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ON ϕ-RECURRENT TYPES OF PARACONTACT METRIC $(\kappa \neq -1, \mu)$-MANIFOLDS

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Abstract. The main aim of the present paper is to investigate geometric properties of hyper-generalized ϕ-recurrent and quasi-generalized ϕ-recurrent paracontact metric $(\kappa \neq -1, \mu)$-manifolds.
Keywords: paracontact metric (κ, μ)-manifold, hyper-generalized ϕ-recurrent manifold, quasi-generalized ϕ-recurrent manifold.

1. Introduction

Symmetry plays a significant role in nature. One of the most useful ways to analyse the symmetry of a semi-Riemannian manifold is to study the curvature conditions arising from the restriction of its curvature. Cartan showed that if all local geodesic symmetry is an isometry for any point on the semi-Riemannian manifold M of dimension $(2n+1)$, then M is called *locally symmetric manifold* [3]. This is equivalent to the following condition

$$\nabla R = 0,$$

where R and ∇ denote the Riemannian curvature tensor and the Levi-Civita connection, resp. The idea of locally symmetric manifolds has been weakened and extensively studied. *Recurrent manifolds* as a generalization that properly includes the set

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of locally symmetric manifolds were introduced by Ruse [15]. A semi-Riemannian manifold is said to be recurrent manifold if it satisfies

$$\nabla R = A \otimes R,$$

where A is a non-vanishing 1-form, $\otimes$ is the tensor product. Geometrically, the recurrent manifolds are related with the study of first order change on sectional curvature of a plane that obtained after the parallel transportation around a curve on M. This work extended to *ϕ-recurrent manifolds* by De [11] as following

$$\phi^2(\nabla R) = A \otimes R.$$

Dubey [12] introduced the notation of *generalized recurrent manifold* which satisfies

$$\nabla R = A \otimes R + B \otimes G.$$

Here A and B are two 1-forms of which B is non-zero. G is defined by $G(X, Y)Z = g(Y, Z)X - g(X, Z)Y$. If $B = 0$, the manifold reduces to a recurrent manifold. This work extended to *generalized ϕ-recurrent manifolds* as follows

$$\phi^2(\nabla R) = A \otimes R + B \otimes G.$$

Afterwards, new kinds of recurrent manifolds were introduced. A semi-Riemannian manifold is said to be *hyper-generalized recurrent manifold* if the condition

$$\nabla R = A \otimes R + B \otimes (g \wedge S)$$

holds where S is the Ricci tensor, A, B are two 1-forms which B is non-zero and $\wedge$ denotes the Kulkarni-Nomizu product. The Kulkarni-Nomizu product $E \wedge F$ of two $(0, 2)$ tensors E and F is defined by

$$\begin{aligned}(E \wedge F)(X_1, X_2, X_3, X_4) = \quad & E(X_1, X_4)F(X_2, X_3) + E(X_2, X_3)F(X_1, X_4) \\ & -E(X_1, X_3)F(X_2, X_4) - E(X_2, X_4)F(X_1, X_3),\end{aligned}$$

for $X_i \in \chi(M)$, $i = 1, 2, 3, 4$.

A semi-Riemannian manifold is called a *quasi-generalized recurrent manifold* if the condition

$$\nabla R = A \otimes R + B \otimes (G + g \wedge H)$$

holds where A and B are two 1-forms of which B is non-zero and $H = \eta \otimes \eta$, η being a non-zero 1-form.

A semi-Riemannian manifold is said to be *Ricci symmetric* if $\nabla S = 0$, where S is a Ricci tensor. The notation of Ricci symmetry has been weakened by many authors such as *Ricci recurrent manifold* which was presented by Patterson [14]. A semi-Riemannian manifold is said to be Ricci recurrent if the condition

$$\nabla S = A \otimes S$$

holds where A is a non-vanishing 1-form.

De et al. introduced the notation of *generalized Ricci-recurrent manifold* [9] which satisfies

$$\nabla S = A \otimes S + B \otimes g,$$

where A and B are two 1-forms of which B is non-zero. If $B = 0$, then it reduces to the notation of Ricci-recurrent manifold.

A semi-Riemannian manifold is said to be *super generalized Ricci-recurrent manifold* if its Ricci tensor S satisfies the condition

$$\nabla S = A \otimes S + B \otimes g + C \otimes (\eta \otimes \eta), \tag{1.1}$$

where A, B and C are non-vanishing 1-forms. In particular, if $B = C$, then it reduces to the notation of a *quasi-generalized Ricci-recurrent manifold.* There have been many studies on recurrent manifolds and paracontact manifolds [1,4–8,10,16].

All the studies mentioned above motivate us to investigate hyper-generalized ϕ-recurrent and quasi-generalized ϕ-recurrent paracontact metric $(\kappa \neq -1, \mu)$-manifolds. This paper is organized in the following way. In Section 2, we recall some notations required for this paper. In Section 3, we present some properties of paracontact metric $(\kappa \neq -1, \mu)$-manifolds. In Section 4, we work on hyper-generalized ϕ-recurrent paracontact metric $(\kappa \neq -1, \mu)$-manifolds and give some relations between 1-forms. We show that, in recurrent-like structures on paracontact metric (κ, μ)-manifolds, either the manifold reduces to $N(\kappa)$-paracontact manifold or the characteristic vector field ξ and ρ_1 (the associated vector field which corresponds to 1- form A) are co-directional. This is also valid for contact cases. We give the necessary conditions for a hyper-generalized ϕ-recurrent paracontact metric $(\kappa \neq -1, 0)$ manifold to be generalized Ricci recurrent and Ricci recurrent. We present some results according to whether the scalar curvature of a hyper-generalized ϕ-recurrent paracontact metric $(\kappa \neq -1, \mu)$-manifold is zero or not. We obtain the necessary condition for a hyper-generalized ϕ-recurrent paracontact metric $(\kappa \neq -1, 0)$-manifold to be an η-Einstein manifold. Moreover, we prove that there does not exist any Einstein hyper-generalized ϕ-recurrent paracontact metric $(\kappa \neq -1, 0)$-manifold of dimension $2n+1$, where $n > 1$. For dimension 3, the manifold is Ricci flat. In the last section, we study quasi-generalized ϕ-recurrent paracontact metric $(\kappa \neq -1, 0)$-manifolds. We obtain some relations between 1-forms. We prove that a quasi-generalized ϕ-recurrent paracontact metric $(\kappa \neq -1, \mu)$-manifold is either $N(\kappa)$-paracontact metric manifold or characteristic vector field ξ and ρ_3 (is a vector field associated with 1-form D) are co-directional. We show that every quasi-generalized ϕ-recurrent paracontact metric $(\kappa \neq -1, 0)$ manifold is a super generalized Ricci recurrent manifold and also can never be a quasi-generalized Ricci recurrent manifold. We prove that the scalar curvature r of a quasi-generalized ϕ-recurrent paracontact metric $(\kappa \neq -1, \mu)$-manifold can never be zero. Also we show that there does not exist any quasi-generalized ϕ-recurrent paracontact metric $(0, \mu)$-manifold. Finally, we give the necessary condition for a quasi-generalized ϕ-recurrent paracontact metric $(\kappa \neq -1, 0)$-manifold to be η-Einstein. Moreover, we prove that there do not exist any Einstein quasi-generalized ϕ-recurrent paracontact metric $(\kappa \neq -1, 0)$-manifold.

2. Preliminaries

A $(2n+1)$-dimensional smooth manifold M is called an *almost paracontact manifold* if it admits a triple (ϕ, ξ, η) satisfying the followings:

$$\eta(\xi) = 1, \quad \phi^2 = I - \eta \otimes \xi \tag{2.1}$$

and ϕ induces on almost paracomplex structure on each fiber of $\mathcal{D} = ker(\eta)$, where ϕ, ξ and η are $(1,1)-$tensor field, vector field and $1-$form, resp. One can easily checked that $\phi\xi = 0, \eta \circ \phi = 0$ and $rank\phi = 2n$, by the definition. Here, ξ is a unique vector field (called *Reeb* or *characteristic vector field*) dual to η and satisfying $d\eta(\xi, X) = 0$ for all $X \in \chi(M)$. When the tensor field $N_\phi := [\phi, \phi] - 2d\eta \otimes \xi$ vanishes identically, the almost paracontact manifold is said to be *normal*. If the structure (M, ϕ, ξ, η) admits a pseudo-Riemannian metric such that

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{2.2}$$

for all $X, Y \in \chi(M)$ then we say that (M, ϕ, ξ, η, g) is an *almost paracontact metric manifold*. Note that any pseudo-Riemannian metric with a given almost paracontact metric manifold structure is necessarily of signature $(n+1, n)$. For an almost paracontact metric manifold, one can always find an orthogonal basis $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, \xi\}$, namely $\phi-$basis, such that $g(X_i, X_j) = -g(Y_i, Y_j) = \delta_{ij}$ and $Y_i = \phi X_i$, for any $i, j \in \{1, \ldots, n\}$. Further, an almost paracontact metric manifold is said to be *paracontact metric manifold* if the following holds for all vector fields $X, Y \in \chi(M)$:

$$d\eta(X, Y) = g(X, \phi Y).$$

In a paracontact metric manifold, one defines a symmetric, trace-free operator $h := \frac{1}{2}L_\xi \phi$, where L denotes the Lie derivative. The operator h also satisfies the followings:

$$h\xi = 0, \quad \phi h = -h\phi, trace(h) = 0, \tag{2.3}$$

$$\nabla_X \xi = -\phi X + \phi h X \tag{2.4}$$

where ∇ is the Levi-Civita connection of the pseudo-Riemannian manifold [17].

An almost paracontact metric manifold is said to be η-*Einstein* if its Ricci tensor S is of the form

$$S = \lambda g + \mu \eta \otimes \eta, \tag{2.5}$$

where λ and μ are smooth functions on the manifold. If $\mu = 0$, then the manifold is said to be *Einstein*.

3. Paracontact metric (κ, μ)-manifolds

The (κ, μ)-nullity distribution on (M, ϕ, ξ, η, g) is a distribution

$$N(\kappa,\mu)\colon p \to N_p(\kappa,\mu) = \{Z \in T_pM : \quad R(X,Y)Z = \kappa[g(Y,Z)X - g(X,Z)Y] \\ +\mu[g(Y,Z)hX - g(X,Z)hY]\} \tag{3.1}$$

for all $X, Y \in T_pM$ and $\kappa, \mu \in \mathbb{R}$. If the ξ belongs to the above distribution, namely,

$$R(X,Y)\xi = \kappa[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY], \tag{3.2}$$

then the paracontact metric manifold is called a paracontact metric (κ, μ)- manifold. When $\mu = 0$, a paracontact metric (κ, μ)-manifold reduces to $N(\kappa)$-paracontact metric manifold [2].

Lemma 3.1. *[2] Let (M, ϕ, ξ, η, g) be a paracontact metric (κ, μ)-manifold such that $\kappa \neq -1$. If $\kappa > -1$ (respectively $\kappa < -1$), then there exists a local orthogonal ϕ-basis $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, \xi\}$ of eigenvectors of h (respectively ϕh) such that $X_1, \ldots, X_n \in \Gamma(\mathcal{D}_h(\lambda))$ (respectively, $\mathcal{D}_{\phi h}(\lambda)$), $Y_1, \ldots, Y_n \in \Gamma(\mathcal{D}_h(-\lambda))$ (respectively, $\mathcal{D}_{\phi h}(-\lambda)$), and*

$$g(X_i, X_i) = -g(Y_i, Y_i) = \begin{cases} 1, \ \textit{for } 1 \leqslant i \leqslant r, \\ -1 \ \textit{for } r+1 \leqslant i \leqslant r+s, \end{cases} \tag{3.3}$$

where $r = index(\mathcal{D}_h(-\lambda))$ (respectively, $r = index(\mathcal{D}_{\phi h}(-\lambda))$) and $s = n - r = index(\mathcal{D}_h(\lambda))$ (respectively, $s = index(\mathcal{D}_{\phi h}(\lambda))$).

Lemma 3.2. *[2] Let (M, ϕ, ξ, η, g) be a paracontact metric (κ, μ)-manifold of dimension $2n+1$, then the following identities hold:*

$$h^2 = (1+\kappa)\phi^2, \tag{3.4}$$

$$Q\xi = 2n\kappa\xi, \tag{3.5}$$

$$(\nabla_X \phi)Y = -g(X - hX, Y)\xi + \eta(Y)(X - hX); \quad \kappa \neq -1, \tag{3.6}$$

$$\nabla_\xi h = \mu h \circ \phi, \quad \nabla_\xi \phi h = -\mu h, \tag{3.7}$$

$$R(\xi, X)Y = \kappa[g(X,Y)\xi - \eta(Y)X] + \mu[g(hX,Y)\xi - \eta(Y)hX], \tag{3.8}$$

for all vector fields $X, Y \in \chi(M)$, where Q denotes the Ricci operator of M.

Corollary 3.1. *[2] In any $(2n+1)$-dimensional paracontact (κ, μ)-manifold (M, ϕ, ξ, η, g) such that $\kappa \neq -1$, the Ricci operator Q is given by*

$$Q = (2(1-n) + n\mu)I + (2(n-1) + \mu)h + (2(n-1) + n(2\kappa - \mu))\eta \otimes \xi. \tag{3.9}$$

For $(2n+1)$-dimensional paracontact $(\kappa \neq -1, \mu)$-manifolds, from (3.9), we have the following:

$$\begin{aligned} S(X,Y) = {} & (2(1-n)+n\mu)g(X,Y)+(2(n-1)+\mu)g(hX,Y) \\ & +(2(n-1)+n(2\kappa-\mu))\eta(X)\eta(Y), \end{aligned} \tag{3.10}$$

$$\begin{aligned} S(\phi X,\phi Y) = {} & S(X,Y)-2(2(1-n)+n\mu)g(X,Y)-2(2(n-1) \\ & +n(\kappa-\mu))\eta(X)\eta(Y), \end{aligned} \tag{3.11}$$

$$\begin{aligned} S(X,hY) = {} & 2(n-1)(1+\kappa)g(X,Y)+2(1-n)g(X,hY) \\ & -2(n-1)(1+\kappa)\eta(X)\eta(Y), \end{aligned} \tag{3.12}$$

$$S(X,\xi) = 2n\kappa\eta(X), \tag{3.13}$$

$$r = 2n(2(1-n)+n\mu+\kappa), \tag{3.14}$$

for all $X, Y \in \chi(M)$.

Proposition 3.1. *For a $(2n+1)$- dimensional paracontact metric $(\kappa \neq -1, \mu)$-manifold, we have the followings*

$$(\nabla_W R)(X,Y)\xi = \mu[\eta(Y)(\nabla_W h)X - \eta(X)(\nabla_W h)Y] \tag{3.15}$$

and

$$g((\nabla_W R)(\xi,Y)Z,\xi) = -\mu\eta(Z)g(\nabla_W hY,\xi), \tag{3.16}$$

for all $X, Y, Z \in \chi(M)$.

Proof. Using (3.1) in

$$(\nabla_W R)(X,Y)\xi = \nabla_W R(X,Y)\xi - R(\nabla_W X,Y)\xi - R(X,\nabla_W Y)\xi - R(X,Y)\nabla_W\xi,$$

we have

$$(\nabla_W R)(X,Y)\xi = \mu[\eta(Y)\nabla_W hX - \eta(X)\nabla_W hY + \eta(X)h\nabla_W Y - \eta(Y)h\nabla_W X],$$

which gives (3.15). The proof of (3.16) is similar to (3.15). □

4. Hyper-generalized ϕ-recurrent paracontact metric $(\kappa \neq -1, \mu)$-manifold

Definition 4.1. [16] A (2n+1)-dimensional paracontact metric (κ, μ)-manifold is said to be a hyper-generalized ϕ-recurrent if its curvature tensor R satisfies

$$\phi^2((\nabla_W R)(X,Y)Z) = A(W)R(X,Y)Z + B(W)H(X,Y)Z \tag{4.1}$$

for all vector fields X, Y and Z, where A, B are two non-vanishing 1-forms such that $A(X) = g(X,\rho_1), B(X) = g(X,\rho_2)$ and the tensor H is defined by

$$H(X,Y)Z = S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY, \tag{4.2}$$

for all $X, Y, Z \in \chi(M)$, where Q is the Ricci operator, ρ_1 and ρ_2 are vector fields associated with 1-forms A and B, resp. If $B = 0$, then it reduces to ϕ-recurrent manifold [11].

Theorem 4.1. *In a $(2n+1)$-dimensional hyper-generalized ϕ-recurrent paracontact metric $(\kappa \neq -1, \mu)$-manifold, the associated vector fields ρ_1 and ρ_2 corresponding to 1-forms A and B satisfy the following relation*

$$[r - 4n\kappa]\eta(\rho_1) + [2(2n-1)(r - 2nk)]\eta(\rho_2) = 0. \tag{4.3}$$

Proof. We consider a $(2n+1)$-dimensional hyper-generalized ϕ-recurrent paracontact metric $(\kappa \neq -1, \mu)$-manifold. Then, from the equations (2.1) and (4.1), we have

$$(\nabla_W R)(X,Y)Z - \eta((\nabla_W R)(X,Y)Z)\xi = A(W)R(X,Y)Z + B(W)H(X,Y)Z. \tag{4.4}$$

Taking the inner product of (4.4) with the vector field U, we get

$$\begin{aligned} g((\nabla_W R)(X,Y)Z, U) - \eta((\nabla_W R)(X,Y)Z)\eta(U) \quad &= A(W)g(R(X,Y)Z,U) \\ &+ B(W)g(H(X,Y)Z,U). \end{aligned} \tag{4.5}$$

Changing X, Y, Z cyclically and using the second Bianchi's identity in (4.5), we derive

$$\begin{aligned} &A(W)g(R(X,Y)Z,U) + B(W)g(H(X,Y)Z,U) + A(X)g(R(Y,W)Z,U) \\ &+ B(X)g(H(Y,W)Z,U) + A(Y)g(R(W,X)Z,U) + B(Y)g(H(W,X)Z,U) = 0. \end{aligned} \tag{4.6}$$

Let $\{e_i\}$ $(1 \leqslant i \leqslant 2n+1)$ be an orthonormal basis. Taking the summation over i for $Y = Z = e_i$ and using (3.1) and (4.2) in the last equation, we obtain

$$\begin{aligned} 0 \quad &= A(W)S(X,U) + B(W)[rg(X,U) + (2n-1)S(X,U)] \\ &- A(X)S(W,U) - B(X)[rg(W,U) + (2n-1)S(W,U)] \\ &+ \sum_{i=1}^{2n+1} \varepsilon_i \{g(e_i,\rho_1)[\kappa(g(X,e_i)g(W,U) - g(W,e_i)g(X,U)) \\ &\mu(g(X,e_i)g(hW,U) - g(W,e_i)g(hX,U))]\} \\ &+ \sum_{i=1}^{2n+1} \varepsilon_i \{g(e_i,\rho_2)[S(X,e_i)g(W,U) - S(W,e_i)g(X,U) \\ &+ g(X,e_i)S(W,U) - g(W,e_i)S(X,U)]\}. \end{aligned} \tag{4.7}$$

From the above equation, we derive

$$\begin{aligned} &A(W)S(X,U) + B(W)[rg(X,U) + (2n-1)S(X,U)] - A(X)S(W,U) \\ &- B(X)[rg(W,U) + (2n-1)S(W,U)] + \kappa A(X)g(W,U) - \kappa A(W)g(X,U) \\ &+ \mu A(hX)g(W,U) - \mu A(hW)g(X,U) + B(QX)g(W,U) \\ &- B(QW)g(X,U) + B(X)S(W,U) - B(W)S(X,U) = 0. \end{aligned} \tag{4.8}$$

Contracting over X and U in (4.8), we have

$$(r - 2n\kappa)A(W) - A(QW) + [4nr - 2r]B(W) - 2n\mu A(hW) + (2-4n)B(QW) = 0. \tag{4.9}$$

Putting $W = \xi$ and using (3.5) in (4.9), we get (4.3). $\square$

Definition 4.2. Two vector fields P and N are said to be co-directional if $P = FN$, where F is a non-zero scalar, that is, $g(P,X) = Fg(N,X)$.

Theorem 4.2. *A $(2n+1)$-dimensional hyper-generalized ϕ-recurrent paracontact metric $(\kappa \neq -1, \mu)$-manifold is either a paracontact metric $(\frac{r+4n^2-4n}{2n}, 0)$-manifold or characteristic vector field ξ and ρ_1 (is vector field associated with 1-form A) are co-directional.*

Proof. Letting $Y = Z = e_i$ and taking the summation over i in the following equation

$$A(Y)g(R(W,X)Z,U) = -A(Y)g(R(W,X)U,Z) \tag{4.10}$$

we have

$$\sum_{i=1}^{2n+1} \varepsilon_i A(e_i)g(R(W,X)e_i,U) = -\sum_{i=1}^{2n+1} \varepsilon_i A(e_i)g(R(W,X)U,e_i). \tag{4.11}$$

Using (3.1), we compute the left hand side of the equation (4.11) as follows

$$\begin{aligned} &\textstyle\sum_{i=1}^{2n+1} \varepsilon_i A(e_i)g(R(W,X)e_i,U) \\ = &\textstyle\sum_{i=1}^{2n+1} \varepsilon_i \{g(e_i,\rho_1)[\kappa(g(X,e_i)g(W,U) - g(W,e_i)g(X,U))] \\ &+\mu[g(X,e_i)g(hW,U) - g(W,e_i)g(hX,U)]\}. \\ = &\kappa[A(X)g(W,U) - A(W)g(X,U)] \\ &+\mu[A(X)g(hW,U) - A(W)g(hX,U)]. \end{aligned} \tag{4.12}$$

For the right hand side, we obtain

$$\begin{aligned} \sum_{i=1}^{2n+1} \varepsilon_i A(e_i)g(R(W,X)U,e_i) = &\kappa[A(W)g(X,U) - A(X)g(W,U)] \\ &+\mu[A(hW)g(X,U) - A(hX)g(W,U)]. \end{aligned} \tag{4.13}$$

In the view of (4.12) and (4.13), from (4.11), we get

$$\mu[A(X)g(hW,U) - A(W)g(hX,U)] = \mu[A(hX)g(W,U) - A(hW)g(X,U)]. \tag{4.14}$$

Contracting (4.14) over X and U, we derive

$$-\mu g(hX,\rho_1) = \mu[(2n+1)A(hX) - g(h\rho_1,X)],$$

which implies

$$\mu(2n+1)A(hW) = 0. \tag{4.15}$$

From the above equation, two cases occur.

Case I: If $\mu = 0$, then from (3.14) we have $\kappa = \frac{r+4n^2-4n}{2n}$.

Case II: If $A(hW) = 0$, then $A(h^2W) = 0$. Using (3.4) in the last equation, we obtain $(1+\kappa)A(\phi^2W) = 0$. Since $\kappa \neq -1$, we have $A(W) - \eta(W)A(\xi) = 0$. This completes the proof. $\square$

Theorem 4.3. *A $(2n+1)$-dimensional hyper-generalized ϕ-recurrent paracontact metric $(\kappa \neq -1, 0)$-manifold is*

(i) Generalized Ricci-recurrent, if the scalar curvature is non-zero and $A(W) \neq (1-2n)B(W)$,

(ii) Ricci-recurrent, if $r = 0$ and $A(W) \neq (1-2n)B(W)$.

Proof. The equation (4.5) holds for a $(2n+1)$-dimensional hyper-generalized ϕ-recurrent paracontact metric $(\kappa \neq -1, \mu)$-manifold. Contracting over X and U in (4.5), we obtain

$$\begin{aligned} &\textstyle\sum_{i=1}^{2n+1} \varepsilon_i [g((\nabla_W R)(e_i, Y)Z, e_i) - \eta((\nabla_W R)(e_i, Y)Z)\eta(e_i)] \\ &= \textstyle\sum_{i=1}^{2n+1} \varepsilon_i [A(W)g(R(e_i, Y)Z, e_i) + B(W)g(H(e_i, Y)Z, e_i)]. \end{aligned} \tag{4.16}$$

Using (4.2) in (4.16), we get

$$\begin{aligned} (\nabla_W S)(Y, Z) - \eta((\nabla_W R)(\xi, Y)Z) = \ & [A(W) + (2n-1)B(W)]S(Y, Z) \\ & + rB(W)g(Y, Z). \end{aligned} \tag{4.17}$$

Then the proof follows from (3.16). $\square$

Theorem 4.4. *In a $(2n+1)$-dimensional hyper-generalized ϕ-recurrent paracontact metric $(\kappa \neq -1, \mu)$-manifold, the 1-forms A and B satisfy the following relation*

$$\kappa A(W) + [n(2\kappa - 2 + \mu) + 2]B(W) = 0. \tag{4.18}$$

Proof. We know that for a $(2n+1)$-dimensional hyper-generalized ϕ-recurrent paracontact metric $(\kappa \neq -1, \mu)$-manifold, (4.17) is valid. Letting $Z = \xi$ and using (3.13) in (4.17), we have

$$\begin{aligned} &(\nabla_W S)(Y, \xi) - \eta((\nabla_W R)(\xi, Y)\xi) \\ &\qquad = 2n\kappa A(W)\eta(Y) + B(W)[(2n-1)2n\kappa + r]\eta(Y). \end{aligned} \tag{4.19}$$

Putting $Y = \xi$ in (4.19) and using (3.16), we obtain

$$2n\kappa[A(W) + (2n-1)B(W)] + rB(W) = 0. \tag{4.20}$$

Using (3.14) in the above equation, we get the relation (4.18). $\square$

Corollary 4.1. *In a $(2n+1)$-dimensional hyper-generalized ϕ-recurrent paracontact metric $(\kappa \neq -1, \mu)$-manifold, if the scalar curvature of the manifold vanishes, then either*

(i) 1-forms A and B are co-directional or

(ii) it is $(0, \frac{2(n-1)}{n})$-paracontact metric manifold.

Proof. If the scalar curvature r is vanishes, then from (4.20), we have

$$2n\kappa[A(W) + (2n-1)B(W)] = 0. \tag{4.21}$$

From the above equation, either $\kappa = 0$ or $A(W) = (1-2n)B(W)$. If $A(W) = (1-2n)B(W)$, then the 1 forms A and B are co-directional. In the case of $\kappa = 0$, since r vanishes, from (3.14) we get $\mu = \frac{2(n-1)}{n}$. □

By virtue of (3.14) and (4.20), we get the following result:

Corollary 4.2. *In a hyper-generalized ϕ-recurrent paracontact metric $(\kappa \neq -1, 0)$-manifold, if the scalar curvature of the manifold vanishes, then either*

(i) The dimension of the manifold is three and it is flat or

(ii) 1-forms A and B are co-directional.

Theorem 4.5. *In a $(2n+1)$-dimensional hyper-generalized ϕ-recurrent paracontact metric $(\kappa \neq -1, \mu)$-manifold, if the scalar curvature of the manifold is a non-zero constant, then*

$$A(W) = -4nB(W), \tag{4.22}$$

for any vector field W.

Proof. Taking the summation over i $(1 \leqslant i \leqslant 2n+1)$ for $Y = Z = e_i$ in (4.17) and using (3.16), we obtain

$$dr(W) = r[A(W) + 4nB(W)]. \tag{4.23}$$

Then (4.22) follows from (4.23). □

Theorem 4.6. *A $(2n+1)$-dimensional hyper-generalized ϕ-recurrent paracontact metric $(\kappa \neq -1, 0)$-manifold is an η-Einstein manifold if $\kappa = \frac{1-n}{n}$.*

Proof. In order to find $(\nabla_W S)(Y, \xi)$, we use (2.4) and (3.13) in the following equation

$$(\nabla_W S)(Y, \xi) = \nabla_W S(Y, \xi) - S(\nabla_W Y, \xi) - S(Y, \nabla_W \xi). \tag{4.24}$$

Hence, we obtain

$$(\nabla_W S)(Y, \xi) = 2n\kappa g(Y, -\phi W + \phi hW) - S(Y, -\phi W + \phi hW). \tag{4.25}$$

If we use (3.16) in (4.19), then compare the obtained equation with (4.25), we get

$$\begin{aligned} \{2n\kappa A(W) + B(W)[(2n-1)2n\kappa + r]\}\eta(Y) = {} & 2n\kappa g(Y, -\phi W + \phi hW) \\ & -S(Y, -\phi W + \phi hW). \end{aligned} \tag{4.26}$$

On the other hand, by letting $Y = \phi Y$ in (4.26), we have

$$2n\kappa g(\phi Y, -\phi W + \phi hW) - S(\phi Y, -\phi W + \phi hW) = 0. \tag{4.27}$$

In the view of (2.2), (3.4) and (3.11), the equation (4.27) becomes

$$\begin{aligned} S(Y,W) - S(Y,hW) = \; & 2(2(1-n) - n\kappa)g(Y,W) - 2(2(1-n) - n\kappa)g(Y,hW) \\ & +2(2(n-1) + 2n\kappa)\eta(Y)\eta(W). \end{aligned} \tag{4.28}$$

Using (3.12) in (4.28), we derive

$$S(Y,W) = \alpha g(Y,W) + \beta g(Y,hW) + \gamma\eta(Y)\eta(W), \tag{4.29}$$

where $\alpha = 2 - 2n - 2\kappa$, $\beta = 2n\kappa - 2(1-n)$ and $\gamma = 2n\kappa + 2n + 2\kappa - 2$. If $\beta = 0$, then $\kappa = \frac{1-n}{n}$.

□

Theorem 4.7. *There does not exist any Einstein hyper-generalized ϕ-recurrent paracontact metric $(\kappa \neq -1, 0)$-manifold of dimension $2n+1$, where $n > 1$. For dimension 3, the manifold is Ricci flat.*

Proof. The Ricci tensor of a $(2n+1)$-dimensional hyper-generalized ϕ-recurrent paracontact metric $(\kappa \neq -1, 0)$-manifold is given in (4.29). For the manifold to be Einstein, β and γ must vanish. If $\gamma = 0$, then $n\kappa + n + \kappa - 1 = 0$. Again, $\beta = 0$, then $\kappa = \frac{1-n}{n}$. Comparing the last two equations, we obtain $n = 1$, which implies $\kappa = 0$. This completes the proof of the theorem. □

5. Quasi-generalized ϕ-recurrent paracontact metric $(\kappa \neq -1, \mu)$-manifold

Definition 5.1. [16] A $(2n+1)$-dimensional paracontact metric (κ, μ)-manifold is said to be quasi-generalized ϕ-recurrent if its curvature tensor R satisfies

$$\phi^2((\nabla_W R)(X,Y)Z) = D(W)R(X,Y)Z + E(W)F(X,Y)Z \tag{5.1}$$

for all vector fields X, Y and Z, where D, E are two non-vanishing 1-forms such that $D(X) = g(X, \rho_3), E(X) = g(X, \rho_4)$ and the tensor F is defined by

$$\begin{aligned} F(X,Y)Z = \; & g(Y,Z)X - g(X,Z)Y + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y \\ & -g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi, \end{aligned} \tag{5.2}$$

for all vector fields $X, Y, Z \in \chi(M)$, where ρ_3 and ρ_4 are vector fields associated with 1-forms D and E resp.

Theorem 5.1. *In a $(2n+1)$-dimensional quasi-generalized ϕ-recurrent paracontact metric $(\kappa \neq -1, \mu)$-manifold, the associated vector fields ρ_3 and ρ_4 corresponding to 1-forms D and E satisfy the following relation*

$$(r - 4n\kappa)\eta(\rho_3) + 2(2n^2 - n)\eta(\rho_4) = 0. \tag{5.3}$$

Proof. We consider a $(2n+1)$-dimensional quasi-generalized ϕ-recurrent paracontact metric $(\kappa \neq -1, \mu)$-manifold. Using (2.2) in (5.1), we obtain

$$(\nabla_W R)(X,Y)Z - \eta((\nabla_W R)(X,Y)Z)\xi = D(W)R(X,Y)Z + E(W)F(X,Y)Z. \tag{5.4}$$

Using the second Bianchi's identity in (5.4), we get

$$\begin{aligned} 0 = \quad & D(W)R(X,Y)Z + E(W)F(X,Y)Z + D(X)R(Y,W)Z \\ & +E(X)F(Y,W)Z + D(Y)R(W,X)Z + E(Y)F(W,X)Z. \end{aligned} \tag{5.5}$$

Contracting over X in (5.5) and using (3.1), we have

$$\begin{aligned} 0 = \quad & D(W)S(Y,Z) - D(Y)S(W,Z) + E(W)[(2n+1)g(Y,Z) + (2n-1)\eta(Y)\eta(Z)] \\ & -E(Y)[(2n+1)g(W,Z) + (2n-1)\eta(W)\eta(Z)] + E(Y)g(W,Z) - E(W)g(Y,Z) \\ & +\mu[D(hY)g(W,Z) - D(hW)g(Y,Z)] + \kappa[D(Y)g(W,Z) - D(W)g(Y,Z)] \\ & +E(Y)\eta(W)\eta(Z) - E(W)\eta(Y)\eta(Z) + g(W,Z)\eta(\rho_4)\eta(Y) \\ & -g(Y,Z)\eta(\rho_4)\eta(W). \end{aligned} \tag{5.6}$$

Let $\{e_i\}$ $(1 \leqslant i \leqslant 2n+1)$ be a local orthonormal basis. Putting $Y = Z = e_i$ in (5.6) and taking the summation over i, we derive

$$\begin{aligned} 0 = \quad & (r - 2n\kappa)D(W) - D(QW) + (\mu - 2n - 1)D(hW) + 2(2n^2 + n - 1)E(W) \\ & -2(2n-1)\eta(\rho_4)\eta(W). \end{aligned} \tag{5.7}$$

Putting $W = \xi$ and using (2.3) and (3.5) in (5.7), we have the relation (5.3). □

Since the proof of the following theorem is quite similar to the Theorem (4.2), so we do not give the proof it.

Theorem 5.2. *A $(2n+1)$-dimensional quasi-generalized ϕ-recurrent paracontact metric $(\kappa \neq -1, \mu)$-manifold is either a paracontact metric $(\frac{r+4n^2-4n}{2n}, 0)$-manifold or characteristic vector field ξ and ρ_3 (is a vector field associated with 1-form D) are co-directional.*

Theorem 5.3. *A $(2n+1)$-dimensional quasi-generalized ϕ-recurrent paracontact metric $(\kappa \neq -1, 0)$-manifold is a super generalized Ricci-recurrent manifold.*

Proof. Taking the inner product of (5.4) with U, we get

$$\begin{aligned} g((\nabla_W R)(X,Y)Z,U) - \eta((\nabla_W R)(X,Y)Z)\eta(U) = \quad & D(W)g(R(X,Y)Z,U) \\ & +E(W)g(F(X,Y)Z,U). \end{aligned} \tag{5.8}$$

Contracting (5.8) over X and U, we get

$$\begin{aligned} & \textstyle\sum_{i=1}^{2n+1} \varepsilon_i[g((\nabla_W R)(e_i,Y)Z,e_i) - \eta((\nabla_W R)(e_i,Y)Z)\eta(e_i)] \\ & = \textstyle\sum_{i=1}^{2n+1} \varepsilon_i[D(W)g(R(e_i,Y)Z,e_i) + E(W)g(F(e_i,Y)Z,e_i)]. \end{aligned} \tag{5.9}$$

Using (5.2) in (5.9), we obtain

$$(\nabla_W S)(Y,Z) - \eta((\nabla_W R)(\xi,Y)Z) = E(W)[(2n+1)g(Y,Z) + (2n-1)\eta(Y)\eta(Z)] + D(W)S(Y,Z). \tag{5.10}$$

Since $\mu = 0$, using (3.16) in (5.10), we derive

$$(\nabla_W S)(Y,Z) = D(W)S(Y,Z) + \Pi_1(W)g(Y,Z) + \Pi_2(W)\eta(Y)\eta(Z), \tag{5.11}$$

where, $\Pi_1(W) = (2n+1)E(W)$ and $\Pi_2(W) = (2n-1)E(W)$. Thus the proof of the theorem is completed from (1.1) □

From (5.11), we can give the following result:

Corollary 5.1. *A $(2n+1)$-dimensional quasi-generalized ϕ-recurrent paracontact metric $(\kappa \neq -1, 0)$-manifold can never be a quasi-generalized Ricci-recurrent manifold.*

Theorem 5.4. *A $(2n+1)$-dimensional quasi-generalized ϕ-recurrent paracontact metric $(\kappa \neq -1, \mu)$-manifold the 1-forms D and E satisfy the following relation*

$$\kappa D(W) + 2E(W) = 0. \tag{5.12}$$

Proof. Letting $Z = \xi$ in (5.10), we obtain

$$(\nabla_W S)(Y,\xi) - \eta((\nabla_W R)(\xi,Y)\xi) = D(W)S(Y,\xi) + 4nE(W)\eta(Y). \tag{5.13}$$

Putting $Y = \xi$ in the above equation and using (3.13) and (3.16), we get (5.12). □

Theorem 5.5. *In a $(2n+1)$-dimensional quasi-generalized ϕ-recurrent paracontact metric $(\kappa \neq -1, \mu)$-manifold, the scalar curvature r can never be zero and the following equation holds*

$$rD(W) + 2n(2n+3)E(W) = 0. \tag{5.14}$$

Proof. Letting $Y = Z = e_i$ in (5.10) and taking the summation over i, we have

$$dr(W) = rD(W) + 2n(2n+3)E(W). \tag{5.15}$$

Since E is a non-vanishing 1-form, r can not be zero from (5.15). From (3.14), since r is constant we get (5.14). □

Corollary 5.2. *In a $(2n+1)$-dimensional quasi-generalized ϕ-recurrent paracontact metric $(\kappa \neq -1, \mu)$-manifold, the scalar curvature can be given by $r = (2n+3)n\kappa$ and also for a quasi-generalized ϕ-recurrent paracontact metric $(\kappa \neq -1, 0)$-manifold, $\kappa = \frac{4(1-n)}{2n+1}$.*

Proof. From (5.12), we have $E(W) = -\frac{\kappa}{2}D(W)$. Using this in (5.14), we get

$$(r - n\kappa(2n+3))D(W) = 0. \tag{5.16}$$

Since D is a non-vanishing 1-form, the equation (5.16) implies

$$r - n\kappa(2n+3) = 0. \tag{5.17}$$

For a $(2n+1)$-dimensional quasi-generalized ϕ-recurrent paracontact metric $(\kappa \neq -1, 0)$-manifold, using (5.17) in (3.14), we derive $\kappa = \frac{4(1-n)}{2n+1}$. $\square$

From Theorem 5.5, the scalar curvature can not be vanish in a quasi-generalized paracontact metric $(\kappa \neq -1, \mu)$-manifold. With the help of this, we observe that κ can not be zero from (5.17). Therefore, we can state the following theorem.

Theorem 5.6. *There does not exist any $(2n+1)$-dimensional quasi-generalized ϕ-recurrent paracontact metric $(0, \mu)$-manifold.*

Theorem 5.7. *A $(2n+1)$-dimensional quasi-generalized ϕ-recurrent paracontact metric $(\kappa \neq -1, 0)$-manifold is an η-Einstein manifold of dimension $2n+1$ if $\kappa = \frac{1-n}{n}$, where $n > 1$.*

Proof. The equation (4.25) also holds for a $(2n+1)$-dimensional quasi-generalized ϕ-recurrent paracontact metric $(\kappa \neq -1, 0)$-manifold. Then comparing (4.25) and (5.10), we obtain

$$2n\kappa g(Y, -\phi W + \phi hW) - S(Y, -\phi W + \phi hW) = [2n\kappa D(W) + 4nE(W)]\eta(Y). \tag{5.18}$$

Letting $Y = \phi Y$ and using (2.2) and (3.11) in (5.18), we have

$$\begin{aligned} S(Y,W) - S(Y,hW) = \ & 2[2(1-n) - nk]g(Y,W) - 2[2(1-n) - n\kappa]g(Y,hW) \\ & 2[2(n-1) + 2n\kappa]\eta(Y)\eta(W). \end{aligned} \tag{5.19}$$

Using (3.12) in (5.19), the Ricci tensor becomes

$$S(Y,W) = \alpha g(Y,W) + \beta g(Y,hW) + \gamma\eta(Y)\eta(W), \tag{5.20}$$

where $\alpha = -2n - 2\kappa + 2$, $\beta = 2n\kappa + 2n - 2$ and $\gamma = 2n\kappa + 2n + 2\kappa - 2$. If $\beta = 0$, then $\kappa = \frac{1-n}{n}$. $\square$

Theorem 5.8. *There does not exist any Einstein quasi-generalized ϕ-recurrent paracontact metric $(\kappa \neq -1, 0)$-manifold.*

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