FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 40, No 3 (2025), 525–533
https://doi.org/10.22190/FUMI240519038O
Original Scientific Paper

NEW CRITERIA FOR STARLIKENESS IN THE UNIT DISC

Milutin Obradović[1] and Nikola Tuneski[2]

[1] **Department of Mathematics, Faculty of Civil Engineering**
University of Belgrade, Bulevar Kralja Aleksandra, 11000 Belgrade, Serbia
[2] **Department of Mathematics and Informatics, Faculty of Mechanical Engineering**
Ss. Cyril and Methodius University in Skopje, Karpoš II b.b.
1000 Skopje, Republic of North Macedonia

ORCID IDs: Milutin Obradović https://orcid.org/0000-0001-8890-0764
Nikola Tuneski https://orcid.org/0000-0003-3889-0048

Abstract. It is well-known that the condition $\operatorname{Re}\left[1+\frac{zf''(z)}{f'(z)}\right]>0$, $z\in\mathbb{D}$, implies that f is starlike function (i.e. convexity implies starlikeness). If the previous condition is not satisfied for every $z\in\mathbb{D}$, then it is possible to get new criteria for starlikeness by using $\left|\arg\left[\alpha+\frac{zf''(z)}{f'(z)}\right]\right|$, $z\in\mathbb{D}$, where $\alpha>1$.
Keywords: Starlike functions, analytic function criteria, argument conditions.

1. Introduction and definitions

Let $\mathcal{A}$ be the class of functions f which are analytic in the open unit disc $\mathbb{D}=\{z:|z|<1\}$ of the form $f(z)=z+a_2z^2+a_3z^3+\cdots$, and let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of functions that are univalent in $\mathbb{D}$.

Also, let

$$\begin{aligned}\mathcal{C} &= \left[f\in\mathcal{A}:\operatorname{Re}\left[1+\frac{zf''(z)}{f'(z)}\right]>0,\ (z\in\mathbb{D})\right],\\ \mathcal{S}^\star &= \left[f\in\mathcal{A}:\operatorname{Re}\left[\frac{zf'(z)}{f(z)}\right]>0,\ (z\in\mathbb{D})\right],\end{aligned}$$

Received: May 19, 2024, revised: February 08, 2025, accepted: February 08, 2025
Communicated by Dragan Đorđević
Corresponding Author: N. Tuneski
E-mail addresses: obrad@grf.bg.ac.rs (M. Obradović), nikola.tuneski@mf.edu.mk (N. Tuneski)
2020 *Mathematics Subject Classification.* Primary 30C45; Secondary 30C50

denote the classes of convex and starlike functions, respectively. It is well-known that

$$f \in \mathcal{C} \quad \Rightarrow \quad f \in \mathcal{S}^\star,$$

(see Duren [1]). If the condition for convexity, $\mathrm{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] > 0$, is not satisfied for every $z \in \mathbb{D}$, we cannot conclude that $f \in \mathcal{C}$, and consequently, $f \in \mathcal{S}^\star$.

In this paper $f \in \mathcal{S}^\star$ in terms of the expression

$$\left|\arg\left[\alpha + \frac{zf''(z)}{f'(z)}\right]\right|, \quad (z \in \mathbb{D})$$

where $\alpha > 1$.

2. Main results

For our consideration, we need the next lemma given by Nunokawa in [3].

Lemma A. *Let $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ be analytic in the unit disc $\mathbb{D}$ and $p(z) \neq 0$ for $z \in \mathbb{D}$. Also, let's suppose that there exists a point $z_0 \in \mathbb{D}$ such that $\mathrm{Re}\, p(z) > 0$ for $|z| < |z_0|$ and $\mathrm{Re}\, p(z_0) = 0$, where $p(z_0) \neq 0$, i.e. $p(z_0) = ia$, a is real and $a \neq 0$. Then we have*

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik,$$

where $k \geq \frac{1}{2}\left(a + \frac{1}{a}\right)$, when $a > 0$, and $k \leq -\frac{1}{2}\left(|a| + \frac{1}{|a|}\right)$, when $a < 0$.

Theorem 2.1. *Let $f \in \mathcal{A}$, $\alpha > 1$, and let*

$$\left|\arg\left[\alpha + \frac{zf''(z)}{f'(z)}\right]\right| < \mathrm{arctg}\frac{\sqrt{3}}{\alpha - 1} \quad (z \in \mathbb{D}). \tag{2.1}$$

Then $f \in \mathcal{S}^\star$.

Proof. If we put that $p(z) = \frac{zf'(z)}{f(z)}$, then $p(0) = 1$ and $\frac{zp'(z)}{p(z)} + p(z) - 1 = \frac{zf''(z)}{f'(z)}$. So, we would need to prove that the following implication holds:

$$\left|\arg\left[\frac{zp'(z)}{p(z)} + p(z) + \alpha - 1\right]\right| \quad < \quad \mathrm{arctg}\frac{\sqrt{3}}{\alpha - 1} \quad (z \in \mathbb{D}) \tag{2.2}$$

$$\Rightarrow \quad \mathrm{Re}\, p(z) \quad > \quad 0 \quad (z \in \mathbb{D}). \tag{2.3}$$

First, we will prove that $p(z) \neq 0, z \in \mathbb{D}$. On the contrary, if there exists $z_1 \in \mathbb{D}$ such that z_1 is the zero of order m of the function p, then $p(z) = (z - z_1)^m p_1(z)$, where m is positive integer, p_1 is analytic in $\mathbb{D}$ with $p_1(z_1) \neq 0$, and further

$$\frac{zp'(z)}{p(z)} = \frac{mz}{z - z_1} + \frac{zp_1'(z)}{p_1(z)}.$$

This means that the real part of the right hand side can tend to $-\infty$ when $z \to z_1$, which is a contradiction to the assumption of the theorem regarding the argument. Thus $p(z) \neq 0$, $z \in \mathbb{D}$.

Now, let's suppose that the implication (2.2) does not hold in the unit disc. It means that there exists a point $z_0 \in \mathbb{D}$ such that $\operatorname{Re} p(z) > 0$ for $|z| < |z_0|$ and $\operatorname{Re} p(z_0) = 0$, where $p(z_0) \neq 0$. If we put $p(z_0) = ia$, a is real and $a \neq 0$, then by Lemma A, we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik,$$

where $k \geq \frac{1}{2}\left(a + \frac{1}{a}\right)$, when $a > 0$, and $k \leq -\frac{1}{2}\left(|a| + \frac{1}{|a|}\right)$, when $a < 0$.

Next, if we put $\Phi(z) = \frac{zp'(z)}{p(z)} + p(z) + \alpha - 1$, then by the previous facts:

$$\operatorname{Re}\Phi(z_0) = \alpha - 1, \quad \operatorname{Im}\Phi(z_0) = k + a,$$

which for $a > 0$ implies:

$$\begin{aligned} \arg\Phi(z_0) &= \operatorname{arctg}\frac{k+a}{\alpha-1} \geq \operatorname{arctg}\frac{\frac{1}{2}\left(a+\frac{1}{a}\right)+a}{\alpha-1} \\ &= \operatorname{arctg}\frac{3a+\frac{1}{a}}{2(\alpha-1)} \geq \operatorname{arctg}\frac{\sqrt{3}}{\alpha-1}, \end{aligned}$$

because the function $\varphi(a) = 3a + \frac{1}{a}$ has its minimum value $\varphi(\frac{1}{\sqrt{3}}) = 2\sqrt{3}$. Similarly, for $a < 0$:

$$\begin{aligned} \arg\Phi(z_0) &= \operatorname{arctg}\frac{k+a}{\alpha-1} \leq \operatorname{arctg}\frac{-\frac{1}{2}\left(|a|+\frac{1}{|a|}\right)-|a|}{\alpha-1} \\ &= -\operatorname{arctg}\frac{3|a|+\frac{1}{|a|}}{2(\alpha-1)} \leq -\operatorname{arctg}\frac{\sqrt{3}}{\alpha-1}. \end{aligned}$$

Combining the cases $a > 0$ and $a < 0$, we receive

$$|\arg\Phi(z_0)| \geq \operatorname{arctg}\frac{\sqrt{3}}{\alpha-1},$$

which is a contradiction to the relation (2.1).

This show that $\operatorname{Re} p(z) = \operatorname{Re}\frac{zf'(z)}{f(z)} > 0$, for all $z \in \mathbb{D}$, i.e. that $f \in \mathcal{S}^\star$. $\square$

Example 2.1. Let $f \in \mathcal{A}$ is defined by the condition

$$1 + \frac{zf''(z)}{f'(z)} = (\sqrt{3}+1)\left(\frac{1+z}{1-z}\right)^{\frac{1}{2}} - \sqrt{3}, \tag{2.4}$$

where we use the principal value of the square root. For real z close to -1 we have $\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < 0$, which means that f is not convex. On the other hand, from (2.4) we have

$$\frac{zf''(z)}{f'(z)} + (\sqrt{3}+1) = (\sqrt{3}+1)\left(\frac{1+z}{1-z}\right)^{\frac{1}{2}}$$

and from here

$$\begin{aligned}\left|\arg\left[\frac{zf''(z)}{f'(z)}+(\sqrt{3}+1)\right]\right| &\leq \frac{1}{2}\left|\arg\frac{1+z}{1-z}\right| \\ &< \frac{\pi}{4}=\operatorname{arctg}\frac{\sqrt{3}}{(\sqrt{3}+1)-1},\end{aligned}$$

which by Theorem 2.1 (with $\alpha=\sqrt{3}+1$) implies that $f\in\mathcal{S}^\star$.

Letting α tend to 1 in Theorem 2.1, we have the following well-known result.

Corollary 2.1. *Let $f\in\mathcal{A}$ and*

$$\left|\arg\left[\frac{zf''(z)}{f'(z)}+1\right]\right|<\frac{\pi}{2}\quad(z\in\mathbb{D})$$

Then $f\in\mathcal{S}^\star$.

Further, it is easy to verify that the disc with center α and radius $\frac{\alpha\sqrt{3}}{\sqrt{3+(\alpha-1)^2}}$ is contained in the angle $|\arg z|<\operatorname{arctg}\frac{\sqrt{3}}{\alpha-1}$, and so by using the result of Theorem 2.1, we have that

$$\left|\frac{zf''(z)}{f'(z)}\right|=\left|\left(\frac{zf''(z)}{f'(z)}+\alpha\right)-\alpha\right|<\frac{\alpha\sqrt{3}}{\sqrt{3+(\alpha-1)^2}}\quad(z\in\mathbb{D})$$

implies $f\in\mathcal{S}^\star$. Since the function $\varphi(\alpha)=:\frac{\alpha\sqrt{3}}{\sqrt{3+(\alpha-1)^2}}$, $\alpha\geqslant 1$, attains its maximal value 2 for $\alpha=4$, we get

Corollary 2.2. *Let $f\in\mathcal{A}$ and*

$$\left|\frac{zf''(z)}{f'(z)}\right|<2\quad(z\in\mathbb{D})$$

Then $f\in\mathcal{S}^\star$.

In a similar way as in Theorem 2.1, we can consider the starlikeness problem in connection with the class defined by

$$\mathcal{G}=\left[f\in\mathcal{A}:\operatorname{Re}\left[1+\frac{zf''(z)}{f'(z)}\right]<\frac{3}{2},\,z\in\mathbb{D}\right].$$

Ozaki in [4] proved that if $f\in\mathcal{G}$, then f is univalent in $\mathbb{D}$. Later, Umezava in [6] showed the functions from $\mathcal{G}$ are convex in one direction. Also, it is shown in the papers [2] and [5] that $\mathcal{G}$ is subclass of $\mathcal{S}^\star$.

If $\operatorname{Re}\left[1+\frac{zf''(z)}{f'(z)}\right] < \frac{3}{2}$ is not satisfied for every $z \in \mathbb{D}$, then we can pose a question if for some $\beta < 1$, such that

$$\operatorname{Re}\left[\beta+\frac{zf''(z)}{f'(z)}\right] < \frac{3}{2} \quad (z \in \mathbb{D}), \tag{2.5}$$

is it possible obtain sufficient condition for starlikeness, in a similar way as in Theorem 2.1. The condition (2.5) is equivalent to

$$\operatorname{Re}\left[\frac{3-2\beta}{2}-\frac{zf''(z)}{f'(z)}\right] > 0 \quad (z \in \mathbb{D}),$$

and we can use similar technique. This lead so

Theorem 2.2. *Let $f \in \mathcal{A}$, $\beta < 1$, and let*

$$\left|\arg\left[\frac{3-2\beta}{2}-\frac{zf''(z)}{f'(z)}\right]\right| < \operatorname{arctg}\frac{2\sqrt{3}}{5-2\beta} \quad (z \in \mathbb{D}).$$

Then $f \in \mathcal{S}^\star$.

Proof. As in the proof of Theorem 2.1, we put that $p(z) = \frac{zf'(z)}{f(z)}$, such that $p(0) = 1$ and $\frac{zp'(z)}{p(z)} + p(z) - 1 = \frac{zf''(z)}{f'(z)}$. Now, our aim is to prove that for some $\beta < 1$ the following implication holds:

$$\left|\arg\left[\frac{5-2\beta}{2}-\frac{zp'(z)}{p(z)}-p(z)\right]\right| < \operatorname{arctg}\frac{2\sqrt{3}}{5-2\beta} \quad (z \in \mathbb{D}) \tag{2.6}$$

$$\Rightarrow \operatorname{Re} p(z) > 0 \quad (z \in \mathbb{D}). \tag{2.7}$$

First, as in Theorem 2.1 we conclude that $p(z) \neq 0, z \in \mathbb{D}$.

Further, if the implication (2.6) is not true, there exists a point $z_0 \in \mathbb{D}$ such that $\operatorname{Re} p(z) > 0$ for $|z| < |z_0|$ and $\operatorname{Re} p(z_0) = 0$, where $p(z_0) \neq 0$. If we put $p(z_0) = ia$, a is real and $a \neq 0$, then by Lemma A we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik,$$

where $k \geq \frac{1}{2}\left(a+\frac{1}{a}\right)$, when $a > 0$, and $k \leq -\frac{1}{2}\left(|a|+\frac{1}{|a|}\right)$, when $a < 0$. Also, for

$$\Psi(z) = \frac{5-2\beta}{2}-\frac{zp'(z)}{p(z)}-p(z),$$

using the previous conclusions, we have

$$\operatorname{Re}\Psi(z_0) = \frac{5-2\beta}{2}, \quad \operatorname{Im}\Psi(z_0) = -(k+a).$$

Using the same method as in the proof of Theorem 2.1, we easily conclude that

$$\arg\Psi(z_0) = -\operatorname{arctg}\frac{2(k+a)}{5-2\beta} \le -\operatorname{arctg}\frac{2\sqrt{3}}{5-2\beta},$$

when $a > 0$, and

$$\arg\Psi(z_0) = -\operatorname{arctg}\frac{2(k+a)}{5-2\beta} \ge \operatorname{arctg}\frac{2\sqrt{3}}{5-2\beta},$$

when $a < 0$. These facts imply a contradiction of the assumption in (2.6).

So, we have the statement of this theorem. □

Letting β tend to 1 in Theorem 2.2, we receive

Corollary 2.3. *Let $f \in \mathcal{A}$ and*

$$\left|\arg\left[\frac{1}{2} - \frac{zf''(z)}{f'(z)}\right]\right| < \operatorname{arctg}\frac{2}{\sqrt{3}} \approx 49.1^\circ \quad (z \in \mathbb{D}).$$

Then $f \in \mathcal{S}^\star$.

As in the case of Theorem 2.1, we can show that the disc with center $\frac{3-2\beta}{2}$ and radius $\frac{(3-2\beta)\sqrt{3}}{\sqrt{12+(5-2\beta)^2}}$ is containing in the angle $|\arg z| \le \operatorname{arctg}\frac{2\sqrt{3}}{5-2\beta}$. So, if

$$\left|\frac{zf''(z)}{f'(z)}\right| = \left|\left(\frac{3-2\beta}{2} - \frac{zf''(z)}{f'(z)}\right) - \frac{3-2\beta}{2}\right| < \frac{(3-2\beta)\sqrt{3}}{\sqrt{12+(5-2\beta)^2}},$$

for all $z \in \mathbb{D}$, then Theorem 2.2 brings that $f \in \mathcal{S}^\star$. Since the function $\psi(\beta) =: \frac{(3-2\beta)\sqrt{3}}{\sqrt{12+(5-2\beta)^2}}$ is decreasing on the interval $(-\infty, 1]$, and $\psi(\beta) \to \sqrt{3}$, when $\beta \to -\infty$, we get

Corollary 2.4. *Let $f \in \mathcal{A}$ and*

$$\left|\frac{zf''(z)}{f'(z)}\right| < \sqrt{3} \quad (z \in \mathbb{D}).$$

Then $f \in \mathcal{S}^\star$.

For a starlike function, f it is not necessary that $Re\frac{f(z)}{z} > 0$, $z \in \mathbb{D}$. For example, for Koebe function $k(z) = \frac{z}{(1-z)^2}$ we have

$$\operatorname{Re}\frac{k(z)}{z}\bigg|_{z=(1+i)/\sqrt{2}} = -(\sqrt{2}+1) < 0,$$

and this also holds for points in $\mathbb{D}$ close enough to $\frac{1+i}{\sqrt{2}}$.

In the next theorem, we give a condition which provides $\operatorname{Re}\frac{f(z)}{z} > 0$, $z \in \mathbb{D}$.

Theorem 2.3. *Let $f \in \mathcal{A}, \gamma \geq 0,$ and let*

$$\left|\arg\left[\frac{zf'(z)}{f(z)} + \gamma\right]\right| < \operatorname{arctg}\frac{1}{1+\gamma} \quad (z \in \mathbb{D}).$$

Then $\operatorname{Re}\frac{f(z)}{z} > 0,\ z \in \mathbb{D}.$

Proof. We apply the same method as in the previous theorems. In this case, we use $p(z) = \frac{f(z)}{z}$, such that $p(0) = 1$ and $\frac{zp'(z)}{p(z)} + 1 = \frac{zf'(z)}{f(z)}$. So, the result we need to prove can be rewritten in the following equivalent form:

$$\left|\arg\left[\frac{zp'(z)}{p(z)} + 1 + \gamma\right]\right| < \operatorname{arctg}\frac{1}{1+\gamma} \quad (z \in \mathbb{D}) \tag{2.8}$$

$$\Rightarrow \quad \operatorname{Re} p(z) > 0 \quad (z \in \mathbb{D}). \tag{2.9}$$

First, $p(z) \neq 0$ for all $z \in \mathbb{D}$.

Next, if the implication (2.8) does not hold in the unit disc, then there exists a point $z_0 \in \mathbb{D}$ such that $\operatorname{Re} p(z) > 0$ for $|z| < |z_0|$ and $\operatorname{Re} p(z_0) = 0$, where $p(z_0) \neq 0$. If we put $p(z_0) = ia$, a is real and $a \neq 0$, then by Lemma A we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik,$$

where $k \geq \frac{1}{2}\left(a + \frac{1}{a}\right)$, when $a > 0$, and $k \leq -\frac{1}{2}\left(|a| + \frac{1}{|a|}\right)$, when $a < 0$. Now, for $a > 0$ we get:

$$\begin{aligned}\arg\left[\frac{z_0 p'(z_0)}{p(z_0)} + 1 + \gamma\right] &= \arg(ik + 1 + \gamma) = \operatorname{arctg}\frac{k}{1+\gamma}\\ &\geq \frac{\frac{1}{2}\left(a + \frac{1}{a}\right)}{1+\gamma} \geq \operatorname{arctg}\frac{1}{1+\gamma},\end{aligned}$$

and for $a < 0$,

$$\arg\left[\frac{z_0 p'(z_0)}{p(z_0)} + 1 + \gamma\right] \leq -\operatorname{arctg}\frac{1}{1+\gamma}.$$

By combining the above conclusions, we receive that

$$\left|\arg\left(\frac{z_0 p'(z_0)}{p(z_0)} + 1 + \gamma\right)\right| \geq \operatorname{arctg}\frac{1}{1+\gamma},$$

which is a contradiction to the assumption in (2.8). □

For $\gamma = 0$ in the previous theorem, we have the following result.

Corollary 2.5. *Let $f \in \mathcal{A}$ and*

$$\left|\arg\frac{zf'(z)}{f(z)}\right| < \frac{\pi}{4} \quad (z \in \mathbb{D}).$$

Then $\operatorname{Re}\frac{f(z)}{z} > 0,\ z \in \mathbb{D}.$

In Theorem 2.3 the function f from $\mathcal{A}$ need not be starlike as the next example shows.

Example 2.2. Let $f \in \mathcal{A}$ is defined by

$$\frac{zf'(z)}{f(z)} = \sqrt{3}\left(\frac{1+z}{1-z}\right)^{\frac{1}{3}} + 1 - \sqrt{3}, \tag{2.10}$$

where we use the principal value for the square root. For real z close to -1 we have that $\mathrm{Re}\left[\frac{zf'(z)}{f(z)}\right] < 0$, which means that f is not starlike. But from (2.10) we have

$$\frac{zf'(z)}{f(z)} + (\sqrt{3}-1) = \sqrt{3}\left(\frac{1+z}{1-z}\right)^{\frac{1}{3}},$$

and from here

$$\begin{aligned}\left|\arg\left[\frac{zf'(z)}{f(z)} + (\sqrt{3}-1)\right]\right| &\le \frac{1}{3}\left|\arg\left(\frac{1+z}{1-z}\right)\right| \\ &< \frac{\pi}{6} = \operatorname{arctg}\frac{1}{(\sqrt{3}-1)+1},\end{aligned}$$

which by Theorem 2.3 (with $\gamma = \sqrt{3}-1$) implies that $\mathrm{Re}\,\frac{f(z)}{z} > 0$, $z \in \mathbb{D}$.

It is possible to show that the disc with center $1+\gamma$ and radius $\frac{1+\gamma}{\sqrt{1+(1+\gamma)^2}}$ lies in the angle $|\arg z| \le \operatorname{arctg}\frac{1}{1+\gamma}$, and so, by using the result of Theorem 2.3 we have that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| = \left|\left(\frac{zf'(z)}{f(z)} + \gamma\right) - (1+\gamma)\right| < \frac{1+\gamma}{\sqrt{1+(1+\gamma)^2}} \quad (z \in \mathbb{D})$$

implies $\mathrm{Re}\,\frac{f(z)}{z} > 0$, $z \in \mathbb{D}$. When $\gamma \to +\infty$ we receive

Corollary 2.6. *Let* $f \in \mathcal{A}$ *and*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 \quad (z \in \mathbb{D}).$$

Then $\mathrm{Re}\,\frac{f(z)}{z} > 0$, $z \in \mathbb{D}$.

REFERENCES

1. P. L. Duren: *Univalent function.* Springer-Verlag, New York, 1983.
2. I. Jovanović and M. Obradović: *A note on certain classes of univalent functions. Filomat,* **9**(1) (1995), 69–72.
3. M. Nunokawa: *On properties of non-Caratheodory functions.* Proc. Japan Acad. **68**, Ser.A (1992), 152–153.

4. S. Ozaki: *On the theory of multivalent functions, II.* Sci. Rep. Tokyo, Bunrika Daigaku, **4** (1941), 45–87.
5. R. Singh and S. Singh: *Some sufficient conditions for univalence and starlikeness.* Colloq. Math. **47** (1982), 309–314.
6. T. Umezava: *Analytic functions convex in one direction.* J. Math. Soc. Japan, **4** (1952), 194–202.