FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 40, No 2 (2025), 421–438
https://doi.org/10.22190/FUMI240515032M
Original Scientific Paper

COMMON FIXED POINT FOR $w-$COMPATIBLE MAPS IN A BIPOLAR METRIC SPACE

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Abstract. The purpose of this paper is to establish some common fixed point theorems of $w-$compatible maps in bipolar metric spaces by employing a comparison function ϕ instead of some altering distance functions. We employ generalized type contraction conditions involving the comparison function ϕ to enunciate common fixed point theorems. Further, we provide illustrative examples to uphold our results.
Keywords: $w-$compatible maps, bipolar metric spaces.

1. Introduction

1n 1905, Fréchet [11] introduced a distance space, that is“ Let X be a non-empty set and $d : X \times X \to [0, \infty)$ be a function, then the pair (X, d) is called a metric space if the following conditions hold:

1. $d(x, y) = 0$ if and only if $x = y$ where $x, y \in X$,
2. $d(x, y) = d(y, x)$ for all $x, y \in X$,
3. $d(x, z) \leqslant d(x, y) + d(y, z)$, for all $x, y, z \in X$ (Triangle Inequality)”.

Received May 15, 2024, accepted: December 26, 2024
Communicated by Dijana Mosić
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2020 *Mathematics Subject Classification.* Primary 47H10, 54H25, 54E50

Metric spaces can be generalized in many ways, one way is to relax some conditions. Some earlier generalizations of the space are pseodometric space, quasimetric space, semimetric space, nontriangular metric space, other generalization includes $b-$metric space [3], dislocated metric space [17], rectangular metric space [7], etc. The second way to generalize metric space is to change the codomain of the function d. For example cone metric space [18], complex-valued metric space [2], quaternion-valued metric space [1], etc. The third way is to change the domain of d of the space. This category includes $2-$metric space [13], $D-$metric space [10], $G-$metric space [25], $S-$metric space [34], Bipolar metric space [26] etc. In rectangular metric, space quadrilateral inequality is satisfied, which is a weaker assumption than triangle inequality. The concept of quadrilateral inequality was used to define the bipolar metric space. The First fixed point theorem has been proved in metric space and this fixed point theorem has been extended by the researchers in different directions, one of these directions is by changing the space also. In line with this, fixed point theorems of many contractive type mappings have been studied in bipolar metric space (see [6, 12, 15, 16, 21, 24, 26–29, 31, 32, 37]). Fixed points of $\phi-$contraction which is a generalization of Banach contraction and Rakotch type contraction [33] have been studied in metric space (see [5]). Gillespie and Williams [14] introduced the expanding map. After that many authors studied the fixed point theorems of expansive type mappings (see [8, 9, 19, 36]). Kishore, Agarwal, Rao and Rao [21] extended the definition of compatible mappings [20] in bipolar metric space to study the common fixed point theorems.
In this paper, we introduce ϕ-contraction and ϕ-expansive maps in a bipolar metric space and study common fixed points of continuous and discontinuous self-mappings. To study the common fixed point theorems, we also introduce the $w-$compatible maps in bipolar metric space which is a generalization of compatible maps and also weak compatibility of mappings. These results are extensions of many existing results, specially from metric fixed point theory.

1.1. Preliminaries

In this section, we recall definition of bipolar metric space with some basic concepts from [26], which will be essential for our results.

Definition 1.1. Let A and B be the two non-empty sets and $\rho : A \times B \to [0, +\infty)$ be a function. The triplet (A, B, ρ) is called bipolar metric space and ρ is called bipolar metric on (A, B), if the following conditions hold:

(BP_1) $\rho(a, b) = 0$ if and only if $a = b$, where $(a, b) \in A \times B$,

(BP_2) If $a, b \in A \cap B$ then $\rho(a, b) = \rho(b, a)$,

(BP_3) $\rho(a_1, b_2) \leq \rho(a_1, b_1) + \rho(a_2, b_1) + \rho(a_2, b_2)$ for all $a_1, a_2 \in A$ and $b_1, b_2 \in B$.

Definition 1.2. Let (A, B, ρ) be a bipolar metric space. Elements of A, B and $A \cap B$ are called left, right and central points, respectively. A sequence in A and a sequence in B are called left and right sequences, respectively. By a sequence, we mean either a left or right sequence.

1. A sequence $\langle t_n \rangle$ is said to be convergent to a point t if and only if $\langle t_n \rangle$ is a left sequence, t is a right point and $\lim_{n \to +\infty} \rho(t_n, t) = 0$; or $\langle t_n \rangle$ is a right sequence, t is a left point and $\lim_{n \to +\infty} \rho(t, t_n) = 0$.
2. A sequence $\langle (a_n, b_n) \rangle$ in $A \times B$ is called a bisequence on (A, B). This sequence is simply denoted by (a_n, b_n). If both the sequences $\langle a_n \rangle$ and $\langle b_n \rangle$ convergent, then the bisequence (a_n, b_n) is said to be convergent. If both the sequences $\langle a_n \rangle$ and $\langle b_n \rangle$ converge to the same point $\upsilon \in A \cap B$, then (a_n, b_n) is called bi-convergent.
3. If $\lim_{n,m \to +\infty} d(a_n, b_m) = 0$, then the bisequence (a_n, b_n) is called a Cauchy bisequence. In a bipolar metric space, every convergent Cauchy bi-sequence is bi-convergent.
4. A bipolar metric space is called complete, if every Cauchy bisequence is convergent, hence biconvergent.

Example 1.1. Let X be the class of singleton subsets of $\mathbb{R}^2$ and Y be the class of nonempty bounded subsets of metric space $(\mathbb{R}^2, d)$ where

$$d(x, y) = |x_1 - y_1| + |x_2 - y_2| \quad \text{for all} \quad x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$$

. We define a function $\rho : X \times Y \to [0, \infty)$ by

$$\rho(\{x\}, A) = \sup\{d(x, y) : y \in A\}$$

We will show that (X, Y, ρ) is a bipolar metric space.

(BP_1) : It is clear that $\rho(\{x\}, \{x\}) = 0$, for every $\{x\} \in X = X \cap Y$. Let $\rho(\{x\}, A) = 0$, then $\sup\{d(x, y) : y \in A\} = 0$. This implies $A = \{x\}$.

(BP_2) : $\rho(\{x\}, \{y\}) = \rho(\{y\}, \{x\})$ for all $\{x\}, \{y\} \in X \cap Y$.

(BP_3) : Let $x = (x_1, x_2), w = (w_1, w_2) \in \mathbb{R}^2$ and $A, B \in Y$ then

$$\begin{aligned}
\rho(\{x\}, A) &= \sup\{d(x, y) : y \in A\} \\
&\leqslant \sup\{d(x, z) + d(w, z) + d(w, y) : y \in A, z \in B\} \\
&\leqslant \sup\{d(x, z) : z \in B\} + \sup\{d(w, z) : z \in B\} + \sup\{d(w, y) : y \in A\} \\
&= \rho(\{x\}, B) + \rho(\{w\}, B) + \rho(\{w\}, A)
\end{aligned}$$

So, (X, Y, ρ) is a bipolar metric space. It can be shown that it is a complete bipolar metric space.

Definition 1.3. Let A_1, B_1, A_2 and B_2 be four sets. A function $f : A_1 \cup B_1 \to A_2 \cup B_2$ is said to be a covariant map if $f(A_1) \subseteq A_2$ and $f(B_1) \subseteq B_2$ and is denoted as $f : (A_1, B_1) \rightrightarrows (A_2, B_2)$. In particular, if (A_1, B_1, ρ_1) and (A_2, B_2, ρ_2) are two bipolar metric spaces then we use the notation $f : (A_1, B_1, \rho_1) \rightrightarrows (A_2, B_2, \rho_2)$ for covariant map f.

Definition 1.4. Let (A_1, B_1, ρ_1) and (A_2, B_2, ρ_2) be two bipolar metric spaces. A map $f : (A_1, B_1) \rightrightarrows (A_2, B_2)$ is said to be continuous at a point $a_0 \in A_1$, if for any given $\varepsilon > 0$, there exists $\delta > 0$ such that $b \in B_1$ and $\rho_1(a_0, b) < \delta$ implies that $\rho_2(f(a_0), f(b)) < \varepsilon$. It is continuous at a point $b_0 \in B_1$ if for any given $\varepsilon > 0$, there exists $\delta > 0$ such that $a \in A_1$ and $\rho_1(a, b_0) < \delta$ implies that $\rho_2(f(a), f(b_0)) < \varepsilon$. If f is continuous at each point $a \in A_1 \cup B_1$, then it is called continuous.

This definition implies that a covariant map $f : (A_1, B_1) \rightrightarrows (A_2, B_2)$ is continuous if and only if $\{t_n\}$ converges to t on (A_1, B_1, ρ_1) implies $\{f(t_n)\}$ converges to $f(t)$ on (A_2, B_2, ρ_2).

Definition 1.5. A function $g : A_1 \cup B_1 \to A_2 \cup B_2$ is said to be a contravariant map if $g(A_1) \subseteq B_2$ and $g(B_1) \subseteq A_2$ and is denoted as $g : (A_1, B_1) \rightleftarrows (A_2, B_2)$.

Definition 1.6. A contravariant map $f : (A_1, B_1, \rho_1) \rightleftarrows (A_2, B_2, \rho_2)$ is continuous if and only if it is continuous as a covariant map $f : (A_1, B_1, \rho_1) \rightrightarrows (B_2, A_2, \bar{\rho_2})$, where $\bar{\rho_2}$ is defined as $\bar{\rho_2}(y, x) = \rho_2(x, y)$, for all $(y, x) \in B_2 \times A_2$.

Definition 1.7. [5] Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function where $\mathbb{R}_+ = [0, +\infty)$. Then ϕ is called a comparison function if it satisfies the following conditions:

1. ϕ is monotonic increasing, i.e., $t_1 \leqslant t_2$ implies $\phi(t_1) \leqslant \phi(t_2)$;
2. $\{\phi^n(t)\}$ converges to 0 for all $t > 0$.

Remark 1.1. if ϕ is a comparison function then it has following properties:

1. $\phi(t) < t$ for all $t > 0$;
2. $\phi(0) = 0$.

Kishore, Agarwal, Rao and Rao [21] gave the definition of compatible mappings in bipolar metric space as follows:

Definition 1.8. [21] Let (X, Y, ρ) be a bipolar metric space and let $S, T : (X, Y) \rightrightarrows (X, Y)$ be two covariant maps then the pair $\{S, T\}$ is said to be compatible if and only if $\rho(TSx_n, STy_n) \to 0$ and $\rho(STx_n, TSy_n) \to 0$, whenever (x_n, y_n) is a sequence in (X, Y) such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Ty_n = t$ for some $t \in X \cap Y$.

Rao and Kishore [32] defined the following version of compatibility of two covariant maps.

Definition 1.9. [32] Let (X, Y, ρ) be a bipolar metric space and let $S, T : (X, Y) \rightrightarrows (X, Y)$ be two covariant maps then the ordered pair (S, T) is said to be compatible if and only if $\rho(TSx_n, STy_n) \to 0$, whenever (x_n, y_n) is a sequence in (X, Y) such that $\lim\limits_{n\to\infty} Sx_n = \lim\limits_{n\to\infty} Ty_n = t$ for some $t \in X \cap Y$.

Remark 1.2. From the above two definitions, it is clear that if the ordered pairs (S,T) and (T,S) both are compatible, then the pair $\{S,T\}$ is compatible.

Definition 1.10. [32] If S and T commute at all of their coincidence points then S and T are called weakly compatible.

Proposition 1.1. *[24] Let (X,Y,ρ) be a bipolar metric space and let $S,T : (X,Y,\rho) \rightrightarrows (X,Y,\rho)$ be two covariant maps satisfying the following condition for all $(x,y) \in X \times Y$:*

$$\rho(Tx,Ty) \leqslant \rho(Sx,Sy)$$

, if S is a continuous function then T is also a continuous function.

2. Main Results

We introduce ϕ-contraction in bipolar metric space which is an extension of ϕ-contraction [5] defined in metric space.

Definition 2.1. Let (A,B,ρ) be a bipolar metric space. A mapping $T : (A,B,\rho) \rightrightarrows (A,B,\rho)$ is said to be a ϕ- contraction if there exists a comparison function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\rho(Ta,Tb) \leqslant \phi(\rho(a,b)) \text{ for all } (a,b) \in A \times B. \tag{2.1}$$

We now generalize the notion of expansive mapping (see [36]) into bipolar metric space and give the following definition.

Definition 2.2. Let (A,B,ρ) be a bipolar metric space. A mapping $T : (A,B,\rho) \rightleftarrows (A,B,\rho)$ is said to be a ϕ-expansive mapping if there exists a comparison function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\phi(\rho(Tb,Ta) \geqslant \rho(a,b) \text{ for all } (a,b) \in A \times B. \tag{2.2}$$

Remark 2.1. If T is ϕ-expansive mapping then it is injective for let $Tx = Ty$ then $0 = \phi(0) = \phi(\rho(Ty,Tx) \geqslant \rho(x,y)$. So $x = y$.

Remark 2.2. If T is ϕ-expansive mapping then T^{-1} is contractive on $T(A \cup B)$ for let $u \in T(B)$ and $v \in T(A)$ such that $u \neq v$ then $Ty = u$ and $Tx = v$ for some $(x,y) \in A \times B$. So $\rho(u,v) > \phi(\rho(u,v)) = \phi(\rho(Ty,Tx)) \geqslant \rho(x,y) = \rho(T^{-1}v, T^{-1}u)$.

We also give the following definition, which is weaker than definition 1.8.

Definition 2.3. Let (X,Y,ρ) be a bipolar metric space and let $S,T : (X,Y) \rightrightarrows (X,Y)$ be two covariant maps, then the pair $\{S,T\}$ is said to be $w-$compatible if and only if $\lim_{n\to\infty} \rho(TSx_n, STy_n) = 0$ or $\lim_{n\to\infty} \rho(STx_n, TSy_n) = 0$, whenever (x_n, y_n) is a bisequence in (X,Y) such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Ty_n = t$ for some $t \in X \cap Y$.

The following example shows that a $w-$compatible mappings need not be compatible.

Example 2.1. Consider the bipolar metric space (X, Y, ρ), where $X = [0, \infty)$, $Y = (-\infty, 1]$ and ρ is defined by $\rho(x, y) = |x - y|$. Let T and S be two covariant maps defined by

$$Tx = \begin{cases} \frac{1}{x}, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \end{cases}$$

$$Ty = \begin{cases} 2y, & \text{if } y \leqslant 0 \\ \frac{1}{y}, & \text{if } 0 < y \leqslant 1 \end{cases}$$

$$Sx = xe^{-x},$$

$$Sy = \begin{cases} \pi y, & \text{if } y \leqslant 0 \\ ye^{-y}, & \text{if } 0 \leqslant y \leqslant 1 \end{cases}$$

where $x \in X, y \in Y$.
Now, let us assume that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Ty_n = t \in [0, 1]$$

for some bisequence (x_n, y_n).
We observe that the possible value of t is 0. So it is sufficient to consider two cases.

Case 1: $x_n = y_n = 0$ for all $n \in \mathbb{N}$. In this case

$$\lim_{n\to+\infty} \rho(TSx_n, STy_n) = 0 \text{ and } \lim_{n\to+\infty} \rho(STx_n, TSy_n) = 0$$

Case 2: $\{x_n\}$ approaches infinity with $x_n > 0$ and $\{y_n\}$ approaches zero with $y_n < 0$. In this case
$\lim_{n\to+\infty} \rho(TSx_n, STy_n) = \lim_{n\to+\infty} \rho(T(x_n e^{-x_n}), S(2y_n)) = \rho(\frac{1}{x_n}e^{x_n}, 2\pi y_n)$ diverges to $+\infty$. but $\lim_{n\to+\infty} \rho(STx_n, TSy_n) = 0$. Therefore the pair $\{S, T\}$ is a $w-$compatible mapping but not a compatible mapping.

The following propositions and lemmas will be used to prove our main theorems.

Proposition 2.1. *If the ordered pair (S, T) or (T, S) is compatible then the pair $\{S, T\}$ is$w-$compatible.*

Proof. The proof follows from definitions (2.3) and (1.9). □

Proposition 2.2. *Let (X, Y, ρ) be a bipolar metric space and let $S, T : (X, Y, \rho) \rightrightarrows (X, Y, \rho)$ be two covariant maps. If the pair $\{S, T\}$ is $w-$compatible or ordered pair (S, T) is compatible, then S and T are weakly compatible.*

Proof. This proposition can be easily proved by taking $x_n = y_n = u$ in the definition (2.3) and (1.9), where u is a coincidence point of S and T. □

Proposition 2.3. *Let (X, Y, ρ) be a bipolar metric space and let $S, T : (X, Y, \rho) \rightrightarrows (X, Y, \rho)$ be two covariant maps satisfying the following conditions for some comparison function ϕ:*

$$\rho(Tx, Ty) \leqslant \phi(\rho(Sx, Sy)) \text{ for all } (x, y) \in X \times Y$$

then

$$\rho(Tx, Ty) < \rho(Sx, Sy) \text{ if } Sx \neq Sy \text{ and}$$

$$\rho(Tx, Ty) \leqslant \rho(Sx, Sy) \text{ for all } x \in X, y \in Y.$$

Proof. The result is direct consequence of the properties of the comparison function ϕ. □

Lemma 2.1. *Let (X, Y, ρ) be a bipolar metric space, $T, S : (X, Y, \rho) \rightrightarrows (X, Y, \rho)$ be two covariant maps and $\{u_n\}$ be a sequence in X or in Y. If T, S and $\{u_n\}$ satisfy the following conditions:*

1. $Tu_n = Su_{n+1}$
2. $\lim_{n\to\infty} Su_n = Sw$ *for some* $w \in X \cap Y$
3. $\rho(Tx, Ty) \leqslant \rho(Sx, Sy)$ *for all* $(x, y) \in X \times Y$

Then w is a coincidence point of S and T. Further if S is injective and condition 3 is replaced by following stronger condition

4. $\rho(Tx, Ty) < \rho(Sx, Sy)$ for all $(x, y) \in X \times Y$ *with* $Sx \neq Sy$ *and* $Sx = Sy$ *implies* $Tx = Ty$.

Then w will be the unique coincidence point of S and T.

Proof. From condition 3, we have

$$\rho(Tu_n, Tw) \leqslant \rho(Su_n, Sw) \text{ if } \{u_n\} \subseteq X$$

or

$$\rho(Tw, Tu_n) \leqslant \rho(Sw, Su_n) \text{ if } \{u_n\} \subseteq Y.$$

So using condition 2, we get

$$\lim_{n\to\infty} Tu_n = Tw \in X \cap Y$$

But $Tu_n = Su_{n+1}$, so

$$\lim_{n\to\infty} Su_n = Tw.$$

Using condition 2, we get

$$Tw = Sw$$

Hence w is a coincidence point of S and T. Further let S be injective and the condition 4 holds, then as before w is a coincidence point. To prove the uniqueness of coincidence point, let v be another coincidence point of S and T. Then using condition 4, we have

$$\begin{aligned} \rho(Sw, Sv) &= \rho(Tw, Tv) < \rho(Sw, Sv) \text{ if } v \in Y \text{ or} \\ \rho(Sv, Sw) &= \rho(Tv, Tw) < \rho(Sv, Sw) \text{ if } v \in X. \end{aligned}$$

In both cases, we arrive at contradiction. So $w = v$. □

Lemma 2.2. *Let (X, Y, ρ) be a bipolar metric space, $T, S : (X, Y, \rho) \rightrightarrows (X, Y, \rho)$ be two covariant maps which are weakly compatible. S and T satisfy the following contractive condition $\rho(Tx, Ty) < \rho(Sx, Sy)$ for all $(x, y) \in X \times Y, Sx \neq Sy$ If $v \in X \cap Y$ is a coincidence point of S and T, then Sv is the unique common fixed point of S and T. Further let S be injective, then v will be the unique common fixed point of S and T.*

Proof. Let $Sv = Tv = u \in X \cap Y$, then by weak compatibility of S and T, we get

$$STv = TSv$$

This implies

$$Su = Tu$$

. That is u is also a coincidence point of S and T. We will show that u is the unique common fixed point of S and T. On the contrary, let us assume that $Su \neq u$,. Then using the given contractive condition, we get

$$\begin{aligned} \rho(Tu, u) &= \rho(TSv, Tv) \\ &< \rho(SSv, Sv) \\ &= \rho(Su, u) \\ &= \rho(Tu, u) \end{aligned}$$

which is a contradiction. So $Su = u = Tu$.
Further, let S be an injective map then we will show that $u = v$. Suppose not, then using contractive condition, we get

$$\rho(Tu, Tv) < \rho(Su, Sv) = \rho(Tu, Tv).$$

This is a contradiction, so $u = v$ and hence v is a common fixed point of S and T. For the uniqueness of fixed point, Assume that p and q are two distinct common fixed points of S and T such that $p \in X \cap Y$. Then, using the given contractive condition, we get

$$\begin{aligned} \rho(Sp, Sq) &= \rho(Tp, Tq) < \rho(Sp, Sq) \text{ if } q \in Y \text{ or} \\ \rho(Sq, Sp) &= \rho(Tq, Tp) < \rho(Sq, Sp) \text{ if } q \in X \end{aligned}$$

In both cases, we have a contradiction. So $p = q$. □

Lemma 2.3. *Let (X, Y, ρ) be a bipolar metric space and let $T, S : (X, Y, \rho) \rightrightarrows (X, Y, \rho)$ be two covariant maps. (x_n, y_n) be a bisequence. If the following conditions hold:*

1. *the pair $\{S, T\}$ is $w-$compatible,*
2. *S and T are continuous,*
3. *$Tx_n = Sx_{n+1}$ and $Ty_n = Sy_{n+1}$,*
4. *$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Ty_n = l$ for some $l \in X \cap Y$.*

Then l is a coincidence point of S and T.

Proof. From condition 3 and 4, we have

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Sy_n = l$$

. Since $\{S, T\}$ is w-compatible, so we get $\rho(TSx_n, STy_n) \to 0$ or $\rho(STx_n, TSy_n) \to 0$ as $n \to \infty$. As S and T are continuous, this implies that

$$\rho(Tl, Sl) = 0 \ \text{ or } \ \rho(Sl, Tl) = 0$$

. So, we get $Sl = Tl$, that is, l is a coincidence point of S and T. □

Our first main result is the following:

Theorem 2.1. *Let (X, Y, ρ) be a complete bipolar metric space and let $T,\ S : (X, Y, \rho) \rightrightarrows (X, Y, \rho)$ be two covariant maps satisfying the following conditions:*

1. *$T(X \cup Y) \subseteq S(X \cup Y)$,*
2. *$\rho(Tx, Ty) \leqslant \phi(\rho(Sx, Sy))$ for all $(x, y) \in X \times Y$,*
   for some comparison function ϕ,
3. *the pair $\{S, T\}$ is $w-$compatible,*
4. *S is continuous,*

then S and T have unique common fixed point.

Proof. We define a bisequence (u_n, v_n) as follows:

$$\begin{aligned} (x_0, y_0) &\in X \times Y \ \text{ be arbitrary} \\ u_n &= Tx_n = Sx_{n+1} \\ v_n &= Ty_n = Sy_{n+1}, \ \ n \in \mathbb{N} \cup \{0\}. \end{aligned}$$

This bisequence can be defined as $T(X \cup Y) \subseteq S(X \cup Y)$. Now

$$\rho(u_n, v_n) = \rho(Tx_n, Ty_n) \leqslant \phi(\rho(Sx_n, Sy_n)) = \phi(\rho(u_{n-1}, v_{n-1})) \leqslant \phi^n(\rho(u_0, v_0))$$

taking the limit as $n \to \infty$ and using the property of ϕ, we get

$$\lim_{n\to\infty} \rho(u_n, v_n) = 0. \tag{2.3}$$

Similarly, we can prove

$$\lim_{n\to\infty} \rho(u_{n+1}, v_n) = 0 \tag{2.4}$$

$$\lim_{n\to\infty} \rho(u_n, v_{n+1}) = 0. \tag{2.5}$$

We will show that (u_n, v_n) is a Cauchy bisequence. For this let $\epsilon > 0$ be arbitrary. Choose $\delta > 0$ such that

$$\delta < \frac{\epsilon - \phi(\epsilon)}{2} \tag{2.6}$$

Using (2.3), (2.4) and (2.5), we can find $n_0 \in \mathbb{N}$ such that

$$\rho(u_n, v_n) \leqslant \delta, \;\; \rho(u_{n+1}, v_n) \leqslant \delta, \;\; \rho(u_n, v_{n+1}) \leqslant \delta \tag{2.7}$$

for all $n \geqslant n_0$.
Now we will show that $\rho(u_n, Sy) \leqslant \epsilon$ implies $\rho(u_n, Ty) \leqslant \epsilon$ if $n \geqslant n_0$. For this let $n \geqslant n_0$ and

$$\rho(u_n, Sy) \leqslant \epsilon. \tag{2.8}$$

Now using (BP_3), (2.6) (2.7), (2.8), contractive condition 2 and property of ϕ, we get

$$\begin{aligned}
\rho(u_n, Ty) &\leqslant \rho(u_n, v_n) + \rho(u_{n+1}, v_n) + \rho(u_{n+1}, Ty) \\
&= \rho(u_n, v_n) + \rho(u_{n+1}, v_n) + \rho(Tx_{n+1}, Ty) \\
&\leqslant \rho(u_n, v_n) + \rho(u_{n+1}, v_n) + \phi(\rho(Sx_{n+1}, Sy)) \\
&\leqslant \delta + \delta + \phi(\rho(u_n, Sy)) \\
&\leqslant 2\delta + \phi(\epsilon) \\
&< \epsilon.
\end{aligned}$$

Now since $\rho(u_n Sy_{n+1}) = \rho(u_n, v_n) \leqslant \delta < \frac{\epsilon - \phi(\epsilon)}{2} < \epsilon$, so this implies that

$$\rho(u_n, v_{n+1}) = \rho(u_n, Ty_{n+1}) \leqslant \epsilon.$$

But $\rho(u_n, v_{n+1}) = \rho(u_n, Sy_{n+2})$. So we get

$$\begin{aligned}
\rho(u_n, Ty_{n+2}) &\leqslant \epsilon \\
\rho(u_n, v_{n+2}) &\leqslant \epsilon
\end{aligned}$$

By induction, we get

$$\rho(u_n, v_{n+p}) \leqslant \epsilon \text{ for all } n \geqslant n_0. \tag{2.9}$$

Similarly, we can prove that

$$\rho(u_{n+p}, v_n) \leqslant \epsilon \text{ for all } n \geqslant n_0. \tag{2.10}$$

From (2.9) and (2.10), we conclude that (u_n, v_n) is a Cauchy bisequence and hence biconverges to some point $l \in X \cap Y$ as (X, Y, ρ) is complete. That is

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = l$$

. By proposition (1.1) and (2.3), T is also continuous, so using lemma (2.3), we conclude that l is a coincidence point of S and T. Let $Sl = Tl = u$. By proposition (2.2) S and T are weakly compatible. Hence using proposition (2.3) and lemma (2.2), we conclude that u is the unique common fixed point of S and T. □

Remark 2.3. In view of proposition (2.1), condition 3 of above theorem can be replaced by the stronger condition that either ordered pair (S, T) or (T, S) is compatible.

Here are some corollaries:

Corollary 2.1. *Let (X, Y, ρ) be a complete bipolar metric space and let $T : (X, Y, \rho) \rightrightarrows (X, Y, \rho)$ be a ϕ-contraction for some comparison function ϕ, then T has unique fixed point in $X \cap Y$.*

Proof. This can be proved by taking $S = I$ in theorem (2.1), where I is the identity mapping on $X \cup Y$. □

We get the theorem 5.1 in [26] as a following corollary.

Corollary 2.2. *Let (X, Y, ρ) be a complete bipolar metric space and let $T : (X, Y, \rho) \rightrightarrows (X, Y, \rho)$ be a mapping satisfying the following condition*

$$\rho(Tx, Ty) \leqslant a\rho(x, y) \text{ for all } (x, y) \in X \times Y$$

for some $a \in [0, 1)$, then T has unique fixed point.

Proof. In corollary (2.1), take $\phi(t) = at$. □

Remark 2.4. In the above corollary, when we take $X = Y$, then we get the Banach fixed point theorem in metric space.

In the next main result, we have relaxed the continuity of S and T discussed as in theorem (2.1).

Theorem 2.2. *Let (X,Y,ρ) be a bipolar metric space and let $T,S:(X,Y,\rho)\rightrightarrows(X,Y,\rho)$ be two covariant maps satisfying the following conditions:*

1. $T(X\cup Y)\subseteq S(X\cup Y)$,
2. $S(X\cup Y)$ *is complete,*
3. $\rho(Tx,Ty)\leqslant\phi(\rho(Sx,Sy))$, for all $(x,y)\in X\times Y$, *for some comparison function* ϕ,
4. S *and* T *are weakly compatible,*
5. S *is an injective map.*

Then S and T have unique common fixed point.

Proof. As in theorem (2.1), we can define a bisequence (u_n,v_n) as follows:

$$\begin{aligned}(x_0,y_0) &\in X\times Y \text{ be arbitrary}\\ u_n &= Tx_n=Sx_{n+1}\\ v_n &= Ty_n=Sy_{n+1},\ \ n\in\mathbb{N}\cup\{0\}\end{aligned}$$

Arguing in the same way given in theorem (2.1), we conclude that the bisequence (u_n,v_n) is a Cauchy bisequence. As $S(X\cup Y)=S(X)\cup S(Y)$ is complete, so bisequence (u_n,v_n) biconverges to a point in $S(X)\cap S(Y)=S(X\cap Y)$. Let the point be Sw for some $w\in X\cap Y$. So that $\lim\limits_{n\to\infty}Sx_n=Sw$. Hence using Proposition (2.3) and Lemma (2.1), w is a unique coincidence point of S and T. Now using proposition (2.3) and Lemma (2.2), w is the unique common fixed point of S and T. □

Now we are going to prove a fixed point theorem for ϕ-expansive map which follows:

Theorem 2.3. *Let (X,Y,ρ) be a complete bipolar metric space and let $T:(X,Y,\rho)\rightleftarrows(X,Y,\rho)$ be a surjective and ϕ-expansive map for some comparison function ϕ, then T has a unique fixed point.*

Proof. First we observe by remark (2.1) that T is injective and hence bijective. Let $x_0\in X$. As T is surjective, we can choose $y_0\in Y$ such that $Ty_0=x_0$, then choose $x_1\in X$ such that $Tx_1=y_0$. Continuing this process we find a bisequence (x_n,y_n) such that $y_n=Tx_{n+1}$ and $x_n=Ty_n, n=0,1,2,.....$
As T is ϕ-expansive, we obtain

$$\phi(\rho(x_n,y_{n-1}))\geqslant\phi(\rho(Ty_n,Tx_n))\geqslant\rho(x_n,y_n)). \tag{2.11}$$

Similarly, we can obtain

$$\phi(\rho(x_n,y_n))\geqslant\rho(x_{n+1},y_n)). \tag{2.12}$$

From (2.11) and (2.12), we obtain

$$\phi^2(\rho(x_n, y_{n-1})) \geqslant \rho(x_{n+1}, y_n)),$$

this implies by induction that

$$\phi^{2n}(\rho(x_1, y_0)) \geqslant \rho(x_{n+1}, y_n)).$$

Taking limit and using the property of ϕ, we obtain

$$\lim_{n\to\infty} \rho(x_{n+1}, y_n)) = 0. \tag{2.13}$$

From (2.11), we have

$$\rho(x_n, y_{n-1}) \geqslant \phi(\rho(x_n, y_{n-1})) \geqslant \rho(x_n, y_n))$$

taking limit, we get

$$\lim_{n\to\infty} \rho(x_n, y_n)) = 0. \tag{2.14}$$

Now using (BP_3), we have

$$\rho(x_n, y_{n+1})) \leqslant \rho(x_n, y_n) + \rho(x_{n+1}, y_n) + \rho(x_{n+1}, y_{n+1}) \tag{2.15}$$

taking limit, we get

$$\lim_{n\to\infty} \rho(x_n, y_{n+1})) = 0. \tag{2.16}$$

We will show that (x_n, y_n) is a Cauchy bi-sequence. For this let $\epsilon > 0$ be arbitrary. Choose $\delta > 0$ such that

$$\delta < \frac{\epsilon - \phi^2(\epsilon)}{2}. \tag{2.17}$$

Using (2.13), (2.14) and (2.16), we can find $n_0 \in \mathbb{N}$ such that

$$\rho(x_n, y_n) \leqslant \delta, \quad \rho(x_{n+1}, y_n) \leqslant \delta, \quad \rho(x_n, y_{n+1}) \leqslant \delta \tag{2.18}$$

for all $n \geqslant n_0$.
Now we show that $\bar{B}(x_n, \epsilon)$ is invariant under T^{-2} if $n \geqslant n_0$. For this let $n \geqslant n_0$ and $y \in \bar{B}(x_n, \epsilon)$. That is

$$\rho(x_n, y) \leqslant \epsilon. \tag{2.19}$$

Now using (BP_3) and (2.18), we get

$$\begin{aligned} \rho(x_n, T^{-2}y) &\leqslant \rho(x_n, y_n) + \rho(x_{n+1}, y_n) + \rho(x_{n+1}, T^{-2}y) \\ &\leqslant 2\delta + \rho(x_{n+1}, T^{-2}y) \end{aligned} \tag{2.20}$$

Now

$$\begin{aligned}\phi(\rho(x_n, y)) &= \phi(\rho(Ty_n, TT^{-1}y)) \\ &\geqslant \rho(T^{-1}y, y_n)\end{aligned}$$

This implies

$$\begin{aligned}\phi^2(\rho(x_n, y)) &\geqslant \phi(\rho(T^{-1}y, y_n)) \\ &= \phi\rho(TT^{-2}y, TT^{-1}y_n) \\ &\geqslant \rho(x_{n+1}, T^{-2}y)\end{aligned}$$

So

$$\rho(x_{n+1}, T^{-2}y) \leqslant \phi^2(\rho(x_n, y)) \leqslant \phi^2(\epsilon). \tag{2.21}$$

Hence from (2.17), (2.20) and (2.21), we get

$$\rho(x_n, T^{-2}y) < 2\delta + \psi^2(\epsilon) < \epsilon.$$

This implies

$$T^{-2}y \in \bar{B}(x_n, \epsilon)$$

So by induction, we conclude that

$$T^{(-2m)}y \in \bar{B}(x_n, \epsilon) \text{ for all } \text{m} \in \mathbb{N}.$$

Taking $y = y_n$ as $y_n \in \bar{B}(x_n, \epsilon)$, we get

$$T^{(-2m)}y_n \in \bar{B}(x_n, \epsilon)$$

But $T^{(-2m)}y_n = y_{n+m}$, so $\rho(x_n, y_{n+m}) \leqslant \epsilon$.
Similarly, we can prove that $\rho(x_{n+m}, y_n) \leqslant \epsilon$ for all $n \geqslant n_0, m \in \mathbb{N}$ and hence the sequence (x_n, y_n) is Cauchy bisequence. So it biconverges to a point $p \in X \cap Y$. As T^{-1} is contractive by remark (2.2), so it is continuous, and hence $\{T^{-1}x_n\}$ converges to $T^{-1}p$, i.e. $\{y_n\}$ converges to $T^{-1}p$. So $T^{-1}p = p$. This implies $Tp = p$. Fixed point p must be unique as T^{-1} is contractive mapping. □

Example 2.2. Let (X, Y, ρ) be the complete bipolar metric space defined in example 1.1. Let $T, S : X \cup Y \rightrightarrows X \cup Y$ be two covariant maps defined by

$$\begin{aligned}T\{(x_1, x_2)\} &= \left\{\left(\frac{x_1}{4}, \frac{x_2}{4}\right)\right\} \\ T(A) &= \left\{\left(\frac{x_1}{4}, \frac{x_2}{4}\right) : (x_1, x_2) \in A\right\} \\ S\{(x_1, x_2)\} &= \left\{\left(\frac{x_1}{2}, \frac{x_2}{2}\right)\right\} \\ S(A) &= \left\{\left(\frac{x_1}{2}, \frac{x_2}{2}\right) : (x_1, x_2) \in A\right\}\end{aligned}$$

for every $\{(x_1, x_2)\} \in X$ and $A \in Y$.
Now we will prove that the pair $\{S, T\}$ is w-compatible. For this let

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} SA_n = \lim_{n\to\infty} TA_n = t$$

for some $t = \{(t_1, t_2)\} \in X \cap Y$ and bisequence (x_n, A_n). Let $x_n = \{(a_n, b_n)\}$.
Now $\lim_{n\to\infty} \rho(Sx_n, t) = 0$ implies that $\lim_{n\to\infty} a_n = 2t$ and $\lim_{n\to\infty} b_n = 2t$. Similarly, $\lim_{n\to\infty} \rho(Tx_n, t) = 0$ implies that $\lim_{n\to\infty} a_n = 4t$ and $\lim_{n\to\infty} b_n = 4t$. Hence $t_1 = t_2 = 0$.
So

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0 \tag{2.22}$$

$$\rho(\{(0,0)\}, SA_n) = 0 \text{ and } \rho(\{(0,0)\}, TA_n) = 0 \tag{2.23}$$

(2.23) implies that

$$\lim_{n\to\infty} M_n = 0 \text{ where } M_n = \sup\{|c_n| + |d_n| : (c_n, d_n) \in A_n\} \tag{2.24}$$

Now

$$\begin{aligned}
\rho(STx_n, TSA_n) &= \frac{1}{8}\sup\{|a_n - c_n| + |b_n - d_n| : \{(c_n, d_n)\} \in A_n\} \\
&\leqslant \frac{1}{8}\sup\{|a_n| + |b_n| + |c_n| + |d_n| : \{(c_n, d_n)\} \in A_n\} \\
&= \frac{1}{8}(|a_n| + |b_n|) + \frac{1}{8}\sup\{|c_n| + |d_n| : \{(c_n, d_n)\} \in A_n\} \\
&= \frac{1}{8}(|a_n| + |b_n|) + \frac{1}{8}M_n
\end{aligned}$$

This implies using (2.22) and (2.24) that

$$\lim_{n\to\infty} \rho(STx_n, TSA_n) = 0$$

So the pair $\{S, T\}$ is w-compatible. Now we can observe the following

$$X \cup Y = Y \text{ and } T(Y) = Y = S(Y)$$

and we can prove the following

$$\rho(Tx, Ty) = \frac{1}{4}\rho(x, y) \text{ for all } (x, y) \in X \times Y$$

and

$$\rho(Sx, Sy) = \frac{1}{2}\rho(x, y) \text{ for all } (x, y) \in X \times Y.$$

So we can write

$$\rho(Tx, Ty) = \frac{1}{2}\rho(Sx, Sy).$$

So all the conditions of theorem (2.1) are satisfied with $\phi(t) = \frac{t}{2}$, So S and T have unique common fixed point.

3. Conclusion

This paper establishes common fixed point theorems for $w-$compatible maps in bipolar metric spaces using a comparison function ϕ in place of altering distance functions. By introducing generalized contraction conditions involving ϕ, we derive new common fixed point results that extend previous theorems in bipolar metric spaces. These results provide a broader, more flexible framework for fixed point theory. Additionally, illustrative examples are provided to validate and support the theoretical findings, demonstrating the practical relevance of our theorems. The results can be explored in other metric-like spaces, such as partial metric spaces, cone metric spaces, or fuzzy metric spaces, to generalize the fixed point theory. Also, investigating the application of $w-$compatible maps and comparison functions in stochastic or probabilistic bipolar metric spaces could lead to new insights in uncertain environments.

Acknowledgements

All authors are thankful to the Department of Science and Technology, New Delhi, India for approving the proposal under the scheme FIST Program (Ref. No. SR/FST/MS/2022/122 dated 19 December 2022). All the authors are grateful to the editor and referees of the journal for their constructive suggestions for the improvement and preparation of the manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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