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JENSEN-TYPE INEQUALITIES FOR A CLASS OF CONVEX FUNCTIONS *

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Abstract. In this paper, we introduce a new class of function to obtain some interesting inequalities of Jensen-type. Our results generalize and refine some existing results in literature. Some applications for Schurs inequality are also included.
Key words: Jensen-type inequalities, Schurs inequality, integral ineauality, convex function.

1. Introduction

The study of Convex functions has been a widely known concept used by several researcher (see [5],[6], [7]). Some previous studies have employed a good number of classes of convex functions to derive new integral inequalities of practical interest (see [23], [24],[27]).

In this paper, I and J will be used to denote intervals of real numbers.

Definition 1.1. [24]. A function $f : [a, b] \to R$ is said to be convex if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y), \tag{1.1}$$

for all $x, y \in [a, b]$ and $t \in [0, 1]$.

Definition 1.2. [8]. A function $f : I \to R$ is a Godunova-Levin function or that f belongs to the class $Q(I)$ if f is non-negative and for all $x, y \in I$ and $t \in (0, 1)$

$$f(tx + (1-t)y) \leq \frac{f(x)}{t} + \frac{f(y)}{1-t}. \tag{1.2}$$

This class of convex functions was first introduced in [2] by Godunova and Levin and has gained attention in literature (see [7], [20], [21]).

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Definition 1.3. [2]. Let s be a real number, $s \in (0,1]$. A function $f : [0,\infty) \to [0,\infty)$ is said to be s-convex (in the second sense), or that f belongs to the class K_s^2, if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y), \tag{1.3}$$

for all $x, y \in [0,\infty)$ and $t \in [0,1]$. This was first introduced by Breckner in [2] and some of its properties given in [9].

Definition 1.4. [7]. We say that $f : I \to R$ is a $P-$function, or that f belongs to the class $P(I)$, if f is a non-negative function and for all $x, y \in I$, $t \in [0,1]$, we have

$$f(tx + (1-t)y) \leq f(x) + f(y). \tag{1.4}$$

Definition 1.5. [28]. Let $h : J \to R$ be a non-negative function, $h \not\equiv 0$. We say that $f : I \to R$ is an h-convex function, or that f belongs to the class $SX(h, I)$, if f is non-negative and for all $x, y \in I$, $t \in (0,1)$ we have

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y). \tag{1.5}$$

Definition 1.6. [28]. A function $f : I \to R$ is said to be super-multiplicative if

$$f(xy) \geq f(x)f(y), \tag{1.6}$$

for all $x, y \in I$.

If the inequality (1.6) is reversed, then f is sub-multiplicative.

Definition 1.7. [28]. If $f, g : I \to R$ are functions such that for $x, y \in I$ the following inequality holds

$$(f(x) - f(y))(g(x) - g(y)) \geq 0,$$

then, f and g are said to be similarly ordered.

By developing a new concept different from those in [3], [4], [11]-[18], [22], we prove interesting inequalities for a new class of convex functions.

2. Main Results

We now define a new class of convex functions.

Definition 2.1. Let $h : J \to (0,\infty)$, $s \in [0,1]$, $t \in (0,1)$ and ϕ be a given real-valued function. Then, $f : I \to [0,\infty)$ is a ϕ_{h-s} convex function if for all $x, y \in I$,

$$f(t\phi(x) + (1-t)\phi(y)) \leq \left(\frac{h(t)}{t}\right)^{-s} f(\phi(x)) + \left(\frac{h(1-t)}{(1-t)}\right)^{-s} f(\phi(y)). \tag{2.1}$$

Observe that:
(i) If $s = 0$ and $\phi(x) = x$, then $f \in P(I)$.
(ii) If $h(t) = t^{\frac{s}{s+1}}$ and $\phi(x) = x$, then $f \in SX(h, I)$.
(iii) If $s = 1$, $h(t) = 1$ and $\phi(x) = x$, then $f \in SX(I)$ i.e f is convex.
(iv) If $h(t) = 1$ and $\phi(x) = x$ then f is *Breckner s-convex* or *s-convex* in the second sense (see [2]).
(v) If $h(t) = t^2$ and $\phi(x) = x$ then f is *s-Godunova-Levin* function (see [5], [6]).

Let us denote by $Q_s(I)$ and $SX(\phi_{h-s}, I)$ the class of *s-Godunova-Levin* functions and ϕ_{h-s} convex functions respectively, then it is easy to see that

$$P(I)=Q_0(I)=SX(\phi_{h-0}, I)\subseteq SX(\phi_{h-s_1}, I)\subseteq SX(\phi_{h-s_2}, I)\subseteq SX(\phi_{h-1}, I)=SX(\phi_h),$$

for $0 \leq s_1 \leq s_2 \leq 1$ whenever ϕ is the identity function.
If the inequality sign in (2.1) is reversed, then f is ϕ_{h-s} concave, i.e. $f\in SV(\phi_{h-s}, I)$.
Next, we give some properties and examples of our new class of convex functions.

Proposition 2.1. *If $f, g \in SX(\phi_{h-s}, I)$ i.e they are ϕ_{h-s} convex, and $c \geq 0$ is a positive real number, then $f + g$ and cf are both ϕ_{h-s} convex.*

Proof. $f, g \in SX(\phi_{h-s}, I)$ implies

$$f(\lambda\phi(a) + (1-\lambda)\phi(b)) \leq \left(\frac{h(\lambda)}{\lambda}\right)^{-s} f(\phi(a)) + \left(\frac{h(1-\lambda)}{1-\lambda}\right)^{-s} f(\phi(b)). \tag{2.2}$$

$$g(\lambda\phi(a) + (1-\lambda)\phi(b)) \leq \left(\frac{h(\lambda)}{\lambda}\right)^{-s} g(\phi(a)) + \left(\frac{h(1-\lambda)}{1-\lambda}\right)^{-s} g(\phi(b)). \tag{2.3}$$

Clearly,
(2.4)
$$(f+g)(\lambda\phi(a) + (1-\lambda)\phi(b)) \leq \left(\frac{h(\lambda)}{\lambda}\right)^{-s} [(f+g)(\phi(a))] + \left(\frac{h(1-\lambda)}{1-\lambda}\right)^{-s} [(f+g)(\phi(b))]$$

and,

$$cf(\lambda\phi(a) + (1-\lambda)\phi(b)) \leq \left(\frac{h(\lambda)}{\lambda}\right)^{-s} cf(\phi(a)) + \left(\frac{h(1-\lambda)}{1-\lambda}\right)^{-s} cf(\phi(b)). \tag{2.5}$$

□

We have just shown that addition and scalar multiplication holds for our newly defined class of convex functions so that $SX(\phi_{h-s}, I)$ is a linear space for $c \geq 0$. Observe that, $SX(\phi_{h-s}, I)$ satisfies the properties (and even more) enjoyed by many known classes of convex function. We give examples in Proposition 2.2.

Proposition 2.2. *Let f be a non-negative convex function on I. If h is a non-negative function such that*

$$h(\lambda) \leq \lambda^{1-\frac{1}{s}}, \qquad s \in (0,1], \lambda \in (0,1).$$

Then, $f \in SX(\phi_{h-s}, I)$.

Proof. Proposition 2.2 implies that all convex functions are examples of our newly defined class of convex function provided that the condition $h(\lambda) \leq \lambda^{1-\frac{1}{s}}$ is satisfied. In particular, an example of such $h(\lambda)$ is $h(\lambda) = \lambda^k$ for $k \geq 1 - \frac{1}{s}$, $s \in (0,1]$. Now,

$$\begin{aligned} f(\lambda\phi(a) + (1-\lambda)\phi(b)) &\leq \lambda f(\phi(a)) + (1-\lambda) f(\phi(b)) \\ &\leq \left(\frac{h(\lambda)}{\lambda}\right)^{-s} f(\phi(a)) + \left(\frac{h(1-\lambda)}{1-\lambda}\right)^{-s} f(\phi(b)). \end{aligned} \tag{2.6}$$

Therefore, $f \in SX(\phi_{h-s}, I)$. $\square$

Similarly, if h satisfies $h(\lambda) \geq \lambda^{1-\frac{1}{s}}$ for any $\lambda \in (0,1)$, then any non-negative concave function f belongs to the class $SV(\phi_{h-s}, I)$ i.e is ϕ_{h-s} concave. This new notion of $\phi(x)_{h-s}$ convex functions generalizes quite a number of classes of convex functions which exist in literature. The immediate implication of this is that $\phi(x)_{h-s}$ convex functions provide us with many inequalities which generalize and extend the Jensen-type inequalities for the classes of convex functions that already exist in literature. In this paper, we obtain some interesting results for our newly introduced class of convex functions.

Theorem 2.1. *Let $f \in SX(\phi_{h-s}, I)$,*

- *(i) If g is a linear function, then $f \circ g$ is $\phi_{h-s}(I)$ convex.*
- *(ii) If f is increasing and g is convex, then $f \circ g$ is $\phi_{h-s}(I)$ convex.*

Proof. Since f is $\phi_{h-s}(I)$ convex then,

$$f\left(\lambda\phi(x) + (1-\lambda)\phi(y)\right) \leq \left(\frac{\lambda}{h(t)}\right)^{s} f(\phi(x)) + \left(\frac{(1-\lambda)}{h(1-\lambda)}\right)^{s} f(\phi(y)). \tag{2.7}$$

Now,

$$\begin{aligned} f \circ g\left(\lambda\phi(a) + (1-\lambda)\phi(b)\right) &= f\left(\lambda g(\phi(a)) + (1-\lambda) g(\phi(b))\right) \\ &\leq \left(\frac{\lambda}{h(\lambda)}\right)^{s} f \circ g(\phi(a)) + \left(\frac{(1-\lambda)}{h(1-\lambda)}\right)^{s} f \circ g(\phi(b)). \end{aligned} \tag{2.8}$$

and *(i)* is proved.

We now prove *(ii)*. Since g is convex then

$$g(\lambda\phi(a) + (1-\lambda)\phi(b)) \leq \lambda g(\phi(a)) + (1-\lambda) g(\phi(b)).$$

Now,

$$
\begin{aligned}
f\circ g(\lambda\phi(a)+(1-\lambda)\phi(b)) &\le f(\lambda g(\phi(a))+(1-\lambda)g(\phi(b)))\\
&\le \left(\frac{\lambda}{h(\lambda)}\right)^{s} f(\phi(a))+\left(\frac{(1-\lambda)}{h(1-\lambda)}\right)^{s} f(\phi(b)). \qquad (2.9)
\end{aligned}
$$

□

Remark 2.1. This result generalizes Theorems 1 and 3 recently obtained by Ardic and Özdemir in [1] with appropriate choices of $h(\lambda), \phi$ and s. In particular, with $h(\lambda) = \lambda^2$ and $s = 1$, we obtain Theorem 3 in [1] and with $h(\lambda) = 1$, we obtain Theorem 1 in [1].

Theorem 2.2. *Let $0 \in I$ and $\phi(x)$ be an identity function. Then the following holds.*

(i). If $f \in SX(\phi_{h-s}, I)$, $f(0) = 0$ and h is super-multiplicative, then the inequality

$$f(\alpha x+\beta y)\le \left(\frac{h(\alpha)}{\alpha}\right)^{-s} f(x)+\left(\frac{h(\beta)}{\beta}\right)^{-s} f(y), \qquad (2.10)$$

holds for all $x, y \in I$ and $\alpha, \beta \ge 0$ such that $\alpha+\beta \le 1$.

(ii). Let h be non-negative function with $\left(\frac{h(\alpha)}{\alpha}\right)^{-s} < \frac{1}{2}$ for some $\alpha \in (0, \frac{1}{2})$, $s \in [0,1]$. If f is a non-negative function satisfying (2.10) for all $x, y \in I$, and $\alpha, \beta > 0$ with $\alpha + \beta \le 1$, then $f(0) = 0$.

(iii). If $f \in SV(\phi_{h-s}, I)$, $f(0) = 0$ and h is sub-multiplicative then the inequality

$$f(\alpha x+\beta y)\ge \left(\frac{h(\alpha)}{\alpha}\right)^{-s} f(x)+\left(\frac{h(\beta)}{\beta}\right)^{-s} f(y), \qquad (2.11)$$

holds for all $x, y \in I$ and $\alpha, \beta > 0$ such that $\alpha+\beta \le 1$.

(iv). Let h be a non-negative function with $\left(\frac{h(\alpha)}{\alpha}\right)^{-s} > \frac{1}{2}$ for some $\alpha \in (0, \frac{1}{2})$, $s \in [0,1]$. If f is a non-negative function satisfying (2.11) for all $x, y \in I$ and $\alpha, \beta > 0$ with $\alpha + \beta \le 1$, then $f(0) = 0$.

Proof. We first prove *(i)*. The proof is trivial for $\alpha, \beta > 0$, $\alpha + \beta = 1$ since (2.10) reduces to (2.1) in definition 2.1. Let $\alpha, \beta > 0$, $\alpha + \beta = r < 1$ and let a and b be numbers such that $a = \frac{\alpha}{r}$ and $b = \frac{\beta}{r}$, then $a + b = 1$ and

$$\begin{aligned}
f(\alpha x + \beta y) &= f(arx + bry) \\
&\leq \left(\frac{h(a)}{a}\right)^{-s} f(\phi(rx)) + \left(\frac{h(b)}{b}\right)^{-s} f(\phi(ry)) \\
&\leq \left(\frac{h(a)}{a}\right)^{-s} \left[\left(\frac{h(r)}{r}\right)^{-s} f(x) + \left(\frac{h(1-r)}{1-r}\right)^{-s} f(0)\right] \\
&+ \left(\frac{h(b)}{b}\right)^{-s} \left[\left(\frac{h(r)}{r}\right)^{-s} f(y) + \left(\frac{h(1-r)}{1-r}\right)^{-s} f(0)\right] \\
&\leq \left(\frac{h(ar)}{ar}\right)^{-s} f(x) + \left(\frac{h(br)}{br}\right)^{-s} f(y) \\
&= \left(\frac{h(\alpha)}{\alpha}\right)^{-s} f(x) + \left(\frac{h(\beta)}{\beta}\right)^{-s} f(y).
\end{aligned}$$

We now prove *(ii)*. Suppose $f(0) \neq 0$, then $f(0) > 0$. By setting $x = y = 0$ in (2.10) we get

$$f(0) \leq \left(\frac{h(\alpha)}{\alpha}\right)^{-s} f(0) + \left(\frac{h(\beta)}{\beta}\right)^{-s} f(0). \tag{2.12}$$

Again, we set $\alpha = \beta$, where $\alpha \in (0, \frac{1}{2})$ and dividing both sides of the inequality (2.12) by $f(0)$ to obtain $2\left(\frac{h(\alpha)}{\alpha}\right)^{-s} \geq 1$, for all $\alpha \in (0, \frac{1}{2})$. This contradicts the assumption that $\left(\frac{h(\alpha)}{\alpha}\right)^{-s} < \frac{1}{2}$, so $f(0)$ must be 0. Hence the proof of *(ii)*.

The proofs of *(iii)* and *(iv)* follow by similar arguments and are thus omitted. $\square$

Theorem 2.3. *Let f and g be similarly ordered on I for all $a, b \in I$. If $f \in SX(\phi_{h_1-s}, I)$ and $g \in SX(\phi_{h_2-s}, I)$ such that $\left(\frac{h(\lambda)}{\lambda}\right)^{-s} + \left(\frac{h(1-\lambda)}{1-\lambda}\right)^{-s} \leq c^{-s}$, $\forall\ \lambda \in (0,1)$ with $h(\lambda) = \max_{\lambda\in(0,1)}\{h_1(\lambda), h_2(\lambda)\}$ and c is a fixed positive number. Then, $fg \in SX(\phi_{ch-s}, I)$.*

Proof. Since f and g are similarly ordered, then

$$(f(\phi(a)) - f(\phi(b)))(g(\phi(a)) - g(\phi(b))) \geq 0, \qquad \forall \phi(a), \phi(b) \in I.$$

Implying,

$$f(\phi(a))g(\phi(a)) + f(\phi(b))g(\phi(b)) \geq f(\phi(a))g(\phi(b)) + f(\phi(b))g(\phi(a)).$$

Now,

$$\begin{aligned}
&fg(\lambda\phi(a)+(1-\lambda)\phi(b)) = f(\lambda\phi(a)+(1-\lambda)\phi(b))g((\lambda\phi(a)+(1-\lambda)\phi(b)))\\
&\le \left[\left(\frac{h_1(\lambda)}{\lambda}\right)^{-s} f(\phi(a)) + \left(\frac{h_1(1-\lambda)}{1-\lambda}\right)^{-s} f(\phi(b))\right]\\
&\quad\times \left[\left(\frac{h_2(\lambda)}{\lambda}\right)^{-s} g(\phi(a)) + \left(\frac{h_2(1-\lambda)}{1-\lambda}\right)^{-s} g(\phi(b))\right]\\
&\le \left(\frac{h(\lambda)}{\lambda}\right)^{-2s} fg(\phi(a)) + \left(\frac{h(\lambda)h(1-\lambda)}{\lambda(1-\lambda)}\right)^{-s} f(\phi(a))g(\phi(b))\\
&+ \left(\frac{h(1-\lambda)}{1-\lambda}\frac{h(\lambda)}{\lambda}\right)^{-s} f(\phi(b))g(\phi(a)) + \left(\frac{h(1-\lambda)}{1-\lambda}\right)^{-2s} fg(\phi(b))\\
&\le \left(\frac{h(\lambda)}{\lambda}\right)^{-2s} fg(\phi(a)) + \left(\frac{h(\lambda)}{\lambda}\frac{h(1-\lambda)}{1-\lambda}\right)^{-s} f(\phi(a))g(\phi(a))\\
&+ \left(\frac{h(\lambda)}{\lambda}\frac{h(1-\lambda)}{1-\lambda}\right)^{-s} f(\phi(b))g(\phi(b)) + \left(\frac{h(1-\lambda)}{1-\lambda}\right)^{-2s} fg(\phi(b))\\
&\le \left(\frac{ch(\lambda)}{\lambda}\right)^{-s} fg(\phi(a)) + \left(\frac{ch(1-\lambda)}{1-\lambda}\right)^{-s} fg(\phi(b)).
\end{aligned}$$

Hence, $fg \in SX(\phi_{ch-s}, I)$. □

Remark 2.2. Theorem 2.4 extends the following result of Varoŝanec on $SX(h, I)$ i.e the class of $h-$convex functions (see [28]). For arbitrary $\lambda \in (0,1)$ and $c \ge 0$, define $\phi(x) = x$ and $\lambda^{\frac{s}{s+1}} = ch(\lambda)$ in Theorem 2.3, then Corollary 2.1 follows immediately.

Corollary 2.1. *Let f and g be similarly ordered functions on I, $\forall\ x, y \in I$. If $f \in SX(h_1, I)$ and $g \in SX(h_2, I)$ such that $h(\lambda) + h(1-\lambda) \le c$ for all $\lambda \in (0,1)$, with $h(\lambda) = \max\{h_1, h_2\}$ and c a fixed positive number. Then the product fg belongs to $SX(ch, I)$.*

Theorem 2.4. *Let $h : J \to R$ be a non-negative super-multiplicative and let $f : I \to R$ be a function such that $f \in SX(\phi_{h-s}, I)$ where $\phi(x) = x$. Then, for all $x_1, x_2, x_3 \in I$ such that $x_1 < x_2 < x_3$ and $x_3 - x_1, x_3 - x_2, x_2 - x_1 \in J$, the following holds:*

$$\begin{aligned}
&[(x_3-x_1)(x_2-x_1)h(x_3-x_2)]^{-s} f(x_1) - [(x_3-x_2)(x_2-x_1)h(x_3-x_1)]^{-s} f(x_2)\\
&\qquad + [(x_3-x_1)(x_3-x_2)h(x_2-x_1)]^{-s} f(x_3) \ge 0. \qquad (2.13)
\end{aligned}$$

Proof. Since $f \in SX(\phi_{h-s}, I)$, then it is easy to see that

$$\frac{x_3-x_2}{x_3-x_1},\ \frac{x_2-x_1}{x_3-x_1} \in J \qquad \textit{and} \qquad \frac{x_3-x_2}{x_3-x_1} + \frac{x_2-x_1}{x_3-x_1} = 1.$$

Also,

$$(h(x_3-x_2))^{-s} = \left(h\left(\frac{x_3-x_2}{x_3-x_1}(x_3-x_1)\right)\right)^{-s} \geq \left(h\left(\frac{x_3-x_2}{x_3-x_1}\right)h(x_3-x_1)\right)^{-s}.$$

Similarly,

$$(h(x_2-x_1))^{-s} \geq \left(h\left(\frac{x_2-x_1}{x_3-x_1}\right)h(x_3-x_1)\right)^{-s}.$$

Setting $\alpha = \frac{x_3-x_2}{x_3-x_1}$, $x = x_1$, $y = x_3$, $\beta = \frac{x_2-x_1}{x_3-x_1}$ in (2.1), we have $x_2 = \alpha x + \beta y$ and

$$\begin{aligned}
(2.14)\qquad f(x_2) &\leq \left(\frac{h\left(\frac{x_3-x_2}{x_3-x_1}\right)}{\frac{x_3-x_2}{x_3-x_1}}\right)^{-s} f(x_1) + \left(\frac{h\left(\frac{x_2-x_1}{x_3-x_1}\right)}{\frac{x_2-x_1}{x_3-x_1}}\right)^{-s} f(x_3)\\
(2.15)\qquad &\leq \left(\frac{\frac{h(x_3-x_2)}{h(x_3-x_1)}}{\frac{x_3-x_2}{x_3-x_1}}\right)^{-s} f(x_1) + \left(\frac{\frac{h(x_2-x_1)}{h(x_3-x_1)}}{\frac{x_2-x_1}{x_3-x_1}}\right)^{-s} f(x_3).
\end{aligned}$$

Multiplying both sides of (2.15) by $\left(\frac{x_3-x_2}{x_3-x_1}\right)^{-s}(h(x_3-x_1))^{-s}$ and rearranging, gives (2.13). This result has several implications for the Schurs inequality (interested reader can see [19] and references therein). □

Theorem 2.5. *Let $w_1, w_2, \cdots, w_n$ be positive real numbers and let (m, M) be an interval in I. If $h : (0,\infty) \to R$ is a non-negative super-multiplicative function and $f \in SX(\phi_{h-s}, I)$ where ϕ is an identity function, then for all $x_1, x_2, \cdots, x_n \in (m, M)$ the following inequality holds,*

$$(2.16)\quad \sum_{i=1}^{n} p_i f(x_i) \leq f(m)\sum_{i=1}^{n} p_i \left(\frac{h\left(\frac{M-x_i}{M-m}\right)}{\frac{M-x_i}{M-m}}\right)^{-s} + f(M)\sum_{i=1}^{n} p_i \left(\frac{h\left(\frac{x_i-m}{M-m}\right)}{\frac{x_i-m}{M-m}}\right)^{-s},$$

where

$$p_i = \left(\frac{h\left(\frac{w_i}{W_n}\right)}{\frac{w_i}{W_n}}\right)^{-s}, \qquad W_n = \sum_{i=1}^{n} w_i.$$

Proof. Setting $x_1 = m,\ x_2 = x_i,\ x_3 = M$ in (2.14) we obtain

$$(2.17)\qquad f(x_i) \leq \left(\frac{h\left(\frac{M-x_i}{M-m}\right)}{\frac{M-x_i}{M-m}}\right)^{-s} f(m) + \left(\frac{h\left(\frac{x_i-m}{M-m}\right)}{\frac{x_i-m}{M-m}}\right)^{-s} f(M).$$

Multiplying both sides of (2.17) by p_i and adding both sides of the resulting inequality for $(i = 1, \cdots, n)$, we obtain (2.16). □

Theorem 2.6. *Let $f \in SX(\phi_{h-s}, I)$ and $\sum_{i=1}^{n} t_i = T_n = 1$, $t_i \in (0,1)$, $i = 1, 2, \cdots, n$. Then*

$$f\left(\sum_{i=1}^{n} t_i \phi(x_i)\right) \leq \left[\frac{h(t_i)}{t_i}\right]^{-s} \sum_{i=1}^{n} f(\phi(x_i)).$$

Proof. Observe that

$$\begin{aligned}
f\left(\sum_{i=1}^{n} t_i \phi(x_i)\right) &= f\left(T_{n-1} \sum_{i=1}^{n-1} \frac{t_i}{T_{n-1}} \phi(x_i) + t_n \phi(x_n)\right) \\
&\leq \left[\frac{h(T_{n-1})}{T_{n-1}}\right]^{-s} f\left(\frac{T_{n-2}}{T_{n-1}} \sum_{i=1}^{n-2} \frac{t_i}{T_{n-2}} \phi(x_i) + t_{n-1}\phi(x_{n-1})\right) \\
&+ \left[\frac{h(t_n)}{t_n}\right]^{-s} f(\phi(x_n)) \\
&\leq \left[\frac{h(T_{n-1})}{T_{n-1}}\right]^{-s} \left[\frac{h(T_{n-2})}{T_{n-1}}\right]^{-s} \left[\frac{T_{n-1}}{T_{n-2}}\right]^{-s} f\left(\sum_{i=1}^{n-2} \frac{t_i}{T_{n-2}} \phi(x_i)\right) \\
&+ \left[\frac{h(T_{n-1})}{T_{n-1}}\right]^{-s} f(\phi(x_{n-1})) + \left[\frac{h(t_n)}{t_n}\right]^{-s} f(\phi(x_n)) \\
&\leq \left[\frac{h(t_i)}{t_i}\right]^{-s} \sum_{i=1}^{n} f(\phi(x_i)).
\end{aligned}$$

□

Remark 2.3. Set $h(t) = 1$ and $s = 1$ in Theorem 2.6, then we obtain what is known in literature as the discrete version of the Jensen's inequality (see [10]).

Theorem 2.7. *Let $t_1, \cdots, t_n$ be positive real numbers ($n \geq 2$). If h is a non-negative super-multiplicative function and if $f \in SX(\phi_{h-s}, I)$ where ϕ is an identity function, for $x_1, \cdots, x_n \in I$, then*

$$f\left(\frac{1}{T_n} \sum_{i=1}^{n} t_i x_i\right) \leq \sum_{i=1}^{n} \left[\frac{h(\frac{t_i}{T_n})}{\frac{t_i}{T_n}}\right]^{-s} f(x_i), \tag{2.18}$$

where $T_n = \sum_{i=1}^{n} t_i$.
If h is sub-multiplicative and $f \in SV(\phi_{h-s}, I)$ then the inequality (2.18) *is reversed.*

Proof. Let us suppose that $f \in SX(\phi_{h-s}, I)$. If n=2, then the inequality (2.18) is equivalent to the definition of ϕ_{h-s} convex functions with $\lambda = \frac{t_1}{T_2}$ and $1 - \lambda = \frac{t_2}{T_2}$. Suppose the inequality holds for $n-1$, then for n-tuples $(x_1, \cdots, x_n)$ and $(t_1, \cdots, t_n)$, we have,

$$\begin{aligned}
& f\left(\frac{1}{T_n}\sum_{i=1}^{n} t_i x_i\right) = f\left(\frac{t_n}{T_n}x_n + \sum_{i=1}^{n-1}\frac{t_i}{T_n}x_i\right) \\
& = f\left(\frac{t_n}{T_n}x_n + \frac{T_{n-1}}{T_n}\sum_{i=1}^{n-1}\frac{t_i}{T_{n-1}}x_i\right) \\
& \leq \left[\frac{h\left(\frac{t_n}{T_n}\right)}{\frac{t_n}{T_n}}\right]^{-s} f(x_n) + \left[\frac{h\left(\frac{T_{n-1}}{T_n}\right)}{\frac{T_{n-1}}{T_n}}\right]^{-s} f\left(\sum_{i=1}^{n-1}\frac{t_i}{T_{n-1}}x_i\right) \\
& \leq \left[\frac{h\left(\frac{t_n}{T_n}\right)}{\frac{t_n}{T_n}}\right]^{-s} f(x_n) + \left[\frac{h\left(\frac{T_{n-1}}{T_n}\right)}{\frac{T_{n-1}}{T_n}}\right]^{-s} \sum_{i=1}^{n-1}\left[\frac{h\left(\frac{t_i}{T_{n-1}}\right)}{\frac{t_i}{T_{n-1}}}\right]^{-s} f(x_i) \\
& \leq \left[\frac{h\left(\frac{t_n}{T_n}\right)}{\frac{t_n}{T_n}}\right]^{-s} f(x_n) + \sum_{i=1}^{n-1}\left[\frac{h\left(\frac{t_i}{T_n}\right)}{\frac{t_i}{T_n}}\right]^{-s} f(x_i) = \sum_{i=1}^{n}\left[\frac{h\left(\frac{t_i}{T_n}\right)}{\frac{t_i}{T_n}}\right]^{-s} f(x_i).
\end{aligned}$$

□

Remark 2.4. If we choose $h(\lambda) = 1$ and $s = 1$, then Theorem 2.11 becomes the well known discrete version of the classical Jensen inequality for convex functions (see [10], [26]). If we choose $h(\lambda) = 1$, then our result is the Jensen-type inequality for s-convex functions (see [26]). If $h(\lambda) = \lambda^2$, then we obtain the very recent result known as the Jensen-type inequality for s-Godunova-Levin functions (see [25]).

3. Open Problems

It is known in literature that several direct and converse results can be obtained for Jensen's inequality by using convex functions. Is it possible to obtain similar results for functions belonging to the class $\mathrm{SX}(\phi_{h-s}, I)$?
Also, recent developments has shown that time scaled inequalities are more general and widely applicable, can new inequalities on time scales be obtained for functions belonging to the class $\mathrm{SX}(\phi_{h-s}, I)$?

REFERENCES

1. M. A. Ardic and M. E. Özdemir: *On some inequalities for different kinds of convexity.* Journal of Inequalities and Applications. Vol. 3 (2012), 1–6.
2. W. W. Breckner: *Stetigkeitsaussagen fur eine Klasse verallgemeinerter konvexer funktionen in topologischen lin-earen Rumen.* Publ. Inst. Math. **23** (1978), 13–20.
3. Deepmala: *A Study on Fixed Point Theorems for Nonlinear Contractions and its Applications.* Ph. D. Thesis (2014), Pt. Ravishankar Shukla University, Raipur (Chhatisgarh) India 492 010.

4. Deepmala and H. K. Pathak: *A study on some problems on existence of solutions for nonlinear functional-integral equations*. Acta Mathematica Scientia, **33** B(5) (2013), 1305–1313.

5. S. S. Dragomir:*Inequalities of Hermite-Hadamard type for $h-$convex functions on linear spaces*. Preprint.

6. S. S. Dragomir:*$n-$points inequalities of Hermite-Hadamard type for $h-$convex functions on linear spaces*. Preprint.

7. S. S. Dragomir, J. Pecaric and L. E. Persson: *Some inequalities of Hadamard type*. Soochow J. Math. 21, (1995), 335–341.

8. E. K. Godunnova and V. I. Levin: *Inequalities for functions of a broad class that contains convex, monotone and some other forms of functions*. Numerical Mathematics and Mathematical Physics (Russian), Moskov. Gos. Ped. Inst., Moscow, **166**(1985), 138-142.

9. H. Hudzik and L. Maligranda:*Some remarks on s-convex functions*. Aequationes Math. **48** (1994), 100-111.

10. J. L. W. V. Jensen: *Sur les fonctions convexes et les inegalites entre les voleurs mogernmes*. Acta. Math., **30**(1906), 175-193.

11. V. N. Mishra: *Some Problems on Approximations of Functions in Banach Spaces*. Ph. D. Thesis, (2007), Indian Institute of Technology, Roorkee - 247 667, Uttarakhand, India.

12. V. N. Mishra , Kejal Khatri and L. N. Mishra: *Using Linear Operators to Approximate Signals of $L_p(\alpha, p), (p \geq 1)$-Class*. Filomat 27:2 (2013), 353-363, DOI10.2298/FIL1302353M.

13. V. N. Mishra and Kejal Khatri: *Degree of approximation of functions $\widetilde{f} \in H_w$ class by the $(N_p.E^1)$ means in the Hölder metric*. International Journal of Mathematics and Mathematical Sciences, Vol. (2014), Article ID 837408, 9 pages.

14. V. N. Mishra , Kejal Khatri and L. N. Mishra: *Approximation of Functions belonging to Lip$(\xi(t), r)$ class by $(N, p_n)(E, q)$ Summability of Conjugate Series of Fourier series*. Journal of Inequalities and Applications - a Springer Open Access Journal, 2012:296 (2012).DOI:10.1186/1029-242X-2012-296.

15. V. N. Mishra , Kejal Khatri and L. N. Mishra: *Product Summability Transform of Conjugate Series of Fourier series*. International Journal of Mathematics and Mathematical Sciences, Vol. 2012 (2012), Article ID 298923, 13 pages, DOI: 10.1155/2012/298923.

16. L. N. Mishra , V. N. Mishra , Kejal Khatri and Deepmala: *On the Trigonometric Approximation of Signals (Functions) Belonging to Generalized Weighted Lipschitz $W(L^r, \zeta(t))(r \geq 1)$-Class by Matrix $(C^1.N_p)$ Operator of Conjugate Series of its Fourier series*. Applied Mathematics and Computation. **237** (2014), 252-263.

17. V. N. Mishra and L. N. Mishra: *Trigonometric Approximation of Signals (Functions) in $L_p(p1)$-norm*. International Journal of Contemporary Mathematical Sciences. Vol. 7, no. 19, 2012, 909-918.

18. V. N. Mishra , M. L. Mittal , U. Singh: *On best approximation in locally convex space*. Varhmihir Journal of Mathematical Sciences India, Vol.6, No.1, (2006), 43-48.

19. D. S. Mitrinovic: *Analytic Inequalities*. Springer, Berlin, 1970.
20. D. S. Mitrinovic and J. Pecaric: *Note on a class of functions of Godunova and Levin*. C. R. Math. Rep. Acad. Sci. Can. **12** (1990), 33-36.
21. D. S. Mitrinovic , J. Pecaric and A. M. Fink: *Classical and new inequalities in analysis*. Kluwer Academic, Dordrecht, 1993.
22. P. P. Murthy , L. N. Mishra and U. D. Patel: *n-tupled fixed point theorems for weak-contraction in partially ordered complete G-metric spaces*. New Trends in Mathematical Sciences, Vol. 3, No. 4 (2015), 50-75.
23. P. O. Olanipekun and A. A. Mogbademu: *Inequalities of the Jensen type for a new class of convex functions*. Nonlinear Functional Analysis and Applications, Vol. 20, No 4 (2015), 645-658.
24. E. Ozdemir , A. Ekinci and O. Akdemir: *Some new integral inequalities for several kinds of convex functions*. (2012), Arxiv.org/pdf/1202.2003. 1-11.
25. M. E. Özdemir: *On the properties of the s_2-Godunova-Levin type function with its inequalities*. RGMIA, Research Report Collection, **17**(2014) Article 46.
26. I. M. R. Pinheiro: *Jensen's inequality in detail and s-convex functions*. Int. Journal of Math. Analysis, vol.3, 2009, no.2, 95-98.
27. M. Z. Sakariya , E. Set and M. E. Özdemir: *On some new inequalities of Hadamard type involving $h-$convex functions*. Acta Math. Univ. Comenianae, Vol. LXXIX, 2 (2010), 265-272.
28. S. Varosanec: *On $h-$convexity*. J. Math. Anal. Appl., **326** (2007), 303-311.

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