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SOME CONSTRUCTIONS OF PROJECTIVELY RELATED SPHERICALLY SYMMETRIC FINSLER METRICS

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Abstract. The class of spherically symmetric Finsler metrics forms a rich and important class of Finsler metrics. In this paper, for an arbitrary spherically symmetric Finsler metric, we construct four classes of Finsler metrics which are projectively related to it.
Keywords: Finsler metrics, spherically symmetric Finsler metrics.

1. Introduction

Two Finsler metrics F and $\bar{F}$ on a manifold M are called projectively related if any geodesic of the first is also geodesic for the second and vice versa. In [7], Rapcsák gave a generalization of the previous conditions for the case of a pair of projectively equivalent Finsler metrics, that is, a pair of metrics having the same geodesics up to reparametrization. Indeed, he proved that F and $\bar{F}$ are projectively related if and only if there is a scalar function $P = P(x, y)$ defined on TM_0 such that

$$G^i = \bar{G}^i + Py^i,$$

where G^i and $\bar{G}^i$ are the geodesic spray coefficients of F and $\bar{F}$, respectively. In this case, P is called the projective factor.

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In order to find some interesting examples of projectively related metrics, one can consider the class of spherically symmetric Finsler metrics [2, 9]. A Finsler metric $F = F(x, y)$ on a domain $\Omega \subseteq \mathbb{R}^n$ is called spherically symmetric metric if it is invariant under any rotations in $\mathbb{R}^n$. Indeed, the class of spherically symmetric metrics in the Finsler setting was first introduced by S.F. Rutz who studied the spherically symmetric Finsler metrics in 4-dimensional timespace and generalized the classic Birkhoff theorem in general relativity to the Finsler case [8].

In [14], and by solving the equations of Killing fields generated by rotations, Zhou showed that (Ω, F) is spherically symmetric if and only if F can be written in a special form. Indeed, according to the equation of Killing fields, there exists a positive function ϕ depending on two variables so that F can be written as

$$F = |y|\phi\Big(|x|, \frac{\langle x, y\rangle}{|y|}\Big),$$

where x is a point in the domain Ω, y is a tangent vector at the point x and $\langle , \rangle$, $| \cdot |$ are standard inner product and norm in $\mathbb{R}^n$, respectively.

Let us put $u := |y|$, $v := \langle x, y\rangle$, $r := |x|$ and $s :=< x, y > /|y|$. It is proved that a Finsler metric F on a convex domain $\Omega \subseteq \mathbb{R}^n$ is spherically symmetric if and only if there exists a positive function $\phi := \phi(r, u, v)$, such that

$$F(x, y) = \phi(|x|, |y|, \langle x, y\rangle),$$

where

$$|x| = \sqrt{\sum_{i=1}^{n}(x^i)^2}, \quad |y| = \sqrt{\sum_{i=1}^{n}(y^i)^2}, \quad \langle x, y\rangle = \sum_{i=1}^{n} x^i y^i.$$

Thus, some of the well-known Finsler metrics are spherically symmetric metrics, namely, Funk metric, Bryant metric, Berwald metric, etc.

In [6], Mo-Zhou classifies the spherically symmetric Landsberg metrics in $\mathbb{R}^n$ and finds some exceptional almost regular metrics that do not belong to Berwald type. Then, Huang-Mo has found the equations that characterize spherically symmetric Finsler metrics of scalar flag curvature [4]. In [13], Zhou investigated projectively flat spherically symmetric Finsler metric with constant flag curvature in $\mathbb{R}^n$. Recently, Guo-Liu-Mo consider spherically symmetric metrics with isotropic Berwald curvature [3].

In this paper, for an arbitrary spherically symmetric Finsler metric F, we construct some classes of Finsler metrics which are projectively related to it.

Theorem 1.1. *Let $F = u\phi(r, s)$ be a spherically symmetric Finsler metric in $\mathbb{R}^n$. Define*

$$\bar{\phi}(r, s) := \phi(r, s) + \frac{\sqrt{r^2 - s^2}}{r^2} + cs, \tag{1.1}$$

$$\hat{\phi}(r, s) := \phi(r, s) + p(r)s, \tag{1.2}$$

where c is a real constant and $p = p(r)$ is a smooth function in $\mathbb{R}^n$. Then $\bar{F} = u\bar{\phi}(r,s)$ and $\hat{F} = u\hat{\phi}(r,s)$ are projectively related to F.

There is a complicated class of Finsler metrics which is projectively related to a spherically symmetric Finsler metric $F = u\phi(r,s)$.

Theorem 1.2. *Let $F = u\phi(r,s)$ be a spherically symmetric Finsler metric in $\mathbb{R}^n$. Define*

$$\bar{\phi}(r,s) := -\int\int \frac{A_s(r,s)}{s} dsds, \tag{1.3}$$

where

$$A(r,s) := c(r^2 - s^2)^m(\phi - s\phi_s)^{2m+1} + k(r^2 - s^2)^n(\phi - s\phi_s)^{2n+1}, \tag{1.4}$$

$A_s := \partial A/\partial s$ and m, n, k are real constants. Then F is projectively related to $\bar{F} = u\bar{\phi}(r,s)$.

Similarly, we can construct another Finsler metrics that is projectively related to a spherically symmetric Finsler metric.

Theorem 1.3. *Let $F = u\phi(r,s)$ be spherically symmetric Finsler metric in $\mathbb{R}^n$. Define*

$$\bar{\phi}(r,s) := -\int\int \frac{B_s(r,s)}{s} dsds, \tag{1.5}$$

where

$$B(r,s) := \Big[(r^2 - s^2)^m + (\phi - s\phi_s)^{-2m}\Big]^{-\frac{1}{2m}}, \tag{1.6}$$

$B_s := \partial B/\partial s$ and m is a real constant. Then F is projectively related to $\bar{F} = u\bar{\phi}(r,s)$.

2. Preliminary

Let M be an n-dimensional C^∞ manifold, $TM = \bigcup_{x\in M} T_xM$ the tangent space and $TM_0 := TM - \{0\}$ the slit tangent space of M. A Finsler structure on the manifold M is a function $F : TM \to [0,\infty)$ with the following properties: (i) F is C^∞ on TM_0; (ii) F is positively 1-homogeneous on the fibers of tangent bundle TM, i.e., $F(x,\lambda y) = \lambda F(x,y)$, $\forall\lambda > 0$; (iii) The quadratic form $\mathbf{g}_y : T_xM \times T_xM \to \mathbb{R}$ is positive-definite on T_xM

$$\mathbf{g}_y(u,v) := \frac{1}{2}\frac{\partial^2}{\partial s\partial t}\Big[F^2(y + su + tv)\Big]_{s=t=0}, \quad u, v \in T_xM.$$

Then, the pair (M, F) is called a Finsler manifold.

Given a Finsler manifold (M, F), then a global vector field $\mathbf{G}$ is induced by F on slit tangent bundle $TM_0 = TM - \{0\}$, which in a standard coordinate (x^i, y^i) for TM_0 is given by

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i},$$

where $G^i = G^i(x, y)$ are called spray coefficients and given by

$$G^i = \frac{1}{4} g^{il} \Big[\frac{\partial^2 F^2}{\partial x^k \partial y^l} y^k - \frac{\partial F^2}{\partial x^l}\Big].$$

$\mathbf{G}$ is called the spray associated to F.

Two Finsler metrics F and $\bar{F}$ on a manifold M are said to be (pointwise) projectively related if they have the same geodesics as point sets. Hereby, there is a function $P(x, y)$ defined on TM_0 such that

$$G^i = \bar{G}^i + P y^i,$$

on coordinates (x^i, y^i) on TM_0 , where G^i and $\bar{G}^i$ are the geodesic spray coefficients of F and $\bar{F}$, respectively, and the projective factor $P : TM \to \mathbb{R}$ is positively homogeneous of degree one with respect to y.

Let

$$D^i_{j\ kl} := \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\big(G^i - \frac{1}{n+1}\frac{\partial G^m}{\partial y^m} y^i\big). \tag{2.1}$$

It is easy to verify that $\mathcal{D} := D^i_{j\ kl} dx^j \otimes \partial_i \otimes dx^k \otimes dx^l$ is a well-defined tensor on slit tangent bundle TM_0. We call $\mathcal{D}$ the Douglas tensor. The Douglas tensor $\mathcal{D}$ is a non-Riemannian projective invariant, namely, if two Finsler metrics F and $\bar{F}$ are projectively equivalent, then the Douglas tensor of F is the same as that of $\bar{F}$. Finsler metrics with vanishing Douglas tensor are called Douglas metrics.

A Finsler metric F on a domain $\Omega \subseteq \mathbb{R}^n$ is called spherically symmetric if it is invariant under any rotations in $\mathbb{R}^n$. According to the equation of Killing fields, there exists a positive function ϕ depending on two variables so that F can be written as $F = |y|\phi(|x|, \frac{\langle x,y\rangle}{|y|})$, where x is a point in the domain Ω, y is a tangent vector at the point x and $\langle , \rangle$, $|\cdot|$ are standard inner product and norm in Euclidean space, respectively.

Lemma 2.1. ([14]) A Finsler metric F on a convex domain $\Omega \subseteq \mathbb{R}^n$ is spherically symmetric if and only if there exists a positive function $\phi(r, u, v)$, such that $F(x, y) = \phi(|x|, |y|, \langle x, y\rangle)$, where $|x| = \sqrt{\Sigma^n_{i=1}(x^i)^2}$, $|y| = \sqrt{\Sigma^n_{i=1}(y^i)^2}$ and $\langle x, y\rangle = \Sigma^n_{i=1} x^i y^i$.

3. Proof of Theorem 1.1

Let $F = |y|\phi(|x|, \langle x, y\rangle/|y|)$ be a spherically symmetric Finsler metric. Let us put

$$r := |x|, \quad u := |y|, \quad v := \langle x, y\rangle, \quad s := \frac{\langle x, y\rangle}{|y|}.$$

Then F has the expression $F = u\phi(r, s)$ and it does not always need to be (α, β)-metrics such as the Bryant metrics.

In [6], Mo-Zhou obtain the geodesic spray coefficients of spherically symmetric Finsler metric $F = u\phi(r, s)$ as follows

$$G^i = uPy^i + u^2Qx^i, \tag{3.1}$$

where

$$P = -\frac{1}{\phi}\Big(s\phi + (r^2 - s^2)\phi_s\Big)Q + \frac{1}{2r\phi}\Big(s\phi_r + r\phi_s\Big), \tag{3.2}$$

$$Q = \frac{1}{2r}\frac{-\phi_r + s\phi_{rs} + r\phi_{ss}}{\phi - s\phi_s + (r^2 - s^2)\phi_{ss}}. \tag{3.3}$$

Then we have the following.

Lemma 3.1. *Let $F = u\phi(r, s)$ be a spherically symmetric Finsler metric in $\mathbb{R}^n$. Suppose that $\tilde{\phi}(r, s) := \phi(r, s) + k(r, s)$, where $k = k(r, s)$ is a smooth function satisfies*

$$k_r - sk_{rs} - rk_{ss} = 0, \tag{3.4}$$

$$k - sk_s + (r^2 - s^2)k_{ss} = 0. \tag{3.5}$$

Then F is projectively related to $\tilde{F} := u\tilde{\phi}(r, s)$.

Proof. Let G^i and $\tilde{G}^i$ be geodesic spray coefficients of F and $\tilde{F}$, respectively. By (3.1), we have

$$\tilde{G}^i = u\tilde{P}y^i + u^2\tilde{Q}x^i, \tag{3.6}$$

where

$$\tilde{P} = -\frac{1}{\tilde{\phi}}\Big[s\tilde{\phi} + (r^2 - s^2)\tilde{\phi}_s\Big]\tilde{Q} + \frac{1}{2r\tilde{\phi}}\Big(s\tilde{\phi}_r + r\tilde{\phi}_s\Big), \tag{3.7}$$

$$\tilde{Q} = \frac{1}{2r}\frac{-\tilde{\phi}_r + s\tilde{\phi}_{rs} + r\tilde{\phi}_{ss}}{\tilde{\phi} - s\tilde{\phi}_s + (r^2 - s^2)\tilde{\phi}_{ss}}. \tag{3.8}$$

Then by assumption, we get

$$\tilde{Q} = \frac{1}{2r}\frac{-\phi_r + s\phi_{rs} + r\phi_{ss} + \Big[- k_r + sk_{rs} + rk_{ss}\Big]}{\phi - s\phi_s + (r^2 - s^2)\phi_{ss} + \Big[k - sk_s + (r^2 - s^2)k_{ss}\Big]}. \tag{3.9}$$

By (3.3), (3.4), (3.5) and (3.9), we get

$$\tilde{Q} = Q. \tag{3.10}$$

Then (3.1), (3.6) and (3.10) imply that

$$G^i - \tilde{G}^i = u(P - \tilde{P})y^i. \tag{3.11}$$

Thus F is projectively related to $\tilde{F}$. □

Proof of Theorem 1.1: Let $F = u\phi(r,s)$ be a spherically symmetric Finsler metric in $\mathbb{R}^n$. Put

$$\bar{\phi}(r,s) := \phi(r,s) + \frac{\sqrt{r^2 - s^2}}{r^2} + cs, \tag{3.12}$$

$$\hat{\phi}(r,s) := \phi(r,s) + p(r)s, \tag{3.13}$$

where c is a real constant and $p = p(r)$ is a smooth function in $\mathbb{R}^n$. Then by direct computations, we find that (3.12) and (3.13) satisfy (3.4) and (3.5) with

$$k := \frac{\sqrt{r^2 - s^2}}{r^2} + cs, \quad \text{and} \quad k := p(r)s.$$

Then by Lemma 3.1, we conclude that F is projectively related to $\bar{F}$ and $\hat{F}$. □

In [5], Mo-Solórzano-Tenenblat considers spherically symmetric Douglas metrics. By Theorem 1.1, we get the following.

Corollary 3.1. *Let $F = u\phi(r,s)$ be a spherically symmetric Douglas metric in $\mathbb{R}^n$. Let us define*

$$\bar{\phi}(r,s) := \phi(r,s) + \frac{\sqrt{r^2 - s^2}}{r^2} + cs, \quad \hat{\phi}(r,s) := \phi(r,s) + p(r)s,$$

where such that c is a constant and $p = p(r)$ is a smooth function in $\mathbb{R}^n$. Then $\bar{F} = u\bar{\phi}(r,s)$ and $\hat{F} = u\hat{\phi}(r,s)$ are Douglas metrics.

4. Proof of Theorem 1.2

Lemma 4.1. *Let $F = u\phi(r,s)$ and $\tilde{F} = u\tilde{\phi}(r,s)$ be a spherically symmetric Finsler metric in $\mathbb{R}^n$. Suppose that the following holds*

$$\frac{-sK_r(r,s) - rK_s(r,s)}{sK(r,s) - (r^2 - s^2)K_s(r,s)} = \frac{-s\tilde{K}_r(r,s) - r\tilde{K}_s(r,s)}{s\tilde{K}(r,s) - (r^2 - s^2)\tilde{K}_s(r,s)} \tag{4.1}$$

where

$$K(r,s) := \phi(r,s) - s\phi_s(r,s),$$
$$\tilde{K}(r,s) := \tilde{\phi}(r,s) - s\tilde{\phi}_s(r,s).$$

Then F is projectively related to $\bar{F}$.

Proof. Let G^i and $\tilde{G}^i$ be geodesic spray coefficients of F and $\tilde{F}$. Then by (3.1), we have

$$G^i = uPy^i + u^2Qx^i, \tag{4.2}$$
$$\tilde{G}^i = u\tilde{P}y^i + u^2\tilde{Q}x^i, \tag{4.3}$$

where

$$Q = \frac{1}{2r}\frac{-\phi_r + s\phi_{rs} + r\phi_{ss}}{\phi - s\phi_s + (r^2-s^2)\phi_{ss}} = \frac{-sK_r(r,s) - rK_s(r,s)}{sK(r,s) - (r^2-s^2)K_s(r,s)} \tag{4.4}$$

$$\tilde{Q} = \frac{1}{2r}\frac{-\tilde{\phi}_r + s\tilde{\phi}_{rs} + r\tilde{\phi}_{ss}}{\tilde{\phi} - s\tilde{\phi}_s + (r^2-s^2)\tilde{\phi}_{ss}} = \frac{-s\tilde{K}_r(r,s) - r\tilde{K}_s(r,s)}{s\tilde{K}(r,s) - (r^2-s^2)\tilde{K}_s(r,s)} \tag{4.5}$$

By (4.1), we get

$$\tilde{Q} = Q. \tag{4.6}$$

Then by (4.2), (4.3) and (4.6) we have

$$G^i - \tilde{G}^i = u(P - \tilde{P})y^i. \tag{4.7}$$

This means that F is projectively related to $\bar{F}$. □

Proof of Theorem 1.2: By (1.3) we have

$$A_s(r,s) = -s\bar{\phi}_{ss}(r,s). \tag{4.8}$$

Then

$$A(r,s) = \bar{\phi}(r,s) - s\bar{\phi}_s(r,s). \tag{4.9}$$

Plugging

$$D(r,s) := \phi(r,s) - s\phi_s(r,s)$$

in (1.4) implies that

$$A(r,s) = c(r^2-s^2)^m D(r,s)^{2m+1} + k(r^2-s^2)^n D(r,s)^{2n+1}. \tag{4.10}$$

By a direct computation, we have

$$-sA_r(r,s) - rA_s(r,s) = \frac{-1}{D(r,s)}\big[sD_r(r,s) + rD_s(r,s)\big]\mathbb{A}, \tag{4.11}$$

where

$$\mathbb{A} := c(1+2m)(r^2-s^2)^m D(r,s)^{2m+1} + k(1+2n)(r^2-s^2)^n D(r,s)^{2n+1}.$$

Similarly, we get

$$sA(r,s) - (r^2-s^2)A_s(r,s) = \frac{1}{D(r,s)}\big[sD(r,s) - (r^2-s^2)D_s(r,s)\big]\mathbf{A}, \tag{4.12}$$

where

$$\mathbf{A} := c(1+2m)(r^2-s^2)^m D(r,s)^{2m+1} + k(1+2n)(r^2-s^2)^n D(r,s)^{2n+1}.$$

By (4.11) and (4.12), it follows that

$$\frac{-sA_r(r,s) - rA_s(r,s)}{sA(r,s) - (r^2-s^2)A_s(r,s)} = \frac{-sD_r(r,s) - rD_s(r,s)}{sD(r,s) - (r^2-s^2)D_s(r,s)}. \tag{4.13}$$

Thus by Lemma 4.1, F is projectively related to $\bar{F}$. □

5. Proof of Theorem 1.3

Proof of Theorem 1.3: By (1.5) we have

$$B_s(r,s) = -s\bar{\phi}_{ss}(r,s). \tag{5.1}$$

(5.1) implies that

$$B(r,s) = \bar{\phi}(r,s) - s\bar{\phi}_s(r,s). \tag{5.2}$$

Plugging $E(r,s) := \phi(r,s) - s\phi_s(r,s)$ in (1.6) implies that

$$B(r,s) = \Big[(r^2-s^2)^m + E(r,s)^{-2m}\Big]^{-\frac{1}{2m}} \tag{5.3}$$

A direct computation shows that

$$-sB_r(r,s) - rB_s(r,s) = \frac{\big[-sE_r(r,s) - rE_s(r,s)\big]B(r,s)}{E(r,s)\Big[(r^2-s^2)^m E(r,s)^{2m} + 1\Big]}, \tag{5.4}$$

$$sB(r,s) - (r^2-s^2)B_s(r,s) = \frac{\big[sE(r,s) - (r^2-s^2)E_s(r,s)\big]B(r,s)}{E(r,s)\Big[(r^2-s^2)^m E(r,s)^{2m} + 1\Big]}. \tag{5.5}$$

By (5.4) and (5.5) we get

$$\frac{-sB_r(r,s) - rB_s(r,s)}{sB(r,s) - (r^2-s^2)B_s(r,s)} = \frac{-sE_r(r,s) - rE_s(r,s)}{sE(r,s) - (r^2-s^2)E_s(r,s)} \tag{5.6}$$

By Lemma 4.1, we conclude that F is projectively related to $\bar{F}$. □

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