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NEW RESULTS ON $(\grave{k}, \grave{\mu})$-CONTACT METRIC MANIFOLDS

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Abstract. In the present paper, we introduce generalized extended C-Bochner curvature tensor on $(\grave{k}, \grave{\mu})$-contact metric manifolds. Also, we study $\grave{h}$-generalized extended C-Bochner semisymmetric and ψ-generalized extended C-Bochner semisymmetric non-Sasakian $(\grave{k}, \grave{\mu})$-contact metric manifolds.
Keywords: contact metric manifolds, curvature tensor, semisymmetry conditions.

1. Introduction

In recent years, some symmetry or some semisymmetry conditions of a Riemannian manifold have been studied by many authors ([7], [8], [15]). A Riemannian manifold is called semisymmetric if its curvature tensor R_{cur} satisfies $R_{cur}(\partial_1, \partial_2) \cdot R_{cur} = 0, \ \partial_1, \partial_2 \in \chi(G)$ ([12], [17]).

The Bochner curvature tensor was introduced by S. Bochner [5]. Geometric properties of the Bochner curvature tensor were given by D. E. Blair [2]. By using the Boothby-Wang's fibration, the C-Bochner curvature tensor was introduced by M. Matsumoto and G. Chūman [13]. They studied its properties in a Sasakian manifold. Also, vanishing C-Bochner curvature tensor in Sasakian manifolds was studied by some authors ([10], [11]). Then extended C-Bochner curvature tensor on a K-contact Riemannian manifold was introduced by H. Endo, and it was called

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the *E-contact Bochner curvature tensor* [9]. He was showed that a K-contact Riemannian manifold with vanishing E-Bochner curvature tensor is a Sasakian manifold. Also *generalized C-Bochner* curvature tensor was defined by Shaikh and Baishya [16].

Motivated by these studies, in the present paper after preliminaries in section 3, firstly, we give generalized the C-Bochner curvature tensor defined by Shaikh and Baishya, then using the H. Endo's definition we define generalized extended C-Bochner (briefly GEC-Bochner) curvature tensor and give some basic results. In the last section we give the main results of the paper.

2. Preliminaries

Let G^{2n+1} be a connected differentiable manifold which is said to admit an *almost contact structure* $(\grave{\psi}, \grave{\zeta}, \eta)$, where $\grave{\psi}$ is a tensor field of type $(1,1)$, $\grave{\zeta}$ is a vector field and η is a 1-form satisfying

$$\grave{\psi}^2 = -I + \eta \otimes \grave{\zeta}, \quad \eta(\grave{\zeta}) = 1, \quad \grave{\psi}\zeta = 0, \ \eta \circ \grave{\psi} = 0. \tag{2.1}$$

Let ρ be a compatible Riemannian metric with $(\grave{\psi}, \grave{\zeta}, \eta)$, that is,

$$\rho(\grave{\psi}\partial_1, \grave{\psi}\partial_2) = \rho(\partial_1, \partial_2) - \eta(\partial_1)\eta(\partial_2) \tag{2.2}$$

or equivalently,

$$\rho(\partial_1, \grave{\psi}\partial_2) = -\rho(\grave{\psi}\partial_1, \partial_2) \qquad \text{and} \qquad \eta(\partial_1) = \rho(\partial_1, \grave{\zeta})$$

for all $\partial_1, \partial_2 \in \Gamma(TG)$. Thus, G is an almost contact metric manifold equipped with $(\grave{\psi}, \grave{\zeta}, \eta, \rho)$. An almost contact metric structure is a contact metric structure if

$$\rho(\partial_1, \grave{\psi}\partial_2) = d\eta(\partial_1, \partial_2).$$

The 1-form η is then a contact form and $\grave{\zeta}$ is its characteristic vector field. A Sasakian manifold satisfies [4]

$$R_{cur}(\partial_1, \partial_2)\grave{\zeta} = \eta(\partial_2)\partial_1 - \eta(\partial_1)\partial_2. \tag{2.3}$$

In a contact metric manifold, the $(1,1)$-tensor field $\grave{\hbar}$ is symmetric and satisfies

$$\grave{\hbar}\grave{\zeta} = 0, \quad \grave{\hbar}\grave{\psi} + \grave{\psi}\grave{\hbar} = 0, \quad \nabla\grave{\zeta} = -\grave{\psi} - \grave{\psi}\grave{\hbar}, \quad tr\grave{\hbar} = tr\psi\grave{\hbar} = 0. \tag{2.4}$$

A contact metric manifold is said to be η-Einstein if

$$R_{op} = aId + b\eta \otimes \grave{\zeta},$$

where a, b are smooth functions on G.

The $(\grave{k}, \grave{\mu})$-nullity distribution $\tilde{N}(\grave{k}, \grave{\mu})$ of a contact metric manifold G is defined by

$$\begin{aligned}\tilde{N}(\grave{k}, \grave{\mu}) \quad : \quad t \longrightarrow \tilde{N}_t(\grave{k}, \grave{\mu}) = \{\partial_3 \in T_tG \mid R_{cur}(\partial_1, \partial_2)\partial_3 &= \grave{k}[\rho(\partial_2, \partial_3)\partial_1 - \rho(\partial_1, \partial_3)\partial_2] \\ &+ \grave{\mu}[\rho(\partial_2, \partial_3)\grave{\hbar}\partial_1 - \rho(\partial_1, \partial_3)\grave{\hbar}\partial_2]\},\end{aligned}$$

for all $\partial_1, \partial_2 \in \Gamma(TG)$, where $\grave{k}, \grave{\mu}$ real constants ([3], [14]). In a $(\grave{k}, \grave{\mu})$-contact metric manifold, we have

$$R_{cur}(\partial_1, \partial_2)\grave{\zeta} = \grave{k}\{\eta(\partial_2)\partial_1 - \eta(\partial_1)\partial_2\} + \grave{\mu}\left\{\eta(\partial_2)\grave{\hbar}\partial_1 - \eta(\partial_1)\grave{\hbar}\partial_2\right\}. \tag{2.5}$$

Such a manifold was studied by Arslan and et. al [1]. If $\grave{\mu} = 0$, then we obtain the condition of $\grave{k}$-nullity distribution was introduced by Tanno [18].

In a $(\grave{k}, \grave{\mu})$-contact metric manifold ([3], [6]), we have

$$R_{cur}(\partial_1, \grave{\zeta}) = 2nk\eta(\partial_1), \tag{2.6}$$

$$R_{op}\grave{\zeta} = 2nk\zeta, \tag{2.7}$$

$$\grave{\hbar}^2 = (\grave{k} - 1)\grave{\psi}^2, \ \grave{k} \le 1, \tag{2.8}$$

$$R_{op}\grave{\psi} - \grave{\psi}R_{op} = 2[2(n-1) + \grave{\mu}]\grave{\hbar}\grave{\psi}, \tag{2.9}$$

$$R_{cur}(\grave{\zeta}, \partial_1)\partial_2 = \grave{k}\{\rho(\partial_1, \partial_2)\grave{\zeta} - \eta(\partial_2)\partial_1\} + \grave{\mu}\left\{\rho(\grave{\hbar}\partial_1, \partial_2)\grave{\zeta} - \eta(\partial_2)\grave{\hbar}\partial_1\right\}, \tag{2.10}$$

where R_{op} is the Ricci operator.

Also from (2.6)-(2.7), we have

$$tr\grave{\hbar}^2 = 2n(1 - \grave{k}),$$
$$R_{cur}(\partial_1, \grave{\psi}\partial_2) + R_{cur}(\grave{\psi}\partial_1, \partial_2) = 2(2(n-1) + \grave{\mu})\rho(\grave{\hbar}\grave{\psi}\partial_1, \partial_2),$$
$$R_{cur}(\grave{\psi}\partial_1, \grave{\psi}\partial_2) = R_{cur}(\partial_1, \partial_2) - 2nk\eta(\partial_1)\eta(\partial_2) - 2(2(n-1) + \grave{\mu})\rho(\grave{\hbar}\partial_1, \partial_2),$$
$$R_{op}\grave{\psi} + \grave{\psi}R_{op} = 2\psi R_{op} + 2(2(n-1) + \grave{\mu})\grave{\hbar}\grave{\psi},$$
$$\grave{\psi}R_{op}\grave{\psi} = 2(2(n-1) + \grave{\mu})\grave{\hbar} - R_{op} + 2nk\eta \otimes \grave{\zeta},$$
$$R_{cur}(\grave{\psi}\partial_1, \grave{\zeta}) = 0,$$
$$tr(R_{op}\grave{\psi}) = tr(\grave{\psi}R_{op}) = 0.$$

If a $(\grave{k}, \grave{\mu})$-manifold is a (non)-Sasakain manifold then the Ricci operator R_{op} is given by

$$R_{op} = (2(n-1) - n\grave{\mu})I + (2(n-1) + \grave{\mu})\grave{\hbar} + (2(1-n) + n(2\grave{k} + \grave{\mu}))\eta \otimes \grave{\zeta}. \tag{2.11}$$

Also a contact metric manifold satisfying $R_{cur}(\partial_1, \partial_2)\grave{\zeta} = 0$ is locally isometric to $E^{n+1} \times S^n(4)$ for $n > 1$ and flat $n = 1$. Thus we have $R_{op}\grave{\psi} - \grave{\psi}R_{op} = 2[2(n-1) + \grave{\mu}]\grave{\hbar}\grave{\psi}$. From the definition of η-Einstein manifold then we have $R_{op}\grave{\psi} = \grave{\psi}R_{op}$. If a (non)-Sasakian $(\grave{k}, \mu)$-contact metric manifold is η-Einstein manifold then we have $R_{op}\grave{\psi} = \grave{\psi}R_{op}$. The conversely is true.

3. Generalized extended C-Bochner curvature tensor

In [16], Shaikh and Baishya defined the *generalized contact Bochner* (briefly GC-Bochner) curvature tensor in a contact metric manifold as follows:

$$
\begin{aligned}
\tilde{B}(\partial_1,\partial_2)\partial_3 \;=\; & R_{cur}(\partial_1,\partial_2)\partial_3+\rho(\grave{\psi}\partial_2,\grave{\hbar}\partial_3)\grave{\hbar}\grave{\psi}\partial_1-\rho(\grave{\psi}\partial_1,\grave{\hbar}\partial_3)\grave{\hbar}\grave{\psi}\partial_2\\
&+\frac{1}{\grave{m}+4}\{\frac{\grave{m}}{2}+\alpha+\frac{tr\grave{\hbar}^2}{\grave{m}+2}\}[\rho(\partial_1,\partial_3)\eta(\partial_2)\grave{\zeta}-\rho(\partial_2,\partial_3)\eta(\partial_1)\grave{\zeta}\\
&-\eta(\partial_2)\eta(\partial_3)\partial_1+\eta(\partial_1)\eta(\partial_3)\partial_2]\\
&-\frac{1}{\grave{m}+4}\{\alpha+\grave{m}+\frac{tr\grave{\hbar}^2}{\grave{m}+2}\}[\rho(\grave{\psi}\partial_1,\partial_3)\grave{\psi}\partial_2\\
&-\rho(\grave{\psi}\partial_2,\partial_3)\grave{\psi}\partial_1+2\rho(\grave{\psi}\partial_1,\partial_2)\grave{\psi}\partial_3]\\
&-\frac{1}{\grave{m}+4}\{\alpha-4+\frac{tr\grave{\hbar}^2}{\grave{m}+2}\}[\rho(\partial_1,\partial_3)\partial_2-\rho(\partial_2,\partial_3)\partial_1]\\
&+\frac{1}{2(\grave{m}+4)}[\rho(\partial_1,\partial_3)R_{op}\partial_2-\rho(\partial_2,\partial_3)R_{op}\partial_1-R_{cur}(\partial_2,\partial_3)\partial_1\\
&+R_{cur}(\partial_1,\partial_3)\partial_2-\rho(\partial_1,\partial_3)\psi R_{op}\grave{\psi}\partial_2+\rho(\partial_2,\partial_3)\psi R_{op}\grave{\psi}\partial_1\\
&-R_{cur}(\grave{\psi}\partial_2,\grave{\psi}\partial_3)\partial_1+R_{cur}(\grave{\psi}\partial_1,\grave{\psi}\partial_3)\partial_2-R_{cur}(\grave{\psi}\partial_2,\partial_3)\grave{\psi}\partial_1\\
&+R_{cur}(\partial_2,\grave{\psi}\partial_3)\grave{\psi}\partial_1+R_{cur}(\grave{\psi}\partial_1,\partial_3)\grave{\psi}\partial_2-R_{cur}(\partial_1,\grave{\psi}\partial_3)\grave{\psi}\partial_2\\
&+2R_{cur}(\grave{\psi}\partial_1,\partial_2)\grave{\psi}\partial_3-2R_{cur}(\partial_1,\grave{\psi}\partial_2)\grave{\psi}\partial_3\\
&+\rho(\grave{\psi}\partial_1,\partial_3)(\psi R_{op}+R_{op}\grave{\psi})\partial_2-\rho(\grave{\psi}\partial_2,\partial_3)(\grave{\psi}R_{op}+R_{op}\grave{\psi})\partial_1\\
&+2\rho(\grave{\psi}\partial_1,\partial_2)(\grave{\psi}R_{op}+R_{op}\grave{\psi})\partial_3-\eta(\partial_1)\eta(\partial_3)R_{op}\partial_2\\
&+\eta(\partial_2)\eta(\partial_3)R_{op}\partial_1+\eta(\partial_1)\eta(\partial_3)\grave{\psi}R_{op}\grave{\psi}\partial_2\\
&-\eta(\partial_2)\eta(\partial_3)\grave{\psi}R_{op}\grave{\psi}\partial_1+R_{cur}(\partial_2,\partial_3)\eta(\partial_1)\grave{\zeta}-R_{cur}(\partial_1,\partial_3)\eta(\partial_2)\grave{\zeta}\\
&+R_{cur}(\grave{\psi}\partial_2,\grave{\psi}\partial_3)\eta(\partial_1)\grave{\zeta}-R_{cur}(\grave{\psi}\partial_1,\grave{\psi}\partial_3)\eta(\partial_2)\grave{\zeta}],
\end{aligned}
\tag{3.1}
$$

where $\alpha=\frac{\tau+\grave{m}}{\grave{m}+2}$, $\grave{m}=2n$. If the manifold is a Sasakian manifold, then we have $\grave{\hbar}=0$, $R_{op}\grave{\psi}=\psi R_{op}$, $tr\grave{\hbar}^2=0$, $R_{cur}(\grave{\psi}\partial_1,\grave{\psi}\partial_2)=R_{cur}(\partial_1,\partial_2)-m\eta(\partial_1)\eta(\partial_2)$ and hence (3.1) reduces to the definition of the C-Bochner curvature tensor in a Sasakian manifold G^{2n+1} defined by Matsumoto and Chūman [13].

From (3.1), we have the followings:

$$\tilde{B}(\partial_1,\partial_2)\partial_3=-\tilde{B}(\partial_2,\partial_3)\partial_1, \tag{3.2}$$

$$\tilde{B}(\partial_1,\partial_2)\partial_3+\tilde{B}(\partial_2,\partial_3)\partial_1+\tilde{B}(\partial_3,\partial_1)\partial_2=0, \tag{3.3}$$

$$\rho(\tilde{B}(\partial_1,\partial_2)\partial_3,\partial_4)=-\rho(\tilde{B}(\partial_1,\partial_2)\partial_4,\partial_3), \tag{3.4}$$

$$\rho(\tilde{B}(\partial_1,\partial_2)\partial_3,\partial_4)=\rho(\tilde{B}(\partial_3,\partial_4)\partial_1,\partial_2), \tag{3.5}$$

for any vector fields $\partial_1, \partial_2, \partial_3, \partial_4 \in \Gamma(G)$. Also

$$\tilde{B}(\partial_1, \partial_2)\grave{\zeta} = \frac{(\grave{k}-1)(\grave{m}+8)}{2(\grave{m}+4)}[\eta(\partial_2)\partial_1 - \eta(\partial_1)\partial_2] + \grave{\mu}[\eta(\partial_2)\grave{\hbar}\partial_1 - \eta(\partial_1)\grave{\hbar}\partial_2], \tag{3.6}$$

and

$$\begin{aligned}\tilde{B}(\grave{\zeta}, \partial_1)\partial_2 &= \frac{(\grave{k}-1)(\grave{m}+8)}{2(\grave{m}+4)}[\rho(\partial_1, \partial_2)\grave{\zeta} - \eta(\partial_2)\partial_1] \\ &\quad + \grave{\mu}[\rho(\grave{\hbar}\partial_1, \partial_2)\grave{\zeta} - \eta(\partial_2)\grave{\hbar}\partial_1],\end{aligned} \tag{3.7}$$

Consequently, we have

$$\tilde{B}(\grave{\zeta}, \partial_1)\grave{\zeta} = \frac{(\grave{k}-1)(\grave{m}+8)}{2(\grave{m}+4)}[\eta(\partial_1)\grave{\zeta} - \partial_1] + \grave{\mu}\grave{\hbar}\partial_1, \tag{3.8}$$

$$\eta(\tilde{B}(\partial_1, \partial_2)\grave{\zeta}) = 0, \tag{3.9}$$

$$\eta(\tilde{B}(\grave{\zeta}, \partial_1)\partial_2) = \frac{(\grave{k}-1)(\grave{m}+8)}{2(\grave{m}+4)}[\rho(\partial_1, \partial_2) - \eta(\partial_1)\eta(\partial_2)] + \mu\rho(\grave{\hbar}\partial_1, \partial_2), \tag{3.10}$$

where $\grave{m} = 2n$.

In [9], H. Endo defined the extended C-Bochner curvature tensor $\tilde{B}^E$ on a contact metric manifold G by

$$\begin{aligned}\tilde{B}^E(\partial_1, \partial_2)\partial_3 &= \tilde{B}(\partial_1, \partial_2)\partial_3 - \eta(\partial_1)\tilde{B}(\grave{\zeta}, \partial_2)\partial_3 \\ &\quad - \eta(\partial_2)\tilde{B}(\partial_1, \grave{\zeta})\partial_3 - \eta(\partial_3)\tilde{B}(\partial_1, \partial_2)\grave{\zeta},\end{aligned} \tag{3.11}$$

Now using t(3.11), we define *generalized extended C-Bochner curvature tensor* $\tilde{B}^{GE}$ on a contact metric manifold M by

$$\begin{aligned}\tilde{B}^{GE}(\partial_1, \partial_2)\partial_3 &= \tilde{B}^E(\partial_1, \partial_2)\partial_3 - \eta(\partial_1)\tilde{B}^E(\grave{\zeta}, \partial_2)\partial_3 \\ &\quad - \eta(\partial_2)\tilde{B}^E(\partial_1, \grave{\zeta})\partial_3 - \eta(\partial_3)\tilde{B}^E(\partial_1, \partial_2)\grave{\zeta}.\end{aligned} \tag{3.12}$$

Using (3.1), (3.6) in (3.12), we have

$$\begin{aligned}\tilde{B}^{GE}(\partial_1, \partial_2)\partial_3 &= \tilde{B}(\partial_1, \partial_2)\partial_3 - \eta(\partial_1)\tilde{B}(\grave{\zeta}, \partial_2)\partial_3 \\ &\quad - \eta(\partial_2)\tilde{B}(\partial_1, \grave{\zeta})\partial_3 - \eta(\partial_3)\tilde{B}(\partial_1, \partial_2)\grave{\zeta} \\ &\quad + 2[\eta(\partial_1)\eta(\partial_3)\tilde{B}(\grave{\zeta}, \partial_2)\grave{\zeta} - \eta(\partial_2)\eta(\partial_3)\tilde{B}(\grave{\zeta}, \partial_1)\grave{\zeta}].\end{aligned} \tag{3.13}$$

Also using (3.1)-(3.8) in (3.13), we write

$$\begin{aligned}\tilde{B}^{GE}(\partial_1, \partial_2)\partial_3 &= \tilde{B}(\partial_1, \partial_2)\partial_3 \\ &\quad + \frac{(1-\grave{k})(\grave{m}+8)}{2(\grave{m}+4)}[\rho(\partial_2, \partial_3)\eta(\partial_1)\grave{\zeta} - \rho(\partial_1, \partial_3)\eta(\partial_2)\grave{\zeta}] \\ &\quad + \grave{\mu}[\rho(\grave{\hbar}\partial_1, \partial_3)\eta(\partial_2)\grave{\zeta} - \rho(\grave{\hbar}\partial_2, \partial_3)\eta(\partial_1)\grave{\zeta}].\end{aligned} \tag{3.14}$$

4. Main results

Definition 4.1. A Riemannian manifold (G^{2n+1}, g), $n > 1$, is said to be $\grave{\hbar}$-generalized extended C-Bochner semisymmetric if

$$\tilde{B}^{GE}(\partial_1, \partial_2) \cdot \grave{\hbar} = 0,$$

holds on G.

Lemma 4.1. [3] *Let $G^{2n+1}(\grave{\psi}, \grave{\zeta}, \eta, \rho)$ be a contact metric manifold with $\grave{\zeta}$ belonging to the $(\grave{k}, \grave{\mu})$-nullity distribution. Then for any vector fields $\partial_1, \partial_2, \partial_3$*

$$\begin{aligned}
& R_{cur}(\partial_1, \partial_2)\grave{\hbar}\partial_3 - \grave{\hbar}R_{cur}(\partial_1, \partial_2)\partial_3 \\
= \; & \{\grave{k}[\rho(\grave{\hbar}\partial_2, \partial_3)\eta(\partial_1) - \rho(\grave{\hbar}\partial_1, \partial_3)\eta(\partial_2)] \\
& + \grave{\mu}(\grave{k} - 1)[\rho(\partial_1, \partial_3)\eta(\partial_2) - \rho(\partial_2, \partial_3)\eta(\partial_1)]\}\grave{\zeta} \\
& + \grave{k}\{\rho(\partial_2, \grave{\psi}\partial_3)\grave{\psi}\grave{\hbar}\partial_1 - \rho(\partial_1, \grave{\psi}\partial_3)\grave{\psi}\grave{\hbar}\partial_2 + \rho(\partial_3, \grave{\psi}\grave{\hbar}\partial_2)\grave{\psi}\partial_1 \\
& - \rho(\partial_3, \grave{\psi}\grave{\hbar}\partial_1)\grave{\psi}\partial_2 + \eta(\partial_3)[\eta(\partial_1)\grave{\hbar}\partial_2 - \eta(\partial_2)\grave{\hbar}\partial_1]\} \\
& - \grave{\mu}\{\eta(\partial_2)[(1 - \grave{k})\eta(\partial_3)\partial_1 + \mu\eta(\partial_1)\grave{\hbar}\partial_3] \\
& - \eta(\partial_1)[(1 - \grave{k})\eta(\partial_3)\partial_2 + \mu\eta(\partial_2)\grave{\hbar}\partial_3] + 2\rho(\partial_1, \grave{\psi}\partial_2)\grave{\psi}\grave{\hbar}\partial_3\}.
\end{aligned} \tag{4.1}$$

Theorem 4.1. *Let $G^{2n+1}(\grave{\psi}, \grave{\zeta}, \eta, \rho)$ be a (non)-Sasakian $(\grave{k}, \grave{\mu})$-contact metric manifold. If M is $\grave{\hbar}$-generalized extended C-Bochner semisymmetric then G is an η-Einstein manifold.*

Proof. Let G be a $(2n+1)$-dimensional $\grave{\hbar}$-generalized extended C-Bochner semisymmetric (non)-Sasakian $(\grave{k}, \grave{\mu})$-contact metric manifold. The condition $\tilde{B}^{GE}(\partial_1, \partial_2) \cdot \grave{\hbar} = 0$ turns into

$$\tilde{B}^{GE}(\partial_1, \partial_2) \cdot \grave{\hbar})\partial_3 = \tilde{B}^{GE}(\partial_1, \partial_2)\grave{\hbar}\partial_3 - \grave{\hbar}\tilde{B}^{GE}(\partial_1, \partial_2)\partial_3 = 0, \tag{4.2}$$

for any vector fields $\partial_1, \partial_2, \partial_3$. Using (3.14) and (4.1) in (4.2), we have

$$\begin{aligned}
& \tilde{B}^{GE}(\partial_1, \partial_2)\grave{\hbar}\partial_3 - \grave{\hbar}\tilde{B}^{GE}(\partial_1, \partial_2)\partial_3 \\
= \; & \tilde{B}(\partial_1, \partial_2)\grave{\hbar}\partial_3 - \grave{\hbar}\tilde{B}(\partial_1, \partial_2)\partial_3 \\
& + \frac{(1 - \grave{k})(\grave{m} + 8)}{2(\grave{m} + 4)}[\rho(\partial_2, \grave{\hbar}\partial_3)\eta(\partial_1)\grave{\zeta} - \rho(\partial_1, \grave{\hbar}\partial_3)\eta(\partial_2)\grave{\zeta}] \\
& + \grave{\mu}[\rho(\grave{\hbar}\partial_1, \grave{\hbar}\partial_3)\eta(\partial_2)\grave{\zeta} - \rho(\grave{\hbar}\partial_2, \grave{\hbar}\partial_3)\eta(\partial_1)\grave{\zeta}] = 0.
\end{aligned} \tag{4.3}$$

Firstly using (3.1) and (4.1) in (4.3), then replacing ∂_1 by $\grave{\hbar}\partial_1$, using symmetry property of $\grave{\hbar}$ and taking the inner product with ∂_4, after very long calculations, we get

$$(\grave{k} - 1)\{\frac{(\grave{k} - 1)(\grave{m} + 8)}{2(\grave{m} + 4)}[\rho(\partial_1, \partial_3) - \eta(\partial_1)\eta(\partial_3)] + \mu\rho(\grave{\hbar}\partial_1, \partial_3)\} = 0. \tag{4.4}$$

Since G is a (non)-Sasakian manifold, from (4.4), we have

$$\frac{(\grave{k}-1)(\grave{m}+8)}{2(\grave{m}+4)}[\rho(\partial_1,\partial_3)-\eta(\partial_1)\eta(\partial_3)]+\mu\rho(\grave{\hbar}\partial_1,\partial_3)=0. \tag{4.5}$$

Using (2.11) in (4.5), we obtain

$$\begin{aligned}R_{cur}(\partial_1,\partial_3) = \{&\frac{(\grave{k}-1)(\grave{m}+8)(\grave{m}-2+\grave{\mu})}{2(\grave{m}+4)}+(\grave{m}-2-\frac{\grave{m}}{2}\grave{\mu})\}\rho(\partial_1,\partial_3)\\ &+\{(2-\grave{m}+mk+\frac{\grave{m}}{2}\grave{\mu})-\frac{(\grave{k}-1)(\grave{m}+8)(\grave{m}-2+\grave{\mu})}{2(\grave{m}+4)}\}\eta(\partial_1)\eta(\partial_3).\end{aligned}$$

Thus G is an η-Einstein manifold. The proof of the Theorem is completed. □

Now from Corollary 1, we can state the following:

Corollary 4.1. *If a 3-dimensional (non)-Sasakian $(\grave{k}, \grave{\mu})$-contact metric manifold is h-generalized extended C-Bochner semisymmetric, then the manifold is a $\tilde{N}(\grave{k})$-contact metric manifold.*

Definition 4.2. A Riemannian manifold (G^{2n+1}, g), $n > 1$, is said to be $\grave{\psi}$-generalized extended C-Bochner semisymmetric if

$$\tilde{B}^{GE}(\partial_1,\partial_2)\cdot\grave{\psi}=0$$

holds on G.

Now we need the following:

Lemma 4.2. *[3] Let $G^{2n+1}(\grave{\psi},\grave{\zeta},\eta,\rho)$ be a contact metric manifold with $\grave{\zeta}$ belonging to the $(\grave{k},\grave{\mu})$-nullity distribution. Then for any vector fields $\partial_1,\partial_2,\partial_3$*

$$\begin{aligned}& R_{cur}(\partial_1,\partial_2)\grave{\psi}\partial_3-\psi R(\partial_1,\partial_2)\partial_3\\ =\ & \{(1-\grave{k})[\rho(\grave{\psi}\partial_2,\partial_3)\eta(\partial_1)-\rho(\grave{\psi}\partial_1,\partial_3)\eta(\partial_2)]\\ & +(1-\grave{\mu})[\rho(\grave{\psi}\grave{\hbar}\partial_2,\partial_3)\eta(\partial_1)-\rho(\grave{\psi}\grave{\hbar}\partial_1,\partial_3)\eta(\partial_2)]\}\grave{\zeta}\\ & -\rho(\partial_2+\grave{\hbar}\partial_2,\partial_3)(\grave{\psi}\partial_1+\grave{\psi}\grave{\hbar}\partial_1)+\rho(\partial_1+\grave{\hbar}\partial_1,\partial_3)(\grave{\psi}\partial_2+\grave{\psi}\grave{\hbar}\partial_2)\\ & -\rho(\grave{\psi}\partial_2+\grave{\psi}\grave{\hbar}\partial_2,\partial_3)(\partial_1+\grave{\hbar}\partial_1)+\rho(\grave{\psi}\partial_1+\grave{\psi}\grave{\hbar}\partial_1,\partial_3)(\partial_2+\grave{\hbar}\partial_2)\\ & -\eta(\partial_3)\{(1-\grave{k})[\eta(\partial_1)\grave{\psi}\partial_2-\eta(\partial_2)\grave{\psi}\partial_1]\\ & +(1-\grave{\mu})[\eta(\partial_1)\grave{\psi}\grave{\hbar}\partial_2-\eta(\partial_2)\grave{\psi}\grave{\hbar}\partial_1]\}.\end{aligned} \tag{4.6}$$

Theorem 4.2. *Let $G^{2n+1}(\grave{\psi},\grave{\zeta},\eta,\rho)$ be a (non)-Sasakian $(\grave{k},\grave{\mu})$-contact metric manifold. If G is $\grave{\psi}$-generalized extended C-Bochner semisymmetric, then G is an η-Einstein manifold.*

Proof. Let G be a $(2n+1)$-dimensional $\grave{\psi}$-generalized extended C-Bochner semisymmetric (non)-Sasakian $(\grave{k},\grave{\mu})$-contact metric manifold. The condition $\tilde{B}^{GE}(\partial_1,\partial_2)\cdot\grave{\psi}=0$ turns into

$$(\tilde{B}^{GE}(\partial_1,\partial_2)\cdot\grave{\psi})\partial_3=\tilde{B}^{GE}(\partial_1,\partial_2)\grave{\psi}\partial_3-\psi B^{GE}(\partial_1,\partial_2)\partial_3=0, \tag{4.7}$$

or any vector fields $\partial_1,\partial_2,\partial_3$. Using (3.14) and (4.1) in (4.7), we have

$$\begin{aligned}&\tilde{B}^{GE}(\partial_1,\partial_2)\grave{\psi}\partial_3-\psi B^{GE}(\partial_1,\partial_2)\partial_3\\ =\;&\tilde{B}(\partial_1,\partial_2)\grave{\psi}\partial_3-\psi B(\partial_1,\partial_2)\partial_3\\ &+\frac{(1-\grave{k})(\grave{m}+8)}{2(\grave{m}+4)}[\rho(\partial_2,\grave{\psi}\partial_3)\eta(\partial_1)\grave{\zeta}-\rho(\partial_1,\grave{\psi}\partial_3)\eta(\partial_2)\grave{\zeta}]\\ &+\grave{\mu}[\rho(\grave{\hbar}\partial_1,\grave{\psi}\partial_3)\eta(\partial_2)\grave{\zeta}-\rho(\grave{\hbar}\partial_2,\grave{\psi}\partial_3)\eta(\partial_1)\grave{\zeta}]=0.\end{aligned} \tag{4.8}$$

Now using (3.1) and (4.6) in (4.8), replacing ∂_1 by $\grave{\psi}\partial_1$, using $\rho(\grave{\psi}\partial_1,\partial_2)=-\rho(\partial_1,\grave{\psi}\partial_2)$, taking the inner product with ∂_4, putting $\partial_2=\partial_4=\grave{\zeta}$, after very long calculations, we get

$$\frac{(1-\grave{k})(\grave{m}+8)}{2(\grave{m}+4)}\{\rho(\partial_1,\partial_3)-\eta(\partial_1)\eta(\partial_3)\}+\grave{\mu}\rho(\grave{\hbar}\partial_1,\partial_3)=0. \tag{4.9}$$

Using (2.11) in (4.9), we obtain

$$\begin{aligned}R_{cur}(\partial_1,\partial_3)\;=\;&\{\frac{(\grave{k}-1)(\grave{m}+8)(\grave{m}-2+\grave{\mu})}{2(\grave{m}+4)}+(\grave{m}-2-\frac{\grave{m}}{2}\grave{\mu})\}\rho(\partial_1,\partial_3)\\ &+\{(2-\grave{m}+\grave{m}k+\frac{\grave{m}}{2}\grave{\mu})-\frac{(\grave{k}-1)(\grave{m}+8)(\grave{m}-2+\grave{\mu})}{2(\grave{m}+4)}\}\eta(\partial_1)\eta(\partial_3).\end{aligned}$$

Hence G is an η-Einstein manifold. □

Thus, we can state the following:

Corollary 4.2. *If a 3-dimensional (non)-Sasakian $(\grave{k},\grave{\mu})$-contact metric manifold is $\grave{\psi}$-generalized extended C-Bochner semisymmetric, then the manifold is a $\tilde{N}(\grave{k})$-contact metric manifold.*

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