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CENTRAL INDEX ORIENTED SOME GENERALIZED GROWTH ANALYSIS OF COMPOSITE ENTIRE FUNCTIONS

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Abstract. In this paper, we have discussed some different growth properties of composite entire functions on the basis of their central index using the concepts of $(p,q,t)L$-th order and $(p,q,t)L$-th type.
Keywords: entire function, central index, growth analysis.

1. Introduction, Definition and Notation

Let $f(z)$ be an entire function defined in the open complex plane $\mathbb{C}$. For entire function $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ on $|z| = r$, the maximum modulus symbolized as $M_f(r)$, the maximum term denoted as $\mu_f(r)$ and the central index indicated as $\nu_f(r)$ are respectively defined as $\max\limits_{|z|=r}|f(z)|$, $\max\limits_{n\geq 0}(|a_n|r^n)$ and $\max\{m : \mu_f(r) = |a_m|r^m\}$. Therefore, the central index $\nu_f(r)$ of an entire function $f(z)$ is the greatest exponent m such that $|a_m|r^m = \mu_f(r)$. Obviously, $M_f(r)$, $\mu_f(r)$ and $\nu_f(r)$ are real and increasing functions of r. For another entire function $g(z)$, $M_g(r)$ and $\mu_g(r)$ are

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also defined and the ratios $\frac{M_f(r)}{M_g(r)}$ when $r \to +\infty$ as well as $\frac{\mu_f(r)}{\mu_g(r)}$ as $r \to +\infty$ are called the comparative growth of $f(z)$ with respect to $g(z)$ in terms of their maximum moduli and the maximum terms respectively. The prime object of the study of the growth investigation of entire function has usually been done through their maximum modulus and maximum term. Though $\nu_f(r)$ is much weaker than $M_f(r)$ and $\mu_f(r)$ in some sense, from another angle of view $\frac{\nu_f(r)}{\nu_g(r)}$ as $r \to +\infty$ is also called the growth of $f(z)$ with respect to $g(z)$ where $\nu_g(r)$ denotes the central index of entire function $g(z)$. The iterations of the exponential and logarithmic functions are defined as $\exp^{[k]} x = \exp\left(\exp^{[k-1]} x\right)$ and $\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$, with convention that $\exp^{[0]} x = x$, $\exp^{[-1]} x = \log x$, $\log^{[0]} x = x$ and $\log^{[-1]} x = \exp x$, where $x \in [0, \infty)$ and $k \in \mathbb{N}$, the set of all positive integers . Further, we assume that throughout the present paper l, p, q, m and n always denote positive integers and $t \in \mathbb{N} \cup \{-1, 0\}$.

To start the paper, we first recall the following definitions:

Definition 1.1. The order $\rho(f)$ and the lower order $\lambda(f)$ of an entire function $f(z)$ are defined as:

$$\rho(f) = \limsup_{r\to+\infty} \frac{\log\log M_f(r)}{\log r} \text{ and } \lambda(f) = \liminf_{r\to+\infty} \frac{\log\log M_f(r)}{\log r}.$$

Later, He et al. [7] gave the alternative definitions of order and lower order of an entire function $f(z)$ in terms of its central index in the following way:

$$\rho(f) = \limsup_{r\to+\infty} \frac{\log \nu_f(r)}{\log r} \text{ and } \lambda(f) = \liminf_{r\to+\infty} \frac{\log \nu_f(r)}{\log r}.$$

On the other hand, Shen et al. [13] defined the (m,n)-φ order and (m,n)-φ lower order of entire function $f(z)$, which are as follows:

Definition 1.2. [13] Let $\varphi : [0,+\infty) \to (0,+\infty)$ be a non-decreasing unbounded function and $m \geq n$. The (m,n)-φ order $\rho^{(m,n)}(f,\varphi)$ and (m,n)-φ lower order $\lambda^{(m,n)}(f,\varphi)$ of entire function $f(z)$ are defined as:

$$\begin{aligned} \rho^{(m,n)}(f,\varphi) &= \limsup_{r\to+\infty} \frac{\log^{[m+1]} M_f(r)}{\log^{[n]} \varphi(r)} \\ \text{and } \lambda^{(m,n)}(f,\varphi) &= \liminf_{r\to+\infty} \frac{\log^{[m+1]} M_f(r)}{\log^{[n]} \varphi(r)}. \end{aligned}$$

If we take $m = p$, $n = 1$ and $\varphi(r) = \log^{[q-1]} r \cdot \exp^{[t+1]} L(r)$, where $L \equiv L(r)$ is a positive continuous function increasing slowly i.e., $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant 'a' i.e., $\lim\limits_{r\to+\infty} \frac{L(ar)}{L(r)} = 1$, then Definition 1.2 turns into the definitions of $(p,q,t)L$-th order and $(p,q,t)L$-th lower order of an entire function $f(z)$ (for details, see [3]) which are as follows:

$$\rho^{(p,q,t)L}(f) = \limsup_{r\to+\infty} \frac{\log^{[p+1]} M_f(r)}{\log^{[q]} r + \exp^{[t]} L(r)}$$

$$\text{and } \lambda^{(p,q,t)L}(f) = \liminf_{r\to+\infty} \frac{\log^{[p+1]} M_f(r)}{\log^{[q]} r + \exp^{[t]} L(r)}.$$

Further, Shen et al. [13] also established the equivalence of the definitions of (m,n)-φ order of entire function in terms of maximum modulus and central index under some conditions. For details about it, one may see [13]. In view of Lemma 3.4 of [13] and Definition 1.2, one may write the following Definition.

Definition 1.3. [4] Let $f(z)$ be an entire function and $\nu_f(r)$ be the central index of $f(z)$, then

$$\rho^{(p,q,t)L}(f) = \limsup_{r\to+\infty} \frac{\log^{[p]} \nu_f(r)}{\log^{[q]} r + \exp^{[t]} L(r)}$$

and

$$\lambda^{(p,q,t)L}(f) = \liminf_{r\to+\infty} \frac{\log^{[p]} \nu_f(r)}{\log^{[q]} r + \exp^{[t]} L(r)}.$$

In order to compare the relative growth of two entire functions having same non-zero finite $(p,q,t)L$-th order, one may introduce the definitions of $(p,q,t)L$-th type (respectively $(p,q,t)L$-th lower type) of entire functions having finite positive $(p,q,t)L$-th order in the following manner:

Definition 1.4. [3] Let f be an entire function with non-zero finite $(p,q,t)L$-th order $\rho^{(p,q,t)L}(f)$. The $(p,q,t)L$-th type denoted by $\sigma^{(p,q,t)L}(f)$ and $(p,q,t)L$-th lower type denoted by $\overline{\sigma}^{(p,q,t)L}(f)$ are respectively defined as follows:

$$\sigma^{(p,q,t)L}(f) = \limsup_{r\to+\infty} \frac{\log^{[p-1]} M_f(r)}{[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)]^{\rho^{(p,q,t)L}(f)}}$$

and

$$\overline{\sigma}^{(p,q,t)L}(f) = \liminf_{r\to+\infty} \frac{\log^{[p-1]} M_f(r)}{[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)]^{\rho^{(p,q,t)L}(f)}}.$$

Analogously in order to determine the relative growth of two entire functions having same non-zero finite $(p,q,t)L$-th lower order, one may introduce the definition of $(p,q,t)L$-th weak type of entire function having finite positive $(p,q,t)L$-th lower order in the following way:

Definition 1.5. [3] The $(p,q,t)L$-th weak type denoted by $\tau^{(p,q,t)L}(f)$ of an entire function $f(z)$ is defined as follows:

$$\tau^{(p,q,t)L}(f) = \liminf_{r\to+\infty} \frac{\log^{[p-1]} M_f(r)}{[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)]^{\lambda^{(p,q,t)L}(f)}},\ 0 < \ \lambda^{(p,q,t)L}(f) < +\infty.$$

Also one may define the growth indicator $\overline{\tau}^{(p,q,t)L}(f)$ of an entire function $f(z)$ in the following manner:

$$\overline{\tau}^{(p,q,t)L}(f) = \limsup_{r\to\infty} \frac{\log^{[p-1]} M_f(r)}{[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)]^{\lambda^{(p,q,t)L}(f)}},\ 0 < \ \lambda^{(p,q,t)L}(f) < +\infty.$$

Considering the above, here in this present paper, we attempt to prove some results related to the growth rates of composite entire functions on the basis of the central index using the ideas of $(p,q,t)L$-th order and $(p,q,t)L$-th type of an entire function. In fact, some works in this field using central index have been already explored in [1, 2, 4, 10, 11]. We have used the standard notations using the theory of entire functions which are available in [9].

2. Lemmas

In this section, we present some lemmas which will be needed in the sequel.

Lemma 2.1. *[5] Let $f(z)$ and $g(z)$ be any two entire functions with $g(0) = 0$. Also let β satisfy $0 < \beta < 1$ and $c(\beta) = \frac{(1-\beta)^2}{4\beta}$. Then for all sufficiently large values of r,*

$$M_f\left(c(\beta)\, M_g(\beta r)\right) \le M_{f\circ g}(r) \le M_f\left(M_g(r)\right).$$

Lemma 2.2. *([[7], Theorems 1.9 and 1.10, or [8], Satz 4.3 and 4.4]) Let $f(z)$ be any entire function, then*

$$\log \mu_f(r) = \log |a_0| + \int_0^r \frac{\nu_f(t)}{t} dt \ \textit{where}\ a_0 \ne 0,$$

and for $r < R$,

$$M_f(r) < \mu_f(r) \left\{ \nu_f(R) + \frac{R}{R-r} \right\}.$$

3. Main results

Theorem 3.1. *Let $f(z)$ and $g(z)$ be any two entire functions such that $0 < \overline{\sigma}^{(p,q,t)L}(f\circ g) \le \sigma^{(p,q,t)L}(f\circ g) < +\infty$, $0 < \overline{\sigma}^{(p,q,t)L}(f) \le \sigma^{(p,q,t)L}(f) < +\infty$,*

$\rho^{(p,q,t)L}(f\circ g) = \rho^{(p,q,t)L}(f)$ and $\exp^{[t+1]} L(ar) \sim \exp^{[t+1]} L(r)$ as $r \to +\infty$ for every positive constant 'a'. If $\log^{[2]}\left(\frac{r}{2}\right) = o\left(\prod\limits_{l=1}^{p-3} \log^{[l]} \nu_{f\circ g}(r)\right)$ as $r \to +\infty$, then

$$\begin{aligned}
&\frac{\overline{\sigma}^{(p,q,t)L}(f\circ g)}{\sigma^{(p,q,t)L}(f)} \leq \liminf_{r\to+\infty} \frac{\log^{[p-2]} \nu_{f\circ g}(r)}{\log^{[p-2]} \nu_f(r)} \\
&\leq \min\left\{\frac{\overline{\sigma}^{(p,q,t)L}(f\circ g)}{\overline{\sigma}^{(p,q,t)L}(f)}, \frac{\sigma^{(p,q,t)L}(f\circ g)}{\sigma^{(p,q,t)L}(f)}\right\} \leq \max\left\{\frac{\overline{\sigma}^{(p,q,t)L}(f\circ g)}{\overline{\sigma}^{(p,q,t)L}(f)}, \frac{\sigma^{(p,q,t)L}(f\circ g)}{\sigma^{(p,q,t)L}(f)}\right\} \\
&\qquad \leq \limsup_{r\to+\infty} \frac{\log^{[p-2]} \nu_{f\circ g}(r)}{\log^{[p-2]} \nu_f(r)} \leq \frac{\sigma^{(p,q,t)L}(f\circ g)}{\overline{\sigma}^{(p,q,t)L}(f)}.
\end{aligned}$$

Proof. For any constant E, one may get from the Lemma 2.2, that

$$\log M_{f\circ g}(r) < \nu_{f\circ g}(r) \log r + \log \nu_{f\circ g}(2r) + E \ \{cf.[6]\}.$$

Therefore from above we obtain that

$$\begin{aligned}
\log M_{f\circ g}(r) &< \nu_{f\circ g}(2r) \log r + \nu_{f\circ g}(2r) + E \\
i.e.,\ \log M_{f\circ g}(r) &< \nu_{f\circ g}(2r)(1 + \log r) + E \\
i.e.,\ \log M_{f\circ g}(r) &< \nu_{f\circ g}(2r) \log(e\cdot r) + E
\end{aligned} \tag{3.1}$$

$$i.e.,\ \log M_{f\circ g}\left(\frac{r}{2}\right) < \nu_{f\circ g}(r) \log\left(e\cdot\frac{r}{2}\right) + E$$

$$i.e.,\ \log^{[2]} M_{f\circ g}\left(\frac{r}{2}\right) < \log \nu_{f\circ g}(r) + \log^{[2]}\left(\frac{r}{2}\right) + O(1)$$

$$i.e.,\ \log^{[2]} M_{f\circ g}\left(\frac{r}{2}\right) < \log \nu_{f\circ g}(r)\left(1 + \frac{\log^{[2]}\left(\frac{r}{2}\right) + O(1)}{\log \nu_{f\circ g}(r)}\right)$$

$$i.e.,\ \log^{[3]} M_{f\circ g}\left(\frac{r}{2}\right) < \log^{[2]} \nu_{f\circ g}(r) + \log\left(1 + \frac{\log^{[2]}\left(\frac{r}{2}\right) + O(1)}{\log \nu_{f\circ g}(r)}\right)$$

Taking $\log\left(1 + \frac{\log^{[2]}\left(\frac{r}{2}\right)+O(1)}{\log \nu_{f\circ g}(r)}\right) \leq \frac{\log^{[2]}\left(\frac{r}{2}\right)+O(1)}{\log \nu_{f\circ g}(r)}$, we get for sufficiently large values of r,

$$\begin{aligned}
\log^{[3]} M_{f\circ g}\left(\frac{r}{2}\right) &< \log^{[2]} \nu_{f\circ g}(r) + \frac{\log^{[2]}\left(\frac{r}{2}\right) + O(1)}{\log \nu_{f\circ g}(r)} \\
i.e.,\ \log^{[3]} M_{f\circ g}\left(\frac{r}{2}\right) &< \log^{[2]} \nu_{f\circ g}(r)\left(1 + \frac{\log^{[2]}\left(\frac{r}{2}\right) + O(1)}{\log \nu_{f\circ g}(r) \cdot \log^{[2]} \nu_{f\circ g}(r)}\right)
\end{aligned}$$

$$i.e.,\ \log^{[4]} M_{f\circ g}\left(\frac{r}{2}\right) < \log^{[3]} \nu_{f\circ g}(r) + \log\left(1 + \frac{\log^{[2]}\left(\frac{r}{2}\right) + O(1)}{\log \nu_{f\circ g}(r) \cdot \log^{[2]} \nu_{f\circ g}(r)}\right)$$

Taking $\log\left(1+\frac{\log^{[2]}\left(\frac{r}{2}\right)+O(1)}{\log\nu_{f\circ g}(r)\cdot\log^{[2]}\nu_{f\circ g}(r)}\right) \leq \frac{\log^{[2]}\left(\frac{r}{2}\right)+O(1)}{\log\nu_{f\circ g}(r)\cdot\log^{[2]}\nu_{f\circ g}(r)}$, we get for sufficiently large values of r,

$$\log^{[4]} M_{f\circ g}\left(\frac{r}{2}\right) < \log^{[3]}\nu_{f\circ g}(r) + \frac{\log^{[2]}\left(\frac{r}{2}\right)+O(1)}{\log\nu_{f\circ g}(r)\cdot\log^{[2]}\nu_{f\circ g}(r)}$$

Continuing this process, we get for sufficiently large values of r,

$$\log^{[p-1]} M_{f\circ g}\left(\frac{r}{2}\right) < \log^{[p-2]}\nu_{f\circ g}(r) + \frac{\log^{[2]}\left(\frac{r}{2}\right)+O(1)}{\prod\limits_{l=1}^{p-3}\log^{[l]}\nu_{f\circ g}(r)}$$

Using given condition $\log^{[2]}\left(\frac{r}{2}\right) = o\left(\prod\limits_{l=1}^{p-3}\log^{[l]}\nu_{f\circ g}(r)\right)$ as $r\to+\infty$, we have

$$\begin{aligned} \log^{[p-1]} M_{f\circ g}\left(\frac{r}{2}\right) &< \log^{[p-2]}\nu_{f\circ g}(r) + o(1) \\ \text{(3.2)} \qquad i.e.,\ \log^{[p-2]}\nu_{f\circ g}(r) &> \log^{[p-1]} M_{f\circ g}\left(\frac{r}{2}\right) + o(1). \end{aligned}$$

Again in view of the first part of Lemma 2.2, one may obtain that

$$\begin{aligned} \log\mu_f(2r) &= \log|a_0| + \int\limits_0^{2r}\frac{\nu_f(t)}{t}dt \\ &\geq \log|a_0| + \int\limits_r^{2r}\frac{\nu_f(t)}{t}dt \\ \text{(3.3)} \qquad &= \log|a_0| + \nu_f(r)\log 2\ \ \{cf.\ [6]\ \}. \end{aligned}$$

Also by Cauchy's inequality, it is well known that

$$\text{(3.4)} \qquad \mu_f(r) \leq M_f(r)\ \ \{cf.\ [12]\ \}.$$

Therefore for any constant D, one may obtain from (3.3) and (3.4) that

$$\text{(3.5)} \qquad \nu_f(r)\log 2 \leq \log M_f(2r) + D\ \ \{cf.\ [6]\ \}.$$

Thus from above, we get that

$$\begin{aligned} \log\nu_f(r) &\leq \log^{[2]} M_f(2r) + O(1), \\ \text{(3.6)} \qquad i.e.,\ \log^{[p-2]}\nu_f(r) &\leq \log^{[p-1]} M_f(2r) + O(1). \end{aligned}$$

From the definitions of $\sigma^{(p,q,t)L}(f)$ and $\overline{\sigma}^{(p,q,t)L}(f\circ g)$ and in view of (3.2) and (3.6), we have for arbitrary positive ε and for all sufficiently large values of r,

$$\log^{[p-2]}\nu_{f\circ g}(r) > (\overline{\sigma}^{(p,q,t)L}(f\circ g)-\varepsilon)[\log^{[q-1]}\left(\frac{r}{2}\right)\cdot\exp^{[t+1]} L(\frac{r}{2})]^{\rho^{(p,q,t)L}(f\circ g)} + o(1),$$

$$(3.7)\quad i.e.,\ \log^{[p-2]}\nu_{f\circ g}(r) > (\overline{\sigma}^{(p,q,t)L}(f\circ g)-\varepsilon)[(\log^{[q-1]}(r)+O(1))\cdot\exp^{[t+1]}L(r)]^{\rho^{(p,q,t)L}(f\circ g)}+o(1),$$

and

$$\log^{[p-2]}\nu_f(r)\le(\sigma^{(p,q,t)L}(f)+\varepsilon)[\log^{[q-1]}(2r)\cdot\exp^{[t+1]}L(2r)]^{\rho^{(p,q,t)L}(f)}+O(1),$$

$$(3.8)\quad i.e.,\ \log^{[p-2]}\nu_f(r) \le (\sigma^{(p,q,t)L}(f)+\varepsilon)[(\log^{[q-1]}(r)+O(1))\cdot\exp^{[t+1]}L(r)]^{\rho^{(p,q,t)L}(f)}+O(1).$$

Now from (3.7), (3.8) and the condition $\rho^{(p,q,t)L}(f\circ g)=\rho^{(p,q,t)L}(f)$, it follows for all sufficiently large values of r,

$$\frac{\log^{[p-2]}\nu_{f\circ g}(r)}{\log^{[p-2]}\nu_f(r)}\geqslant\frac{\left(\overline{\sigma}^{(p,q,t)L}(f\circ g)-\varepsilon\right)+o(1)}{\left(\sigma^{(p,q,t)L}(f)+\varepsilon\right)+o(1)}.$$

As $\varepsilon(>0)$ is arbitrary, we obtain from above

$$(3.9)\qquad \liminf_{r\to+\infty}\frac{\log^{[p-2]}\nu_{f\circ g}(r)}{\log^{[p-2]}\nu_f(r)}\geqslant\frac{\overline{\sigma}^{(p,q,t)L}(f\circ g)}{\sigma^{(p,q,t)L}(f)}.$$

Again in view of (3.6) , for a sequence of values of r tending to infinity,

$$(3.10)\quad \log^{[p-2]}\nu_{f\circ g}(r) \le (\overline{\sigma}^{(p,q,t)L}(f\circ g)+\varepsilon)[(\log^{[q-1]}(r)+O(1))\cdot\exp^{[t+1]}L(r)]^{\rho^{(p,q,t)L}(f\circ g)}+O(1),$$

and in view of (3.2) , for all sufficiently large values of r,

$$(3.11)\quad \log^{[p-2]}\nu_f(r) > (\overline{\sigma}^{(p,q,t)L}(f)-\varepsilon)[(\log^{[q-1]}(r)+O(1))\cdot\exp^{[t+1]}L(r)]^{\rho^{(p,q,t)L}(f)}+o(1).$$

Combining (3.10) and (3.11) and the condition $\rho^{(p,q,t)L}(f\circ g)=\rho^{(p,q,t)L}(f)$, we get for a sequence of values of r tending to infinity

$$\frac{\log^{[p-2]}\nu_{f\circ g}(r)}{\log^{[p-2]}\nu_f(r)}\le\frac{\left(\overline{\sigma}^{(p,q,t)L}(f\circ g)+\varepsilon\right)+o(1)}{\left(\overline{\sigma}^{(p,q,t)L}(f)-\varepsilon\right)+o(1)}.$$

Since $\varepsilon(>0)$ is arbitrary, it follows from above that

$$(3.12)\qquad \liminf_{r\to+\infty}\frac{\log^{[p-2]}\nu_{f\circ g}(r)}{\log^{[p-2]}\nu_f(r)}\le\frac{\overline{\sigma}^{(p,q,t)L}(f\circ g)}{\overline{\sigma}^{(p,q,t)L}(f)}.$$

Further in view of (3.6) for a sequence of values of r tending to infinity, it follows that

$$(3.13)\qquad \log^{[p-2]}\nu_f(r)$$

$$\leq \quad (\overline{\sigma}^{(p,q,t)L}(f)+\varepsilon)[(\log^{[q-1]}(r)+O(1))\cdot \exp^{[t+1]}L(r)]^{\rho^{(p,q,t)L}(f)}+O(1).$$

Now from (3.7), (3.13) and the condition $\rho^{(p,q,t)L}(f\circ g)=\rho^{(p,q,t)L}(f)$, we obtain for a sequence of values of r tending to infinity

$$\frac{\log^{[p-2]}\nu_{f\circ g}(r)}{\log^{[p-2]}\nu_{f}(r)}\geq \frac{\left(\overline{\sigma}^{(p,q,t)L}(f\circ g)-\varepsilon\right)+o(1)}{\left(\overline{\sigma}^{(p,q,t)L}(f)+\varepsilon\right)+o(1)}.$$

As $\varepsilon(>0)$ is arbitrary, we get from above

$$\limsup_{r\to+\infty}\frac{\log^{[p-2]}\nu_{f\circ g}(r)}{\log^{[p-2]}\nu_{f}(r)}\geq \frac{\overline{\sigma}^{(p,q,t)L}(f\circ g)}{\overline{\sigma}^{(p,q,t)L}(f)}. \tag{3.14}$$

Also for all sufficiently large values of r,

$$\begin{aligned}(3.15)\quad &\log^{[p-2]}\nu_{f}(r)\\ &\leq (\sigma^{(p,q,t)L}(f\circ g)+\varepsilon)[(\log^{[q-1]}(r)+O(1))\cdot \exp^{[t+1]}L(r)]^{\rho^{(p,q,t)L}(f\circ g)}+O(1).\end{aligned}$$

In view of the condition $\rho^{(p,q,t)L}(f\circ g)=\rho^{(p,q,t)L}(f)$, it follows from (3.11) and (3.15) for all sufficiently large values of r,

$$\frac{\log^{[p-2]}\nu_{f\circ g}(r)}{\log^{[p-2]}\nu_{f}(r)}\leq \frac{\left(\sigma^{(p,q,t)L}(f\circ g)+\varepsilon\right)+o(1)}{\left(\overline{\sigma}^{(p,q,t)L}(f)-\varepsilon\right)+o(1)}.$$

Since $\varepsilon(>0)$ is arbitrary, we obtain

$$\limsup_{r\to+\infty}\frac{\log^{[p-2]}\nu_{f\circ g}(r)}{\log^{[p-2]}\nu_{f}(r)}\leq \frac{\sigma^{(p,q,t)L}(f\circ g)}{\overline{\sigma}^{(p,q,t)L}(f)}. \tag{3.16}$$

Again in view of (3.2) , we get for a sequence of values of r tending to infinity

$$\begin{aligned}(3.17)\quad &\log^{[p-2]}\nu_{f}(r)\\ &\geq (\sigma^{(p,q,t)L}(f)-\varepsilon)[(\log^{[q-1]}(r)+O(1))\cdot \exp^{[t+1]}L(r)]^{\rho^{(p,q,t)L}(f)}+o(1).\end{aligned}$$

Now from (3.15), (3.17) and the condition $\rho^{(p,q,t)L}(f\circ g)=\rho^{(p,q,t)L}(f)$, it follows for a sequence of values of r tending to infinity

$$\frac{\log^{[p-2]}\nu_{f\circ g}(r)}{\log^{[p-2]}\nu_{f}(r)}\leq \frac{\left(\sigma^{(p,q,t)L}(f\circ g)+\varepsilon\right)+o(1)}{\left(\sigma^{(p,q,t)L}(f)-\varepsilon\right)+o(1)}.$$

As $\varepsilon(>0)$ is arbitrary, we obtain

$$\liminf_{r\to\infty}\frac{\log^{[p-2]}\nu_{f\circ g}(r)}{\log^{[p-2]}\nu_{f}(r)}\leq \frac{\sigma^{(p,q,t)L}(f\circ g)}{\sigma^{(p,q,t)L}(f)}. \tag{3.18}$$

Again in view of (3.2) , for a sequence of values of r tending to infinity

$$(3.19)\quad \log^{[p-2]}\nu_{f\circ g}(r)$$

$$\geqslant \quad (\sigma^{(p,q,t)L}(f\circ g)-\varepsilon)[(\log^{[q-1]}(r)+O(1))\cdot\exp^{[t+1]}L(r)]^{\rho^{(p,q,t)L}(f\circ g)}+o(1).$$

Combining (3.8) and (3.19) and in view of the condition $\rho^{(p,q,t)L}(f\circ g)=\rho^{(p,q,t)L}(f)$, we get for a sequence of values of r tending to infinity

$$\frac{\log^{[p-2]}\nu_{f\circ g}(r)}{\log^{[p-2]}\nu_f(r)}\geqslant\frac{\left(\sigma^{(p,q,t)L}(f\circ g)-\varepsilon\right)+o(1)}{\left(\sigma^{(p,q,t)L}(f)+\varepsilon\right)+o(1)}.$$

Since $\varepsilon(>0)$ is arbitrary, it follows that

$$\limsup_{r\to+\infty}\frac{\log^{[p-2]}\nu_{f\circ g}(r)}{\log^{[p-2]}\nu_f(r)}\geqslant\frac{\sigma^{(p,q,t)L}(f\circ g)}{\sigma^{(p,q,t)L}(f)}. \tag{3.20}$$

Thus the theorem follows from (3.9), (3.12), (3.14), (3.16), (3.18) and (3.20). □

Remark 3.1. In Theorem 3.1, if we replace the conditions "$0<\overline{\sigma}^{(p,q,t)L}(f)\leq\sigma^{(p,q,t)L}(f)<+\infty$" and "$\rho^{(p,q,t)L}(f\circ g)=\rho^{(p,q,t)L}(f)$" by "$0<\overline{\sigma}^{(m,q,t)L}(g)\leq\sigma^{(m,q,t)L}(g)<+\infty$" and "$\rho^{(p,q,t)L}(f\circ g)=\rho^{(m,q,t)L}(g)$" and other conditions remain same, then Theorem 3.1 remains valid with "$\sigma^{(m,q,t)L}(g)$", "$\log^{[m-2]}\nu_g(r)$" and "$\overline{\sigma}^{(m,q,t)L}(g)$" instead of "$\sigma^{(p,q,t)L}(f)$", "$\log^{[p-2]}\nu_f(r)$" and "$\overline{\sigma}^{(p,q,t)L}(f)$" respectively in the denominators.

Remark 3.2. In Theorem 3.1, if we replace the conditions "$0<\overline{\sigma}^{(p,q,t)L}(f)\leq\sigma^{(p,q,t)L}(f)<+\infty$" and "$\rho^{(p,q,t)L}(f\circ g)=\rho^{(p,q,t)L}(f)$" by "$0<\tau^{(p,q,t)L}(f)\leq\overline{\tau}^{(p,q,t)L}(f)<+\infty$" and "$\rho^{(p,q,t)L}(f\circ g)=\lambda^{(p,q,t)L}(f)$" and other conditions remain same, then Theorem 3.1 remains valid with "$\overline{\tau}^{(p,q,t)L}(f)$" and "$\tau^{(p,q,t)L}(f)$" instead of "$\sigma^{(p,q,t)L}(f)$" and "$\overline{\sigma}^{(p,q,t)L}(f)$" respectively in the denominators.

Remark 3.3. In Theorem 3.1, if we replace the conditions "$0<\overline{\sigma}^{(p,q,t)L}(f)\leq\sigma^{(p,q,t)L}(f)<+\infty$" and "$\rho^{(p,q,t)L}(f\circ g)=\rho^{(p,q,t)L}(f)$" by "$0<\tau^{(m,q,t)L}(g)\leq\overline{\tau}^{(m,q,t)L}(g)<+\infty$" and "$\rho^{(p,q,t)L}(f\circ g)=\lambda^{(m,q,t)L}(g)$" and other conditions remain same, then Theorem 3.1 remains valid with "$\overline{\tau}^{(m,q,t)L}(g)$", "$\log^{[m-2]}\nu_g(r)$" and "$\tau^{(m,q,t)L}(g)$" instead of "$\sigma^{(p,q,t)L}(f)$", "$\log^{[p-2]}\nu_f(r)$" and "$\overline{\sigma}^{(p,q,t)L}(f)$" respectively in the denominators.

The following theorem can be proved in the line of Theorem 3.1 and so its proof is omitted.

Theorem 3.2. *Let $f(z)$ and $g(z)$ be any two entire functions such that $0<\tau^{(p,q,t)L}(f\circ g)\leq\overline{\tau}^{(p,q,t)L}(f\circ g)<+\infty$, $0<\tau^{(p,q,t)L}(f)\leq\overline{\tau}^{(p,q,t)L}(f)<\infty$, $\lambda^{(p,q,t)L}(f\circ g)=\lambda^{(p,q,t)L}(f)$ and $\exp^{[t+1]}L(ar)\sim\exp^{[t+1]}L(r)$ as $r\to+\infty$ for every positive constant 'a'. If $\log^{[2]}\left(\frac{r}{2}\right)=o\left(\prod_{l=1}^{p-3}\log^{[l]}\nu_{f\circ g}(r)\right)$ as $r\to+\infty$, then*

$$\begin{aligned}&\frac{\tau^{(p,q,t)L}(f\circ g)}{\overline{\tau}^{(p,q,t)L}(f)}\leq\liminf_{r\to+\infty}\frac{\log^{[p-2]}\nu_{f\circ g}(r)}{\log^{[p-2]}\nu_f(r)}\\&\leq\min\left\{\frac{\tau^{(p,q,t)L}(f\circ g)}{\tau^{(p,q,t)L}(f)},\frac{\overline{\tau}^{(p,q,t)L}(f\circ g)}{\overline{\tau}^{(p,q,t)L}(f)}\right\}\leq\max\left\{\frac{\tau^{(p,q,t)L}(f\circ g)}{\tau^{(p,q,t)L}(f)},\frac{\overline{\tau}^{(p,q,t)L}(f\circ g)}{\overline{\tau}^{(p,q,t)L}(f)}\right\}\\&\leq\limsup_{r\to+\infty}\frac{\log^{[p-2]}\nu_{f\circ g}(r)}{\log^{[p-2]}\nu_f(r)}\leq\frac{\overline{\tau}^{(p,q,t)L}(f\circ g)}{\tau^{(p,q,t)L}(f)}.\end{aligned}$$

Remark 3.4. In Theorem 3.2, if we replace the conditions "$0 < \tau^{(p,q,t)L}(f) \le \overline{\tau}^{(p,q,t)L}(f) < \infty$" and "$\lambda^{(p,q,t)L}(f \circ g) = \lambda^{(p,q,t)L}(f)$" by "$0 < \tau^{(m,q,t)L}(g) \le \overline{\tau}^{(m,q,t)L}(g) < +\infty$" and "$\lambda^{(p,q,t)L}(f \circ g) = \lambda^{(m,q,t)L}(g)$" and other conditions remain same, then Theorem 3.2 remains valid with "$\overline{\tau}^{(m,q,t)L}(g)$", "$\log^{[m-2]} \nu_g(r)$" and "$\tau^{(m,q,t)L}(g)$" instead of "$\overline{\tau}^{(p,q,t)L}(f)$", "$\log^{[p-2]} \nu_f(r)$" and "$\tau^{(p,q,t)L}(f)$" respectively in the denominators.

Remark 3.5. In Theorem 3.2, if we replace the conditions "$0 < \tau^{(p,q,t)L}(f) \le \overline{\tau}^{(p,q,t)L}(f) < \infty$" and "$\lambda^{(p,q,t)L}(f \circ g) = \lambda^{(p,q,t)L}(f)$" by "$0 < \overline{\sigma}^{(p,q,t)L}(f) \le \sigma^{(p,q,t)L}(f) < +\infty$" and "$\lambda^{(p,q,t)L}(f \circ g) = \rho^{(p,q,t)L}(f)$" and other conditions remain same, then Theorem 3.2 remains valid with "$\sigma^{(p,q,t)L}(f)$" and "$\overline{\sigma}^{(p,q,t)L}(f)$" instead of "$\overline{\tau}^{(p,q,t)L}(f)$" and "$\tau^{(p,q,t)L}(f)$" respectively in the denominators.

Remark 3.6. In Theorem 3.2, if we replace the conditions "$0 < \tau^{(p,q,t)L}(f) \le \overline{\tau}^{(p,q,t)L}(f) < \infty$" and "$\lambda^{(p,q,t)L}(f \circ g) = \lambda^{(p,q,t)L}(f)$" by "$0 < \overline{\sigma}^{(m,q,t)L}(g) \le \sigma^{(m,q,t)L}(g) < +\infty$" and "$\lambda^{(p,q,t)L}(f \circ g) = \rho^{(m,q,t)L}(g)$" and other conditions remain same, then Theorem 3.2 remains valid with "$\sigma^{(m,q,t)L}(g)$", "$\log^{[m-2]} \nu_g(r)$" and "$\overline{\sigma}^{(m,q,t)L}(g)$" instead of "$\overline{\tau}^{(p,q,t)L}(f)$", "$\log^{[p-2]} \nu_f(r)$" and "$\tau^{(p,q,t)L}(f)$" respectively in the denominators.

Theorem 3.3. *Let $f(z)$ and $g(z)$ be any two entire functions such that $\rho^{(p,q,t)L}(f) < +\infty$, $\rho^{(p,q,t)L}(f) = \rho^{(m,n,t)L}(g)$, $0 < \sigma^{(m,n,t)L}(g) < +\infty$ and $\overline{\sigma}^{(p,q,t)L}(f) > 0$ where $m - 1 = n = q$. Also let $\log^{[2]}\left(\frac{r}{2}\right) = o\left(\prod_{l=1}^{p-3} \log^{[l]} \nu_{f\circ g}(r)\right)$ as $r \to +\infty$. Then for any constant E,*

$$\limsup_{r\to+\infty} \frac{\log^{[p]} \nu_{f\circ g}(r)}{\log^{[p-2]} \nu_f(r) + \exp^{[t]}[L(\exp[\nu_g(4r)\log(e\cdot 2r) + E])]}$$
$$\le \begin{cases} \frac{\rho^{(p,q,t)L}(f)\sigma^{(m,n,t)L}(g)}{\overline{\sigma}^{(p,q,t)L}(f)} & \text{if } \exp^{[t]}[L(\exp[\nu_g(4r)\log(e\cdot 2r)+E])] = o\{\log^{[p-2]}\nu_f(r)\}, \\ \rho^{(p,q,t)L}(f) & \text{if } \log^{[p-2]}\nu_f(r) = o\{\exp^{[t]}[L(\exp[\nu_g(4r)\log(e\cdot 2r)+E])]\}. \end{cases}$$

Proof. From (3.1) we obtain that

$$M_g(r) < \exp[\nu_g(2r)\log(e\cdot r) + E]. \tag{3.21}$$

In view of (3.5) and the second part of Lemma 2.1, we obtain for all sufficiently large values of r that

$$\begin{aligned} \nu_{f\circ g}(r)\log 2 &\le \log M_{f\circ g}(2r) + D \le \log M_f(M_g(2r)) + D, \\ i.e.,\ \log^{[p]}(\nu_{f\circ g}(r)\log 2) &\le \log^{[p]}(\log M_f(M_g(2r)) + D), \\ i.e.,\ \log^{[p]}\nu_{f\circ g}(r) &\le \log^{[p+1]} M_f(M_g(2r)) + O(1), \end{aligned} \tag{3.22}$$

$$\begin{aligned} & i.e.,\ \log^{[p]}\nu_{f\circ g}(r) \\ \le\ & (\rho^{(p,q,t)L}(f) + \varepsilon)[\log^{[q]} M_g(2r) + \exp^{[t]} L(M_g(2r))] + O(1). \end{aligned} \tag{3.23}$$

Now in view of (3.21) we get for all sufficiently large values of r that

$$\begin{aligned}\log^{[p]}\nu_{f\circ g}(r) \quad &\leq \quad (\rho^{(p,q,t)L}(f)+\varepsilon)\\ &\times[\log^{[m-1]}M_g(2r)+\exp^{[t]}[L(\exp[\nu_g\,(4r)\log{(e\cdot 2r)}+E])]]+O(1),\end{aligned}$$

$i.e.,$ $\log^{[p]}\nu_{f\circ g}(r) \leq (\rho^{(p,q,t)L}(f)+\varepsilon)$

$$\begin{aligned}&\times[(\sigma^{(m,n,t)L}(g)+\varepsilon)[\log^{[n-1]}2r\cdot\exp^{[t+1]}L(2r)]^{\rho^{(m,n,t)L}(g)}\\ &+\exp^{[t]}[L(\exp[\nu_g\,(4r)\log{(e\cdot 2r)}+E])]]+O(1).\end{aligned}$$

Since $\rho^{(p,q,t)L}(f)=\rho^{(m,n,t)L}(g)$, we obtain from above for all sufficiently large values of r,

$$\log^{[p]}\nu_{f\circ g}(r) \leq (\rho^{(p,q,t)L}(f)+\varepsilon)\cdot$$

$$\begin{aligned}&[(\sigma^{(m,n,t)L}(g)+\varepsilon)[\log^{[n-1]}2r\cdot\exp^{[t+1]}L(2r)]^{\rho^{(p,q,t)L}(f)}\\ &+\exp^{[t]}[L(\exp[\nu_g\,(4r)\log{(e\cdot 2r)}+E])]]+O(1),\end{aligned}$$

$$(3.24)\quad i.e.,\ \log^{[p]}\nu_{f\circ g}(r) \leq (\rho^{(p,q,t)L}(f)+\varepsilon)\cdot$$

$$\begin{aligned}&[(\sigma^{(m,n,t)L}(g)+\varepsilon)[(\log^{[n-1]}r+O(1))\cdot\exp^{[t+1]}L(r)]^{\rho^{(p,q,t)L}(f)}\\ &+\exp^{[t]}[L(\exp[\nu_g\,(4r)\log{(e\cdot 2r)}+E])]]+O(1).\end{aligned}$$

Again in view of (3.2), we get for all sufficiently large values of r,

$$\log^{[p-2]}\nu_f\,(r) \geq (\overline{\sigma}^{(p,q,t)L}(f)-\varepsilon)[(\log^{[q-1]}(r)+O(1))\cdot\exp^{[t+1]}L(r)]^{\rho^{(p,q,t)L}(f)}+o(1),$$

$$i.e.,\ [(\log^{[q-1]}(r)+O(1))\cdot\exp^{[t+1]}L(r)]^{\rho^{(p,q,t)L}(f)} \leq \frac{\log^{[p-2]}\nu_f\,(r)+o(1)}{\overline{\sigma}^{(p,q,t)L}(f)-\varepsilon},$$

$$(3.25)\quad i.e.,\ [(\log^{[n-1]}(r)+O(1))\cdot\exp^{[t+1]}L(r)]^{\rho^{(p,q,t)L}(f)} \leq \frac{\log^{[p-2]}\nu_f\,(r)+o(1)}{\overline{\sigma}^{(p,q,t)L}(f)-\varepsilon}.$$

Now from (3.24) and (3.25) it follows for all sufficiently large values of r that

$$\log^{[p]}\nu_{f\circ g}(r) \leq (\rho^{(p,q,t)L}(f)+\varepsilon)\cdot\exp^{[t]}[L(\exp[\nu_g\,(4r)\log{(e\cdot 2r)}+E])]+O(1)+$$

$$(\rho^{(p,q,t)L}(f)+\varepsilon)(\sigma^{(m,n,t)L}(g)+\varepsilon)\cdot\frac{\log^{[p-2]}\nu_f\,(r)+o(1)}{\overline{\sigma}^{(p,q,t)L}(f)-\varepsilon},$$

$$(3.26)\qquad i.e.,\ \frac{\log^{[p]}\nu_{f\circ g}(r)}{\log^{[p-2]}\nu_f\,(r)+\exp^{[t]}[L(\exp[\nu_g\,(4r)\log{(e\cdot 2r)}+E])]}$$

$$\leq \quad o(1)+\frac{\rho^{(p,q,t)L}(f)+\varepsilon}{1+\frac{\log^{[p-2]}\nu_f(r)}{\exp^{[t]}[L(\exp[\nu_g(4r)\log(e\cdot 2r)+E])]}}$$

$$+\frac{\frac{(\rho^{(p,q,t)L}(f)+\varepsilon)(\sigma^{(m,n,t)L}(g)+\varepsilon)}{(\overline{\sigma}^{(p,q,t)L}(f)-\varepsilon)}+o(1)}{1+\frac{\exp^{[t]}[L(\exp[\nu_g(4r)\log(e\cdot 2r)+E])]}{\log^{[p-2]}\nu_f(r)}}.$$

If $\exp^{[t]}[L(\exp[\nu_g(4r)\log(e\cdot 2r)+E])]=o\{\log^{[p-2]}\nu_f(r)\}$ then from (3.26) we get

$$\limsup_{r\to+\infty}\frac{\log^{[p]}\nu_{f\circ g}(r)}{\log^{[p-2]}\nu_f(r)+\exp^{[t]}[L(\exp[\nu_g(4r)\log(e\cdot 2r)+E])]}$$
$$\leq \quad \frac{(\rho^{(p,q,t)L}(f)+\varepsilon)(\sigma^{(m,n,t)L}(g)+\varepsilon)}{\overline{\sigma}^{(p,q,t)L}(f)-\varepsilon}.$$

Since $\varepsilon(>0)$ is arbitrary, it follows from above that

$$\limsup_{r\to+\infty}\frac{\log^{[p]}\nu_{f\circ g}(r)}{\log^{[p-2]}\nu_f(r)+\exp^{[t]}[L(\exp[\nu_g(4r)\log(e\cdot 2r)+E])]}$$
$$\leq \quad \frac{\rho^{(p,q,t)L}(f)\sigma^{(m,n,t)L}(g)}{\overline{\sigma}^{(p,q,t)L}(f)}.$$

Again if $\log^{[p-2]}\nu_f(r)=o\{\exp^{[t]}[L(\exp[\nu_g(4r)\log(e\cdot 2r)+E])]\}$ then from (3.26) it follows that

$$\limsup_{r\to+\infty}\frac{\log^{[p]}\nu_{f\circ g}(r)}{\log^{[p-2]}\nu_f(r)+\exp^{[t]}[L(\exp[\nu_g(4r)\log(e\cdot 2r)+E])]}\leq\rho^{(p,q,t)L}(f)+\varepsilon.$$

As $\varepsilon(>0)$ is arbitrary, we obtain from above

$$\limsup_{r\to+\infty}\frac{\log^{[p]}\nu_{f\circ g}(r)}{\log^{[p-2]}\nu_f(r)+\exp^{[t]}[L(\exp[\nu_g(4r)\log(e\cdot 2r)+E])]}\leq\ \rho^{(p,q,t)L}(f).$$

Thus the theorem is established. □

Remark 3.7. In Theorem 3.3, if we replace the conditions "$\rho^{(p,q,t)L}(f)<+\infty$" by "$\lambda^{(p,q,t)L}(f)<+\infty$" and other conditions remain the same, then Theorem 3.3 remains valid with "$\lambda^{(p,q,t)L}(f)$" and "$\liminf$" instead of "$\rho^{(p,q,t)L}(f)$" and "$\limsup$" respectively.

Remark 3.8. In Theorem 3.3, if we replace the conditions "$\overline{\sigma}^{(p,q,t)L}(f)>0$" by "$\sigma^{(p,q,t)L}(f)>0$" and other conditions remain the same, then Theorem 3.3 remains valid with "$\sigma^{(p,q,t)L}(f)$" and "$\liminf$" instead of "$\overline{\sigma}^{(p,q,t)L}(f)$" and "$\limsup$" respectively.

Remark 3.9. In Theorem 3.3, if we replace the conditions "$0<\sigma^{(m,n,t)L}(g)<+\infty$" by "$0<\overline{\sigma}^{(m,n,t)L}(g)<+\infty$" and other conditions remain the same, then Theorem 3.3 remains valid with "$\overline{\sigma}^{(m,n,t)L}(g)$" and "$\liminf$" instead of "$\sigma^{(m,n,t)L}(g)$" and "$\limsup$" respectively.

Remark 3.10. In Theorem 3.3, if we replace the conditions "$\rho^{(p,q,t)L}(f)=\rho^{(m,n,t)L}(g)$", "$0<\sigma^{(m,n,t)L}(g)<+\infty$" and "$\overline{\sigma}^{(p,q,t)L}(f)>0$" by "$\lambda^{(p,q,t)L}(f)=\lambda^{(m,n,t)L}(g)$", "$0<\overline{\tau}^{(m,n,t)L}(g)<+\infty$" and "$\tau^{(p,q,t)L}(f)>0$" and other conditions remain the same, then Theorem 3.3 remains valid with "$\tau^{(p,q,t)L}(f)$" and "$\overline{\tau}^{(m,n,t)L}(g)$" instead of "$\overline{\sigma}^{(p,q,t)L}(f)$" and "$\sigma^{(m,n,t)L}(g)$" respectively.

Remark 3.11. In Theorem 3.3, if we replace the conditions "$\rho^{(p,q,t)L}(f)=\rho^{(m,n,t)L}(g)$" and "$\overline{\sigma}^{(p,q,t)L}(f)>0$" by "$\lambda^{(p,q,t)L}(f)=\rho^{(m,n,t)L}(g)$" and "$\tau^{(p,q,t)L}(f)>0$" and other conditions remain the same, then Theorem 3.3 remains valid with "$\tau^{(p,q,t)L}(f)$" instead of "$\overline{\sigma}^{(p,q,t)L}(f)$".

Remark 3.12. In Theorem 3.3, if we replace the conditions "$\rho^{(p,q,t)L}(f)=\rho^{(m,n,t)L}(g)$" and "$0<\sigma^{(m,n,t)L}(g)<+\infty$" by "$\rho^{(p,q,t)L}(f)=\lambda^{(m,n,t)L}(g)$" and "$0<\overline{\tau}^{(m,n,t)L}(g)<+\infty$" and other conditions remain the same, then Theorem 3.3 remains valid with "$\overline{\tau}^{(m,n,t)L}(g)$" instead of "$\sigma^{(m,n,t)L}(g)$".

The following theorem can be proved in the line of Theorem 3.3 and so its proof is omitted.

Theorem 3.4. *Let $f(z)$ and $g(z)$ be any two entire functions such that $\lambda^{(p,q,t)L}(f)<+\infty$, $\lambda^{(p,q,t)L}(f)=\lambda^{(m,n,t)L}(g)$, $0<\overline{\tau}^{(m,n,t)L}(g)<+\infty$ and $\tau^{(p,q,t)L}(f)>0$ where $m-1=n=q$. Also let $\log^{[2]}\left(\frac{r}{2}\right)=o\left(\prod\limits_{l=1}^{p-3}\log^{[l]}\nu_{f\circ g}(r)\right)$ as $r\to+\infty$. Then for any constant E,*

$$\liminf_{r\to+\infty}\frac{\log^{[p]}\nu_{f\circ g}(r)}{\log^{[p-2]}\nu_f(r)+\exp^{[t]}[L(\exp[\nu_g(4r)\log(e\cdot 2r)+E])]}$$
$$\leq\begin{cases}\frac{\lambda^{(p,q,t)L}(f)\overline{\tau}^{(m,n,t)L}(g)}{\tau^{(p,q,t)L}(f)} & \text{if } \exp^{[t]}[L(\exp[\nu_g(4r)\log(e\cdot 2r)+E])]=o\{\log^{[p-2]}\nu_f(r)\}\\ \lambda^{(p,q,t)L}(f) & \text{if } \log^{[p-2]}\nu_f(r)=o\{\exp^{[t]}[L(\exp[\nu_g(4r)\log(e\cdot 2r)+E])]\}.\end{cases}$$

Remark 3.13. In Theorem 3.4, if we replace the conditions "$\lambda^{(p,q,t)L}(f)<+\infty$" and "$\tau^{(p,q,t)L}(f)>0$" by "$\rho^{(p,q,t)L}(f)<+\infty$" and "$\overline{\tau}^{(p,q,t)L}(f)>0$" and other conditions remain same, then Theorem 3.4 remains valid with "$\rho^{(p,q,t)L}(f)$" and "$\overline{\tau}^{(p,q,t)L}(f)$" instead of "$\lambda^{(p,q,t)L}(f)$" and "$\tau^{(p,q,t)L}(f)$" respectively.

Remark 3.14. In Theorem 3.4, if we replace the conditions "$\lambda^{(p,q,t)L}(f)<+\infty$" and "$0<\overline{\tau}^{(m,n,t)L}(g)<+\infty$" by "$\rho^{(p,q,t)L}(f)<+\infty$" and "$0<\tau^{(m,n,t)L}(g)<+\infty$" and other conditions remain the same, then Theorem 3.4 remains valid with "$\rho^{(p,q,t)L}(f)$" and "$\tau^{(m,n,t)L}(g)$" instead of "$\lambda^{(p,q,t)L}(f)$" and "$\overline{\tau}^{(m,n,t)L}(g)$" respectively.

Remark 3.15. In Theorem 3.4, if we replace the conditions "$\lambda^{(p,q,t)L}(f)=\lambda^{(m,n,t)L}(g)$" and "$0<\overline{\tau}^{(m,n,t)L}(g)<+\infty$" by "$\lambda^{(p,q,t)L}(f)=\rho^{(m,n,t)L}(g)$" and "$0<\sigma^{(m,n,t)L}(g)<+\infty$" and other conditions remain same, then Theorem 3.4 remains valid with "$\sigma^{(m,n,t)L}(g)$" instead of "$\overline{\tau}^{(m,n,t)L}(g)$".

Remark 3.16. In Theorem 3.4, if we replace the conditions "$\lambda^{(p,q,t)L}(f)=\lambda^{(m,n,t)L}(g)$" and "$\tau^{(p,q,t)L}(f)>0$" by "$\rho^{(p,q,t)L}(f)=\lambda^{(m,n,t)L}(g)$" and "$\overline{\sigma}^{(p,q,t)L}(f)>0$" and other conditions remain the same, then Theorem 3.4 remains valid with "$\overline{\sigma}^{(p,q,t)L}(f)$" instead of "$\tau^{(p,q,t)L}(f)$".

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