t}, n_T \right\rangle = \kappa_T \tau_T, \\ M &= \langle (\delta_T)_{tt}, n_T \rangle = (\kappa_T)_{s_T s_T} - \kappa_T {\tau_T}^2. \end{aligned} \tag{3.10}$$

So, from the equations (3.8) and (3.10), the Gaussian and mean curvatures of the Hasimoto surface $\delta_T = \delta_T(s_T, t)$ are

$$K_T = \frac{KM - L^2}{EG - F^2} = \frac{-(\kappa_T)_{s_T s_T}}{\kappa_T}$$

and

$$H_T = \frac{1}{2}\frac{EM - 2FL + GK}{EG - F^2} = \frac{1}{2\kappa_T}\left(\frac{(\kappa_T)_{s_T s_T}}{\kappa_T} - {\kappa_T}^2 - {\tau_T}^2\right).$$

□

From Theorem 3.2, one can see that the Hasimoto surface $\delta_T = \delta_T(s_T, t)$ is developable if and only if $(\kappa_T)_{s_T}$ is constant for each fixed parameter t. As a result, the Hasimoto surface $\delta_T = \delta_T(s_T, t)$ is a developable surface if and only if the curve $\delta(s,t)$ provides the condition $\frac{\tau\left(\tau'\kappa-\tau\kappa'\right)}{\kappa^3\sqrt{\kappa^2+\tau^2}}$ to be constant for each fixed parameter t. So the following corollary is obvious.

Corollary 3.1. *If a curve $\delta(s,t)$ is helix for each fixed parameter t, then the Hasimoto surface $\delta_T = \delta_T(s_T, t)$ constructed with the tangent spherical indicatrix of $\delta(s,t)$ is a developable surface.*

Corollary 3.2. *The Hasimoto surface $\delta_T = \delta_T(s_T, t)$ is a minimal surface if and only if $(\kappa_T)_{s_T s_T} = {\kappa_T}^3 + \kappa_T {\tau_T}^2$.*

Now, let us underline the geometric interpretation of the parametric curves of the Hasimoto surfaces and give some related theorems.

Theorem 3.3. *Let $\delta_T = \delta_T(s_T, t)$ be a Hasimoto surface, then the following statements are satisfied:*
i. The s_T-parameter curves of the Hasimoto surface $\delta_T = \delta_T(s_T, t)$ are geodesics,
ii. The $t-$parameter curves of the Hasimoto surface $\delta_T = \delta_T(s_T, t)$ are geodesics iff is κ_T constant.

Proof. In order for the parameter curves to be geodesic curves, the acceleration vector of the curves must be perpendicular to the surface and, therefore, parallel to the normal vector of the surface.
i. From the equations (3.7) and (3.9), we get

$$n_T \wedge (\delta_T)_{s_T s_T} = -N_T \wedge \kappa_T N_T = 0.$$

In that case, s_T-parameter curves of the Hasimoto surface $\delta_T = \delta_T(s_T, t)$ are geodesics.
ii. From the equations (3.7) and (3.9), we get

$$\begin{aligned} n_T \wedge (\delta_T)_{tt} &= -N_T \wedge \left(-\kappa_T(\kappa_T)_{s_T} T_T + \left(\kappa_T {\tau_T}^2 - (\kappa_T)_{s_T s_T}\right) N_T + (\kappa_T)_t B_T\right) \\ &= -(\kappa_T)_t T_T - \kappa_T(\kappa_T)_{s_T} B_T. \end{aligned}$$

Since T_T and B_T are linearly independent, there are $(\kappa_T)_t = 0$ and $\kappa_T(\kappa_T)_{s_T} = 0$. In that case, $n_T \wedge (\delta_T)_{tt} = 0$ iff κ_T is constant. Thus, we call that $t-$parameter curves of the Hasimoto surface $\delta_T = \delta_T(s_T, t)$ are geodesic curves. □

By considering Theorem 3.3 (ii) and the equality $\kappa_T = \sqrt{1+f^2} = \sqrt{1+\left(\frac{\tau}{\kappa}\right)^2}$, we can give the following corollary:

Corollary 3.3. *If $\delta(s,t)$ is a helix for each fixed parameter t, then $t-$parameter curves of the Hasimoto surface $\delta_T = \delta_T(s_T, t)$ constructed with the tangent spherical indicatrix of $\delta(s,t)$ are geodesic curves.*

Theorem 3.4. *Let $\delta_T = \delta_T(s_T, t)$ be a Hasimoto surface, then the following statements are satisfied:*
i. The s_T-parameter curves of the Hasimoto surface are not asymptotic,
ii. The $t-$parameter curves of the Hasimoto surface are asymptotic if and only if $(\kappa_T)_{s_T s_T} = \kappa_T {\tau_T}^2$.

Proof. In order for the parameter curves on the surface to be asymptotic curves, the normal curvature of the parameter curves must be zero everywhere. Therefore, $\left\langle (\delta_T)_{s_T s_T}, n_T \right\rangle = 0$ and $\langle (\delta_T)_{tt}, n_T \rangle = 0$ must be provided for the s_T and $t-$parameter curves.
i. From the equations (3.7) and (3.9), we know that

$$\left\langle n_T, (\delta_T)_{s_T s_T} \right\rangle = \langle -N_T, \kappa_T N_T \rangle = -\kappa_T.$$

Since $\kappa_T \neq 0$, s_T-parameter curves of the Hasimoto surfaces $\delta_T = \delta_T(s_T, t)$ are not asymptotic.
ii. From the equations (3.7) and (3.9), we get

$$\begin{aligned} \langle n_T, (\delta_T)_{tt} \rangle &= \left\langle -N_T, -\kappa_T (\kappa_T)_{s_T} T_T + \left(\kappa_T {\tau_T}^2 - (\kappa_T)_{s_T s_T} \right) N_T + (\kappa_T)_t B_T \right\rangle \\ &= (\kappa_T)_{s_T s_T} - \kappa_T {\tau_T}^2. \end{aligned}$$

From this equation, there is $\langle n_T, (\delta_T)_{tt} \rangle = 0$ if and only if $(\kappa_T)_{s_T s_T} = \kappa_T {\tau_T}^2$. In that case $t-$parameter curves of the Hasimoto surface $\delta_T = \delta_T(s_T, t)$ are asymptotic curves iff $(\kappa_T)_{s_T s_T} = \kappa_T {\tau_T}^2$. □

Theorem 3.5. *Let $\delta_T = \delta_T(s_T, t)$ be a Hasimoto surface. The s_T and $t-$parameter curves of the Hasimoto surfaces $\delta_T = \delta_T(s_T, t)$ are lines of curvature under the necessary and sufficient condition of $\tau_T = 0$.*

Proof. The parameter curves of a surface are lines of curvature, provided that $F = L = 0$. Thus, from the equations (3.8) and (3.10), we get $F = 0$ and $L = \kappa_T \tau_T$. Thus, the s_T and $t-$parameter curves of the Hasimoto surface are lines of curvature if $\tau_T = 0$. □

From Theorem 3.5 and the equality $\tau_T = \frac{\tau' \kappa - \tau \kappa'}{\kappa(\kappa^2 + \tau^2)}$, the following corollary is obvious.

Corollary 3.4. *A curve $\delta(s, t)$ is a slant helix for each fixed parameter t if and only if the s_T and $t-$parameter curves of the Hasimoto surface $\delta_T = \delta_T(s_T, t)$ constructed with the tangent spherical indicatrix of $\delta(s, t)$ are lines of curvature.*

Example 3.1. Let $\delta(s)$ be a curve in the Euclidean 3-space with arc-length parameter s given by the parameter equation

$$\delta(s) = \left(\tfrac{7}{25} \cos(s), \tfrac{32}{25} - \sin(s), -\tfrac{24}{25} \cos(s) \right).$$

The tangent vector and the curvature of $\delta(s)$ are

$$T(s) = \left(-\frac{7}{25}\sin s, -\cos s, \frac{24}{25}\sin s\right), \quad \kappa(s) = 1$$

respectively. Obviously, the arc-length reparameterization of the tangent spherical indicatrix of this curve $\delta(s)$ is $\delta_T(s_T) = T(s)$ by the arc-length function

$$s_T = \int_0^s \kappa(u)\, du = s.$$

The Frenet frame $\{T_T, N_T, B_T\}$ and the curvature of the tangent spherical indicatrix $\delta_T = T(s)$ are found as follows:

$$\begin{aligned} T_T(s) &= \left(-\tfrac{7}{25}\cos s, \sin s, \tfrac{24}{25}\cos s\right), \\ N_T(s) &= \left(\tfrac{7}{25}\sin s, \cos s, -\tfrac{24}{25}\sin s\right), \\ B_T(s) &= \left(-\tfrac{24}{25}, 0, -\tfrac{7}{25}\right), \\ \kappa_T(s) &= 1. \end{aligned}$$

If we consider the variable t as a time parameter together with the arc length parameter s, the curve evolves or varies with time, and we get the families of curves $\delta(s,t)$ and $\delta_T(s,t)$. The motion of these curve evolutions satisfies the following equations

$$(\delta)_t = (\delta)_s \wedge (\delta)_{ss} = \kappa B = \left(-\frac{24}{25}, 0, -\frac{7}{25}\right)$$

and

$$(\delta_T)_t = (\delta_T)_s \wedge (\delta_T)_{ss} = \kappa_T B_T = \left(-\frac{24}{25}, 0, -\frac{7}{25}\right)$$

generates the Hasimoto surfaces. For example,

$$\delta(s,t) = \frac{1}{25}\left(24t + 7\cos s, 32 - 25\sin s, -7t - 24\cos s\right)$$

and

$$\delta_T(s,t) = \frac{1}{25}\left(-24t - 7\sin s, -25\cos s, -7t + 24\sin s\right)$$

are the Hasimoto surfaces constructed via the evolution curve $\delta(s,t)$ and its the tangent spherical indicatrix $\delta_T(s,t)$, respectively, see Figure 3.1.

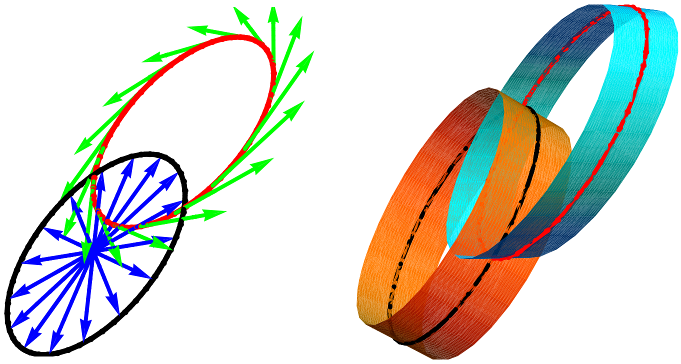

Fig. 3.1: The tangent spherical indicatrix $\delta_T(s,0)$ (black) of the curve $\delta(s,0)$ (red) with its tangent vectors T (green). The position vectors of T (blue). The Hasimoto surfaces $\delta(s,t)$ (cyan) and $\delta_T(s,t)$ (orange) where $s \in [0, 2\pi]$ and $t \in [-0.3, 0.3]$.

4. Conclusion

In this study, we investigated the Hasimoto surfaces satisfying the vortex filament equation formed by the tangent spherical indicatrices of a unit-speed moving curve for all time parameters. We presented the first and second fundamental forms, as well as the Gaussian and mean curvatures of these Hasimoto surfaces. Additionally, we provided the conditions for their parameter curves to be geodesics, asymptotic curves, and lines of curvature with respect to the character of both the tangent indicatrix and the original curve for each fixed time parameter. Finally, an example of the Hasimoto surfaces constructed via the tangent spherical indicatrix and the original curve was given, and their graphs were illustrated.

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