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A NOTE ON POINTWISE QUASI BI-SLANT SUBMERSIONS IN COMPLEX GEOMETRY

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Abstract. In this article, we are acquainted with the notion of pointwise quasi bi-slant (PQBS, in brief)submersions in complex geometry. We present the concept of pointwise quasi bi-slant submersions from almost Hermitian manifolds onto Riemannian manifolds as a generalization of pointwise hemi-slant submersions and pointwise semi-slant submersions and specially, we study such submersions from Kähler manifolds onto Riemannian manifolds. We explore the geometry of leaves of distribution which are involved in the discussed submersions and furnished with a characterization theorem for pointwise quasi bi-slant submersions to be totally umbilical fibers.
Keywords: Pointwise Riemannian submersions, Pointwise semi-invariant submersions, Pointwise quasi bi-slant submersions.

1. Introductions

The theory of smooth maps between Riemannian manifolds has been widely studied in Riemannian geometry. This theory has several important applications in both mathematics and physics.

Let N_1 be a Riemannian manifold endowed with a Riemannian metric g_{N_1}. An almost Hermitian manifold is a subclass of almost complex manifold. Since

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the Riemannian submersions have many applications [4] in science and technology, especially in the theory of relativity, robotics and cosmology, therefore it attracts many researchers to do the research in this area.

The theory of Riemannian submersion was initiated and studied by O' Neill [16] and Gray [9] in $1966-67$, respectively. The theory of Riemannian submersions motivates the researchers to define and study the various types of Riemannian submersions. Watson [30] firstly defined an almost complex type of Riemannian submersions and studied almost Hermitian submersions between almost Hermitian manifolds. In 1985, a new class of Riemannian submersions (almost contact metric submersions) was discussed by D. Chinea [5] which was an extension of almost Hermitian submersion.

B. Sahin introduced the notion of semi-invariant submersions [21] which was a generalization of holomorphic submersions and anti-invariant submersions [20]. Additionally, he also defined slant submersions from almost Hermitian manifolds onto arbitrary Riemannian manifolds [22]. Many geometers studied different types of Riemannian submersions between Riemannian manifolds and found good results in ([3], [8], [10], [11], [18], [19], [23], [28]). In 2013, Park and Prasad [17] defined semi-slant submersions and in 2016, Tastan et al. [29] defined hemi-slant Riemannian submersions from almost Hermitian manifolds onto Riemannian manifolds. Hemi-slant Riemannian submersions from cosymplectic manifolds are also studied in [14]. Akyol and others defined conformal anti-invariant submersions [1] and semi-invariant $\xi^{\perp}$ Riemannian submersions [2] in $2016-17$.

The notion of pointwise slant submersions was introduced by Lee et al. [15] in 2014 and further studied by S. Kumar et al. ([12], [13]) between different Riemannian manifolds in $2017-18$. Recently, Sepet et al. introduced pointwise slant submersions [27] and pointwise semi-slant submersions [26] and on the other hand, C. Sayer et al. introduced pointwise semi-slant submersions [24] whose total manifolds are locally product Riemannian manifolds and Generic submersions from Kaehler manifolds [25] . The above studies inspire us to introduce the notion of PQBS submersions from the almost Hermitian manifolds to the Riemannian manifolds and characterize their geometrical properties. We exhibit our work as follows: after introduction, in the second section, we mention some definitions and properties related to the main topic. The third section deals with the definition of PQBS submersions and some results satisfied by a PQBS submersion. The necessary and sufficient conditions for PQBS submersions to be integrable and totally geodesic are given in the fourth section. Finally, the last section is concerned with some non-trivial examples of PQBS submersion from an almost Hermitian manifold.

2. Preliminaries

[31] An even-dimensional differentiable manifold N_1 with a $(1,1)$ tensor field J in such a manner

$$J^2 = -I, \tag{2.1}$$

(where I is identity operator) is called an an almost complex manifold with an almost complex structure J. It is well known that an almost complex manifold is necessarily orientable. Nijenhuis tensor N of an almost complex structure is defined as:

$$N(V_1, V_2) = [JV_1, JV_2] - [V_1, V_2] - J[JV_1, V_2] - J[V_1, JV_2], \quad \textit{for all} \;\; V_1, V_2 \in \Gamma(TN_1).$$

If the Nijenhuis tensor field N on an almost complex manifold N_1 is zero, then the almost complex manifold N_1 is called a complex manifold.

Let g_{N_1} be a Riemannian metric on N_1 such that

$$g_{N_1}(JZ_1, JZ_2) = g_{N_1}(Z_1, Z_2), \quad \textit{for all} \;\; Z_1, Z_2 \in \Gamma(TN_1). \tag{2.2}$$

Then g_{N_1} is called an almost Hermitian metric on N_1 and manifold N_1 with Hermitian metric g_{N_1} is called almost Hermitian manifold. The Riemannian connection ∇ of the almost Hermitian manifold N_1 can be extended to the whole tensor algebra on N_1. Tensor fields $(\nabla_{Z_1} J)$ are defined as

$$(\nabla_{Z_1} J)Z_2 = \nabla_{Z_1} JZ_2 - J\nabla_{Z_1} Z_2,$$

for all $Z_1, Z_2 \in \Gamma(TN_1)$.

An almost Hermitian manifold (N_1, g_{N_1}, J) is called a Kähler manifold if

$$(\nabla_{Z_1} J)Z_2 = 0, \tag{2.3}$$

for all $Z_1, Z_2 \in \Gamma(TN_1)$ ([6],[7]).

Let (N_1, g_{N_1}) and (N_2, g_{N_2}) be Riemannian manifolds, where g_{N_1} and g_{N_2} are Riemannian metrics on C^∞-manifolds N_1 and N_2 respectively. Let $\digamma : (N_1, g_{N_1}) \to (N_2, g_{N_2})$ be a Riemannian submersions.

Define O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$ by

$$\mathcal{A}_E F = \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F, \tag{2.4}$$

$$\mathcal{T}_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F, \tag{2.5}$$

for any vector fields E, F on N_1, where ∇ is the Levi-Civita connection of g_{N_1}. It is easy to see that $\mathcal{T}_E$ and $\mathcal{A}_E$ are skew-symmetric operators on the tangent bundle of N_1 reversing the vertical and the horizontal distributions.

From equations (2.4) and (2.5), we have

$$\nabla_{Z_1} Z_2 = \mathcal{T}_{Z_1} Z_2 + \mathcal{V}\nabla_{Z_1} Z_2, \tag{2.6}$$

$$\nabla_{Z_1} X_1 = \mathcal{T}_{Z_1} X_1 + \mathcal{H}\nabla_{Z_1} X_1, \tag{2.7}$$

$$\nabla_{X_1} Z_1 = \mathcal{A}_{X_1} Z_1 + \mathcal{V}\nabla_{X_1} Z_1, \tag{2.8}$$

$$\nabla_{X_1} X_2 = \mathcal{H}\nabla_{X_1} X_2 + \mathcal{A}_{X_1} X_2, \tag{2.9}$$

for $Z_1, Z_1 \in \Gamma(\ker \digamma_*)$ and $X_1, X_2 \in \Gamma(\ker \digamma_*)^\perp$, where $\mathcal{H}\nabla_{Z_1} X_1 = \mathcal{A}_{X_1} Z_1$, if X_1 is basic. It is not difficult to observe that $\mathcal{T}$ acts on the fibers as the second fundamental form, while $\mathcal{A}$ acts on the horizontal distribution and measures the obstruction to the integrability of this distribution [4].

We recall that the notation of the second fundamental form of a map between two Riemannian manifolds. Let (N_1, g_{N_1}) and (N_2, g_{N_2}) be Riemannian manifolds and $F : (N_1, g_{N_1}) \to (N_2, g_{N_2})$ be a C^∞ map, then the second fundamental form of F is given by

$$(\nabla F_*)(Y_1, Y_2) = \nabla^F_{Y_1} F_*(Y_2) - F_*(\nabla_{Y_1} Y_2), \tag{2.10}$$

for $Y_1, Y_2 \in \Gamma(TN_1)$, where ∇^F is the pullback connection and we denote for convenience by ∇ the Riemannian connections of the metrics g_{N_1} and g_{N_2}.

Finally, we also recall that a differentiable map F between two Riemannian manifolds is totally geodesic if

$$(\nabla F_*)(Y_1, Y_2) = 0, for\ all \ \ Y_1, Y_2 \in \Gamma(TN_1). \tag{2.11}$$

Now, we can easily prove the following lemma as in [26].

Lemma 2.1. *Let (N_1, g_{N_1}) and (N_2, g_{N_2}) are Riemannian manifolds. If $F : N_1 \to N_2$ be a Riemannian submersion, then for any horizontal vector fields W_1, W_2 and vertical vector fields Z_1, Z_2, we have*

(i) $(\nabla F_*)(W_1, W_2) = 0$,
(ii) $(\nabla F_*)(Z_1, Z_2) = -F_*(\mathcal{T}_{Z_1} Z_2) = -F_*(\nabla_{Z_1} Z_2)$,
(iii) $(\nabla F_*)(W_1, Z_1) = -F_*(\nabla_{W_1} Z_1) = -F_*(\mathcal{A}_{W_1} Z_1)$.

3. PQBS Submersions

Definition 3.1. Let (N_1, g_{N_1}, J) be an almost Hermitian manifold and (N_2, g_{N_2}) be a Riemannian manifold. A Riemannian submersion $F : (N_1, g_{N_1}, J) \to (N_2, g_{N_2})$ is called a PQBS submersion if there exists three mutually orthogonal distributions D, D_1 and D_2 such that

(i) $\ker F_* = D \oplus D_1 \oplus D_2$,
(ii) $J(D) = D$ i.e., D is invariant,
(iii) $J(D_1) \perp D_2$,
(iv) for any non-zero vector field $Y_1 \in (D_1)_p$, $p \in N_1$, the angle θ_1 between JY_1 and $(D_1)_p$ is a slant function and is independent of the choice of point p and Y_1 in $(D_1)_p$,
(v) for any non-zero vector field $Y_2 \in (D_2)_q$, $q \in N_1$, the angle θ_2 between JY_2 and $(D_2)_q$ is a slant function and is independent of the choice of point q and Y_2 in $(D_2)_q$,

These angles θ_1 and θ_2 are called slant functions of the pointwise quasi-bi-slant submersion.

Let F be pointwise quasi bi-slant submersion from an almost Hermitian manifold $(N_1, g_{N_1} J)$ onto a Riemannian manifold (N_2, g_{N_2}). Then, we have

$$TN_1 = \ker F_* \oplus (\ker F_*)^{\perp}. \tag{3.1}$$

Now, for any vector field $Z_1 \in \Gamma(\ker F_*)$, we put

$$Z_1 = PZ_1 + QZ_1 + RZ_1, \tag{3.2}$$

where P, Q and R are projection morphisms of $\ker F_*$ onto D, D_1 and D_2, respectively.

For $Y_1 \in (\Gamma \ker F_*)$, we set

$$JY_1 = \phi Y_1 + \omega Y_1, \tag{3.3}$$

where $\phi Y_1 \in (\Gamma \ker F_*)$ and $\omega Y_1 \in (\Gamma \ker F_*)^{\perp}$.

From equations (3.2) and (3.3), we have

$$\begin{aligned} JZ_1 &= J(PZ_1) + J(QZ_1) + J(RZ_1), \\ &= \phi(PZ_1) + \omega(PZ_1) + \phi(QZ_1) + \omega(QZ_1) + \phi(RZ_1) + \omega(RZ_1). \end{aligned}$$

Since $JD = D$, we get $\omega PZ_1 = 0$.

Hence above equation reduces to

$$JZ_1 = \phi(PZ_1) + \phi QZ_1 + \omega QZ_1 + \phi RZ_1 + \omega RZ_1. \tag{3.4}$$

Thus, we have the following decomposition

$$J(\ker F_*) = D \oplus (\phi D_1 \oplus \phi D_2) \oplus (\omega D_1 \oplus \omega D_2), \tag{3.5}$$

where $\oplus$ denotes orthogonal direct sum.

Further, let $Z_1 \in \Gamma(D_1)$ and $Z_2 \in \Gamma(D_2)$. Then, $g_{N_1}(Z_1, Z_2) = 0$.

From definition (3.1), we have

$$g_{N_1}(JZ_1, Z_2) = g_{N_1}(Z_1, JZ_2) = 0.$$

Now, consider

$$\begin{aligned} g_{N_1}(\phi Z_1, Z_2) &= g_{N_1}(JZ_1 - \omega Z_1, Z_2), \\ &= g_{N_1}(JZ_1, Z_2), \\ &= 0. \end{aligned}$$

Similarly, we have

$$g_{N_1}(Z_1, \phi Z_2) = 0.$$

Let $X_1 \in \Gamma(D)$ and $X_2 \in \Gamma(D_1)$. Then we have

$$\begin{aligned} g_{N_1}(\phi X_2, X_1) &= g_{N_1}(JX_2 - \omega X_2, X_1), \\ &= g_{N_1}(JX_2, X_1), \\ &= -g(X_2, JX_1), \\ &= 0, \end{aligned}$$

as D is invariant i.e., $JX_1 \in \Gamma(D)$.

Similarly, for $Y_1 \in \Gamma(D)$ and $Y_2 \in \Gamma(D_2)$, we obtain

$$g_{N_1}(\phi Y_2, Y_1) = 0.$$

From above equations, we have $g_{N_1}(\phi Y_1, \phi Y_2) = 0$, and $g_{N_1}(\omega Y_1, \omega Y_2) = 0$, for all $Y_1 \in \Gamma(D_1)$ and $Y_2 \in \Gamma(D_2)$.

So, we can write

$$\phi D_1 \cap \phi D_2 = \{0\}, \omega D_1 \cap \omega D_2 = \{0\}.$$

If $\theta_2 = \frac{\pi}{2}$, then $\phi R = 0$ and D_2 is anti-invariant, i.e., $J(D_2) \subseteq (\ker F_*)^\perp$. In this case we denote D_2 by $D^\perp$.

We also have

$$J(\ker F_*) = D \oplus \phi D_1 \oplus \omega D_1 \oplus JD^\perp. \tag{3.6}$$

Since $\omega D_1 \subseteq (\ker F_*)^\perp$, $\omega D_2 \subseteq (\ker F_*)^\perp$. So we can write

$$(\ker F_*)^\perp = \omega D_1 \oplus \omega D_2 \oplus \mathcal{V},$$

where $\mathcal{V}$ is an orthogonal complement of $(\omega D_1 \oplus \omega D_2)$ in $(\ker F_*)^\perp$.

Also for any non-zero vector field $V_1 \in \Gamma(\ker F_*)^\perp$, we have

$$JV_1 = BV_1 + CV_1, \tag{3.7}$$

where $BV_1 \in \Gamma(\ker F_*)$ and $CV_1 \in \Gamma(\ker F_*)^\perp$.

Lemma 3.1. *Let F be a PQBS submersion from an almost Hermitian manifold (N_1, g_{N_1}, J) onto a Riemannian manifold (N_2, g_{N_2}). Then, we have*

$$\phi^2 Z_1 + B\omega Z_1 = -Z_1, \omega\phi Z_1 + C\omega Z_1 = 0,$$

$$\omega B U_1 + C^2 U_1 = -U_1, \phi B U_1 + BCU_1 = 0,$$

for all $Z_1 \in \Gamma(\ker F_*)$ and $U_1 \in \Gamma(\ker F_*)^\perp$.

Proof. Using equations (2.1), (3.3) and (3.7), we have Lemma 3.1. □

Lemma 3.2. *Let F be a PQBS submersion from an almost Hermitian manifold (N_1, g_{N_1}, J) onto a Riemannian manifold (N_2, g_{N_2}). Then, we have*

(i) $\phi^2 Z_1 = -(\cos^2\theta_1)Z_1$
(ii) $g_{N_1}(\phi Z_1, \phi Z_2) = (\cos^2\theta_1)g_{N_1}(Z_1, Z_2)$,
(iii) $g_{N_1}(\omega Z_1, \omega Z_2) = (\sin^2\theta_1)g_{N_1}(Z_1, Z_2)$,
for all $Z_1, Z_2 \in \Gamma(D_1)$.

Proof. *(i)* Let F be a PQBS submersion from an almost Hermitian manifold (N_1, g_{N_1}, J) onto a Riemannian manifold (N, g_N) with the quasi bi-slant function θ_1. Then for a non-vanishing vector field $Z_1 \in \Gamma(D_1)$, we have

$$\cos\theta_1 = \frac{|\ \phi Z_1\ |}{|\ JZ_1\ |} \tag{3.8}$$

and

$$\cos\theta_1 = \frac{g_{N_1}(JZ_1, \phi Z_1)}{|\ JZ_1\ ||\ \phi Z_1\ |}$$

By using equation (3.3), we have

$$\cos\theta_1 = \frac{g_{N_1}(\phi Z_1, \phi Z_1)}{|\ JZ_1\ ||\ \phi Z_1\ |}$$

$$\cos\theta_1 = -\frac{g_{N_1}(Z_1, \phi^2 Z_1)}{|\ JZ_1\ ||\ \phi Z_1\ |} \tag{3.9}$$

From equations (3.8) and (3.9), we get

$$\phi^2 Z_1 = -(\cos^2\theta_1)Z_1, \qquad for Z_1 \in \Gamma(D_1).$$

(ii) For all $Z_1, Z_2 \in \Gamma(D_1)$, using equations (3.3) and Lemma 3(*i*), we have

$$\begin{aligned} g_{N_1}(\phi Z_1, \phi Z_2) &= g_{N_1}(\phi Z_1 + \omega Z_1, \phi Z_2), \\ &= -g_{N_1}(Z_1, \phi^2 Z_2), \\ &= \cos^2\theta_1 g_{N_1}(Z_1, Z_2). \end{aligned}$$

(iii) Using equations (3.3) and Lemma 3(*i*), (*ii*), we have Lemma 3(*iii*). □

In a similar way to above, we obtain the following Lemma:

Lemma 3.3. *Let F be a PQBS submersion from an almost Hermitian manifold (N_1, g_{N_1}, J) onto a Riemannian manifold (N_2, g_{N_2}). Then, we have*

(*i*) $\phi^2 X_1 = -(\cos^2\theta_2)X_1$
(*ii*) $g_{N_1}(\phi X_1, \phi X_2) = (\cos^2\theta_2) g_{N_1}(X_1, X_2)$,
(*iii*) $g_{N_1}(\omega X_1, \omega X_2) = (\sin^2\theta_2) g_{N_1}(X_1, X_2)$,
for all $X_1, X_2 \in \Gamma(D_2)$.

Lemma 3.4. *Let F be a PQBS submersion from a Kähler manifold (N_1, g_{N_1}, J) onto a Riemannian manifold (N_2, g_{N_2}). Then, we have*

$$\mathcal{V}\nabla_{Z_1}\phi V_1 + \mathcal{T}_{Z_1}\omega V_1 = \phi\mathcal{V}\nabla_{Z_1}V_1 + B\mathcal{T}_{Z_1}V_1, \tag{3.10}$$
$$\mathcal{T}_{Z_1}\phi V_1 + \mathcal{H}\nabla_{Z_1}\omega V_1 = \omega\mathcal{V}\nabla_{Z_1}V_1 + C\mathcal{T}_{Z_1}V_1, \tag{3.11}$$
$$\mathcal{V}\nabla_{Y_1}BY_2 + \mathcal{A}_{Y_1}CY_2 = \phi\mathcal{A}_{Y_1}Y_2 + B\mathcal{H}\nabla_{Y_1}Y_2, \tag{3.12}$$
$$\mathcal{A}_{Y_1}BY_2 + \mathcal{H}\nabla_{Y_1}CY_2 = \omega\mathcal{A}_{Y_1}Y_2 + C\mathcal{H}\nabla_{Y_1}Y_2, \tag{3.13}$$
$$\mathcal{V}\nabla_{Z_1}BY_1 + \mathcal{T}_{Z_1}CY_1 = \phi\mathcal{T}_{Z_1}Y_1 + B\mathcal{H}\nabla_{Z_1}Y_1, \tag{3.14}$$
$$\mathcal{T}_{Z_1}BY_1 + \mathcal{H}\nabla_{Z_1}CY_1 = \omega\mathcal{T}_{Z_1}Y_1 + C\mathcal{H}\nabla_{Z_1}Y_1, \tag{3.15}$$
$$\mathcal{V}\nabla_{Y_1}\phi Z_1 + \mathcal{A}_{Y_1}\omega Z_1 = \phi\mathcal{V}\nabla_{Y_1}Z_1 + B\mathcal{A}_{Y_1}Z_1, \tag{3.16}$$
$$\mathcal{A}_{Y_1}\phi Z_1 + \mathcal{H}\nabla_{Y_1}\omega Z_1 = \omega\mathcal{V}\nabla_{Y_1}Z_1 + C\mathcal{A}_{Y_1}Z_1 \tag{3.17}$$

for any $Z_1, V_1 \in \Gamma(\ker F_*)$ and $Y_1, Y_2 \in \Gamma(\ker F_*)^\perp$.

Proof. Using equations (2.6)-(2.9), (3.3) and (3.7), we get equations (3.10)-(3.17). □

Now, we define

$$(\nabla_{Z_1}\phi)V_1 = \mathcal{V}\nabla_{Z_1}\phi V_1 - \phi\mathcal{V}\nabla_{Z_1}V_1, \tag{3.18}$$
$$(\nabla_{Z_1}\omega)V_1 = \mathcal{H}\nabla_{Z_1}\omega V_1 - \omega\mathcal{V}\nabla_{Z_1}V_1, \tag{3.19}$$
$$(\nabla_{Y_1}C)Y_2 = \mathcal{H}\nabla_{Y_1}CY_2 - C\mathcal{H}\nabla_{Y_1}Y_2, \tag{3.20}$$
$$(\nabla_{Y_1}B)Y_2 = \mathcal{V}\nabla_{Y_1}BY_2 - B\mathcal{H}\nabla_{Y_1}Y_2, \tag{3.21}$$

for any $Z_1, V_1 \in \Gamma(\ker F_*)$ and $Y_1, Y_2 \in \Gamma(\ker F_*)^\perp$.

Lemma 3.5. *Let F be a PQBS submersion from a Kähler manifold (N_1, g_{N_1}, J) onto a Riemannian manifold (N_2, g_{N_2}). Then, we have*

$$(\nabla_{Z_1}\phi)V_1 = B\mathcal{T}_{Z_1}V_1 - \mathcal{T}_{Z_1}\omega V_1,$$

$$(\nabla_{Z_1}\omega)V_1 = C\mathcal{T}_{Z_1}V_1 - \mathcal{T}_{Z_1}\phi V_1,$$

$$(\nabla_{Y_1}C)Y_2 = \omega\mathcal{A}_{Y_1}Y_2 - \mathcal{A}_{Y_1}BY_2,$$

$$(\nabla_{Y_1}B)Y_2 = \phi\mathcal{A}_{Y_1}Y_2 - \mathcal{A}_{Y_1}CY_2,$$

for any vectors $Z_1, V_1 \in \Gamma(\ker F_*)$ and $Y_1, Y_2 \in \Gamma(\ker F_*)^\perp$.

Proof. Using equations (3.10)-(3.13) and (3.18)-(3.21), we get all equations of Lemma 3.5. □

If the tensors ϕ and ω are parallel with respect to the linear connection ∇ on N_1, respectively, then

$$B\mathcal{T}_{Z_1}V_1 = \mathcal{T}_{Z_1}\omega V_1, C\mathcal{T}_{Z_1}V_1 - \mathcal{T}_{Z_1}\phi V_1$$

for any $Z_1, V_1 \in \Gamma(TN_1)$.

Theorem 3.1. *Let F be a pointwise proper quasi bi-slant submersion from a Kähler manifold (N_1, g_{N_1}, J) onto a Riemannian manifold (N_2, g_{N_2}). Then, the invariant distribution D is integrable if and only if*

$$\begin{aligned} &(\cos^2\theta_1)g_{N_1}([U_1,U_2],QW_1) + (\cos^2\theta_2)g_{N_1}([U_1,U_2],RW_1) \\ = \;& g_{N_1}(\mathcal{T}_{U_1}U_2 - \mathcal{T}_{U_2}U_1, \omega\phi QW_1 + \omega\phi RW_1) + g_{N_1}(\mathcal{T}_{U_2}JU_1 - \mathcal{T}_{U_1}JU_2, \omega W_1), \end{aligned}$$

for $U_1, U_2 \in \Gamma(D)$, and $W_1 \in \Gamma(D_1 \oplus D_2)$.

Proof. For $U_1, U_2 \in \Gamma(D)$, and $W_1 \in \Gamma(D_1 \oplus D_2)$, using equations (2.2), (2.6), (3.2), (3.3) and Lemmas 3.2 and 3.3, we have

$$\begin{aligned} & g_{N_1}(U_1,U_2],W_1) \\ = \;& g_{N_1}(J\nabla_{U_1}U_2, JQW_1 + JRW_1) - g_{N_1}(J\nabla_{U_2}U_1, JQW_1 + JRW_1), \\ = \;& (\cos^2\theta_1)g_{N_1}(\nabla_{U_1}U_2, QW_1) + (\cos^2\theta_2)g_{N_1}(\nabla_{U_1}U_2, RW_1) - \\ & g_{N_1}(\nabla_{U_1}U_2, \omega\phi QW_1 + \omega\phi RW_1) - (\cos^2\theta_1)g_{N_1}(\nabla_{U_2}U_1, QW_1) - \\ & (\cos^2\theta_2)g_{N_1}(\nabla_{U_2}U_1, RW_1) + g_{N_1}(\nabla_{U_1}U_2, \omega\phi QW_1 + \omega\phi RW_1) + \\ & g_{N_1}(\nabla_{U_1}JU_2, \omega W_1) - g_{N_1}(\nabla_{U_2}JU_1, \omega W_1), \\ = \;& (\cos^2\theta_1)g_{N_1}([U_1,U_2],QW_1) + (\cos^2\theta_2)g_{N_1}([U_1,U_2],RW_1) - \\ & g_{N_1}(\mathcal{T}_{U_1}U_2 - \mathcal{T}_{U_2}U_1, \omega\phi QW_1 + \omega\phi RW_1) - \\ & g_{N_1}(\mathcal{T}_{U_2}JU_1 - \mathcal{T}_{U_1}JU_2, \omega W_1), \end{aligned}$$

which completes the proof. □

Theorem 3.2. *Let F be a PQBS submersion from a Kähler manifold (N_1, g_{N_1}, J) onto a Riemannian manifold (N_2, g_{N_2}). Then, the slant distribution D_1 is integrable if and only if*

$$\begin{aligned} & g_{N_1}(\mathcal{T}_{X_1}\omega\phi QX_2 - \mathcal{T}_{X_2}\omega\phi QX_1, Y_1) \\ = \;& g_{N_1}(\mathcal{T}_{X_1}\omega QX_2 - \mathcal{T}_{X_2}\omega QX_1, JPY_1 + \phi RY_1) + \\ & g_{N_1}(\mathcal{H}\nabla_{X_1}\omega QX_2 - \mathcal{H}\nabla_{X_2}\omega QX_1, \omega RY_1), \end{aligned}$$

for all $X_1, X_2 \in \Gamma(D_1)$ and $Y_1 \in \Gamma(D \oplus D_2)$.

Proof. For $X_1, X_2 \in \Gamma(D_1)$ and $Y_1 \in \Gamma(D \oplus D_2)$, using equations (2.2), (2.7), (3.2), (3.3), (3.7) and Lemma 3.2, we have

$$\begin{aligned}
& g_{N_1}([X_1, X_2], Y_1) \\
= \; & g_{N_1}(\nabla_{X_1} JX_2, JY_1) - g_{N_1}(\nabla_{X_2} JX_1, JY_1), \\
= \; & g_{N_1}(\nabla_{X_1} \phi QX_2, JY_1) + g_{N_1}(\nabla_{X_1} \omega QX_2, JY_1) - g_{N_1}(\nabla_{X_2} \phi QX_1, JY_1) - \\
& g_{N_1}(\nabla_{X_2} \omega QX_1, JY_1), \\
= \; & (\cos^2 \theta_1) g_{N_1}(\nabla_{X_1} X_2 - \nabla_{X_2} X_1, Y_1) - g_{N_1}(\nabla_{X_1} \omega\phi QX_2 - \nabla_{X_2} \omega\phi QX_1, Y_1) - \\
& g_{N_1}(\nabla_{X_1} \omega QX_2 - \nabla_{X_2} \omega QX_1, JPY_1 + \phi RY_1) \\
& + g_{N_1}(\nabla_{X_1} \omega QX_2 - \nabla_{X_2} \omega QX_1, \omega RY_1).
\end{aligned}$$

Now, we have

$$\begin{aligned}
& (\sin^2 \theta_1) g_{N_1}([X_1, X_2], Y_1) \\
= \; & -g_{N_1}(\mathcal{T}_{X_1} \omega\phi QX_2 - \mathcal{T}_{X_2} \omega\phi QX_1, Y_1) + \\
& g_{N_1}(\mathcal{T}_{X_1} \omega QX_2 - \mathcal{T}_{X_2} \omega QX_1, JPY_1 + \phi RY_1) + \\
& g_{N_1}(\mathcal{H}\nabla_{X_1} \omega QX_2 - \mathcal{H}\nabla_{X_2} \omega QX_1, \omega RY_1),
\end{aligned}$$

which completes the proof. □

Theorem 3.3. *Let F be a PQBS submersion from a Kähler manifold (N_1, g_{N_1}, J) onto a Riemannian manifold (N_2, g_{N_2}). Then, the slant distribution D_2 is integrable if and only if*

$$\begin{aligned}
& g_{N_1}(\mathcal{T}_{Z_1} \omega\phi QZ_2 - \mathcal{T}_{Z_2} \omega\phi QZ_1, W_1) \\
= \; & g_{N_1}(\mathcal{T}_{Z_1} \omega QZ_2 - \mathcal{T}_{Z_2} \omega QZ_1, JPW_1 + \phi RW_1) + \\
& g_{N_1}(\mathcal{H}\nabla_{Z_1} \omega QZ_2 - \mathcal{H}\nabla_{Z_2} \omega QZ_1, \omega RW_1),
\end{aligned}$$

for $Z_1, Z_2 \in \Gamma(D_2)$ and $W_1 \in \Gamma(D \oplus D_1)$.

Theorem 3.4. *Let F be a PQBS submersion from a Kähler manifold (N_1, g_{N_1}, J) onto a Riemannian manifold (N_2, g_{N_2}). Then the horizontal distribution $(\ker F_*)$ defines a totally geodesic foliation on N_1 if and only if*

$$\begin{aligned}
& g_{N_1}([U_1, W_1], U_2) \\
= \; & g_{N_1}(\mathcal{V}\nabla_{W_1} PU_1, U_2) + g_{N_1}(\mathcal{A}_{W_1} \omega\phi PU_1, U_2) - \\
& (\cos^2 \theta_1) g_{N_1}(\mathcal{V}\nabla_{W_1} QU_1, U_2) + \sin 2\theta_1 W_1[\theta_1] g_{N_1}(QU_1, U_2) + \\
& g_{N_1}(\mathcal{A}_{W_1} \omega\phi QU_1, U_2) - (\cos^2 \theta_2) g_{N_1}(\mathcal{V}\nabla_{W_1} RU_1, U_2) + \\
& \sin 2\theta_1 W_1[\theta_1] g_{N_1}(QU_1, U_2) + g_{N_1}(\mathcal{A}_{W_1} \omega\phi RU_1, U_2) - \\
& g_{N_1}(\mathcal{H}\nabla_{W_1} \omega U_1, \omega U_2) - g_{N_1}(\mathcal{A}_{W_1} \omega RU_1, \phi U_2),
\end{aligned}$$

for $U_1, U_2 \in \Gamma(\ker F_*)$ and $W_1 \in (\ker F_*)^\perp$.

Proof. For $U_1, U_2 \in \Gamma(\ker F_*)$ and $W_1 \in (\ker F_*)^\perp$, using equations (2.2), (2.6), (2.9), (3.2), (3.3) and Lemmas 3.2 and 3.3 we have

$$\begin{aligned}
& g_{N_1}(\nabla_{U_1}U_2, W_1) \\
= \; & -g_{N_1}(\nabla_{U_1}W_1, U_2), \\
= \; & -g_{N_1}([U_1, W_1], U_2) - g_{N_1}(\nabla_{W_1}U_1, U_2), \\
= \; & -g_{N_1}([U_1, W_1], U_2) - g_{N_1}(\nabla_{W_1}PU_1, JU_2) - (\cos^2\theta_1)g_{N_1}(\nabla_{W_1}QU_1, U_2) + \\
& \sin 2\theta_1 W_1[\theta_1]g_{N_1}(QU_1, U_2) + g_{N_1}(\nabla_{W_1}\omega\phi QU_1, U_2) - \\
& g_{N_1}(\nabla_{W_1}\omega QU_1, JU_2) - (\cos^2\theta_2)g_{N_1}(\nabla_{W_1}RU_1, U_2) + \\
& \sin 2\theta_1 W_1[\theta_1]g_{N_1}(QU_1, U_2) + g_{N_1}(\nabla_{W_1}\omega\phi RU_1, U_2) - \\
& g_{N_1}(\nabla_{W_1}\omega RU_1, JU_2), \\
= \; & -g_{N_1}([U_1, W_1], U_2) + g_{N_1}(\mathcal{V}\nabla_{W_1}PU_1, U_2) + g_{N_1}(\mathcal{A}_{W_1}\omega\phi PU_1, U_2) - \\
& (\cos^2\theta_1)g_{N_1}(\mathcal{V}\nabla_{W_1}QU_1, U_2) + \sin 2\theta_1 W_1[\theta_1]g_{N_1}(QU_1, U_2) + \\
& g_{N_1}(\mathcal{A}_{W_1}\omega\phi QU_1, U_2) - (\cos^2\theta_2)g_{N_1}(\mathcal{V}\nabla_{W_1}RU_1, U_2) + \\
& \sin 2\theta_1 W_1[\theta_1]g_{N_1}(QU_1, U_2) + g_{N_1}(\mathcal{A}_{W_1}\omega\phi RU_1, U_2) - \\
& g_{N_1}(\mathcal{H}\nabla_{W_1}\omega U_1, \omega U_2) - g_{N_1}(\mathcal{A}_{W_1}\omega RU_1, \phi U_2).
\end{aligned}$$

□

Theorem 3.5. *Let F be a PQBS submersion from a Kähler manifold (N_1, g_{N_1}, J) onto a Riemannian manifold (N_2, g_{N_2}). Then the vertical distribution $(\ker F_*)^\perp$ defines a totally geodesic foliation on N_1 if and only if*

$$\begin{aligned}
& g_{N_1}(\mathcal{H}\nabla_{X_1}\omega Z_1 + \mathcal{H}\nabla_{X_1}\omega Z_1 + \mathcal{H}\nabla_{X_1}\omega Z_1, X_2) \\
= \; & g_{N_1}(\mathcal{A}_{X_1}PZ_1 + \cos^2\theta_1\mathcal{A}_{X_1}QZ_1 + \cos^2\theta_2\mathcal{A}_{X_1}RZ_1, X_2) + \\
& g_{N_1}(\mathcal{H}\nabla_{X_1}\omega Z_1, CX_2) + g_{N_1}(\mathcal{A}_{X_1}\omega Z_1, BX_2),
\end{aligned}$$

for $X_1, X_2 \in (\ker F_*)^\perp$ and $Z_1 \in (\ker F_*)$.

Proof. For $X_1, X_2 \in (\ker F_*)^\perp$ and $Z_1 \in (\ker F_*)$, using equations (2.2), (2.8), (2.9), (3.2), (3.3) and Lemmas 3.2 and 3.3 we have

$$\begin{aligned}
& g_{N_1}(\nabla_{X_1}X_2, Z_1) \\
= \; & -g_{N_1}(\nabla_{X_1}Z_1, X_2), \\
= \; & -g_{N_1}(\nabla_{X_1}JZ_1, JX_2), \\
= \; & -g_{N_1}(\nabla_{X_1}PZ_1 + \cos^2\theta_1\nabla_{X_1}QZ_1 + \cos^2\theta_2\nabla_{X_1}RZ_1, X_2) - \\
& g_{N_1}(\nabla_{X_1}\omega Z_1, JX_2) + g_{N_1}(\nabla_{X_1}\omega Z_1 + \nabla_{X_1}\omega Z_1 + \nabla_{X_1}\omega Z_1, X_2), \\
= \; & -g_{N_1}(\mathcal{A}_{X_1}PZ_1 + \cos^2\theta_1\mathcal{A}_{X_1}QZ_1 + \cos^2\theta_2\mathcal{A}_{X_1}RZ_1, X_2) - \\
& g_{N_1}(\mathcal{H}\nabla_{X_1}\omega Z_1, CX_2) - g_{N_1}(\mathcal{A}_{X_1}\omega Z_1, BX_2) + \\
& g_{N_1}(\mathcal{H}\nabla_{X_1}\omega\phi PZ_1 + \mathcal{H}\nabla_{X_1}\omega\phi QZ_1 + \mathcal{H}\nabla_{X_1}\omega\phi RZ_1, X_2), \\
= \; & -g_{N_1}(\mathcal{A}_{X_1}PZ_1 + \cos^2\theta_1\mathcal{A}_{X_1}QZ_1 + \cos^2\theta_2\mathcal{A}_{X_1}RZ_1, X_2) - \\
& g_{N_1}(\mathcal{H}\nabla_{X_1}\omega Z_1, CX_2) - g_{N_1}(\mathcal{A}_{X_1}\omega Z_1, BX_2) + \\
& g_{N_1}(\mathcal{H}\nabla_{X_1}\omega Z_1, CX_2) + g_{N_1}(\mathcal{A}_{X_1}\omega Z_1, BX_2),
\end{aligned}$$

which completes the proof. □

Theorem 3.6. *Let F be a PQBS submersion from a Kähler manifold (N_1, g_{N_1}, J) onto a Riemannian manifold (N_2, g_{N_2}). Then, the invariant distribution D defines a totally geodesic foliation on N_1 if and only if*

$$\begin{aligned} & g_{N_1}(\mathcal{V}\nabla_{U_1}U_2, (\cos^2\theta_1)QY_1 + (\cos^2\theta_2)RY_1) \\ = \; & g_{N_1}(\mathcal{T}_{U_1}U_2, \omega\phi QY_1 + \omega\phi RY_1) + g_{N_1}(\mathcal{T}_{U_1}JPU_2, QY_1), \end{aligned}$$

$$g_{N_1}(\mathcal{V}\nabla_{U_1}BY_2, JPU_2) = -g_{N_1}(\mathcal{T}_{U_1}CY_2, JPU_2),$$

for $U_1, U_2 \in \Gamma(D), Y_1 \in \Gamma(D_1 \oplus D_2)$ and $Y_2 \in \Gamma(\ker F_*)^\perp$.

Proof. For $U_1, U_2 \in \Gamma(D), Y_1 \in \Gamma(D_1 \oplus D_2)$ and $Y_2 \in \Gamma(\ker F_*)^\perp$, using equations $(2.2), (2.6), (3.2), (3.3)$ and Lemmas 3.2 and 3.3, we have

$$\begin{aligned} & g_{N_1}(\nabla_{U_1}U_2, Y_1) \\ = \; & g_{N_1}(J\nabla_{U_1}U_2, JY_1), \\ = \; & g_{N_1}(\mathcal{V}\nabla_{U_1}U_2, (\cos^2\theta_1)QY_1 + (\cos^2\theta_2)RY_1) - \\ & g_{N_1}(\mathcal{T}_{U_1}U_2, \omega\phi QY_1 + \omega\phi RY_1) + g_{N_1}(\mathcal{T}_{U_1}JPU_2, QY_1). \end{aligned}$$

Now, again using equations $(2.2), (2.6), (2.7), (3.2), (3.7)$ and (3.7), we have

$$\begin{aligned} & g_{N_1}(\nabla_{U_1}U_2, Y_2) \\ = \; & -g_{N_1}(\nabla_{U_1}Y_2, U_2), \\ = \; & -g_{N_1}(\nabla_{U_1}JY_2, JU_2), \\ = \; & -g_{N_1}(\mathcal{V}\nabla_{U_1}BY_2, JPU_2) - g_{N_1}(\mathcal{T}_{U_1}CY_2, JPU_2), \end{aligned}$$

which completes the proof. □

Theorem 3.7. *Let F be a PQBS submersion from a Kähler manifold (N_1, g_{N_1}, J) onto a Riemannian manifold (N_2, g_{N_2}). Then, the slant distribution D_1 defines a totally geodesic foliation on N_1 if and only if*

$$\begin{aligned} & g_{N_1}(\mathcal{T}_{X_1}\omega\phi QX_2, Z_1) \\ = \; & g_{N_1}(\mathcal{T}_{X_1}\omega\phi X_2, JPZ_1 + \phi RZ_1) + g_{N_1}(\mathcal{H}_{X_1}\omega QX_2, \omega RZ_1), \end{aligned}$$

$$\begin{aligned} & g_{N_1}([X_1, Z_2], X_2) \\ = \; & (\sin 2\theta_1)Z_2[\theta_1]g_{N_1}(X_1, X_2) + g_{N_1}(\mathcal{A}_{Z_2}\omega\phi X_1, X_2) - \\ & g_{N_1}(\mathcal{A}_{Z_2}\omega X_1, \phi X_2) - g_{N_1}(\mathcal{H}\nabla_{Z_2}\omega X_1, \omega X_2), \end{aligned}$$

for $X_1, X_2 \in \Gamma(D_1), Z_1 \in \Gamma(D \oplus D_2)$ and $Z_2 \in \Gamma(\ker F_*)^\perp$.

Proof. For $X_1, X_2 \in \Gamma(D_1), Z_1 \in \Gamma(D \oplus D_2)$ and $Z_2 \in \Gamma(\ker F_*)^{\perp}$, using equations $(2.2), (2.7), (3.2), (3.3)$ and Lemma 3.2, we have

$$\begin{aligned} & g_{N_1}(\nabla_{X_1} X_2, Z_1) \\ = \;& g_{N_1}(\nabla_{X_1} J X_2, J Z_1), \\ = \;& (\cos^2\theta_1) g_{N_1}(\nabla_{X_1} Q X_2, Z_1) - g_{N_1}(\nabla_{X_1} \omega\phi Q X_2, Z_1) + \\ & + g_{N_1}(\nabla_{X_1} \omega\phi X_2, JPZ_1 + \phi R Z_1) + g_{N_1}(\nabla_{X_1} \omega Q X_2, \omega R Z_1). \end{aligned}$$

Now, we have

$$\begin{aligned} & \sin^2\theta_1 g_{N_1}(\nabla_{X_1} X_2, Z_1) \\ = \;& -g_{N_1}(\mathcal{T}_{X_1} \omega\phi Q X_2, Z_1) + g_{N_1}(\mathcal{T}_{X_1} \omega\phi X_2, JPZ_1 + \phi R Z_1) \\ & + g_{N_1}(\mathcal{H}_{X_1} \omega Q X_2, \omega R Z_1) \end{aligned}$$

Next, from equations $(2.2), (2.9), (3.2), (3.3)$ and Lemma 3.2, we have

$$\begin{aligned} & g_{N_1}(\nabla_{X_1} X_2, Z_2) \\ = \;& -g_{N_1}(\nabla_{X_1} Z_2, X_2), \\ = \;& -g_{N_1}([X_1, Z_2], X_2) - g_{N_1}(\nabla_{Z_2} X_1, X_2), \\ = \;& -g_{N_1}([X_1, Z_2], X_2) - (\cos^2\theta_1) g_{N_1}(\nabla_{Z_2} X_1, X_2) + \\ & (\sin 2\theta_1) Z_2[\theta_1] g_{N_1}(Q X_1, X_2) + g_{N_1}(\nabla_{Z_2} \omega\phi Q X_1, X_2) - \\ & g_{N_1}(\nabla_{Z_2} \omega Q X_1, \phi Q X_2) - g_{N_1}(\nabla_{Z_2} \omega Q X_1, \omega Q X_2). \end{aligned}$$

Now, we have

$$\begin{aligned} & (\sin^2\theta_1) g_{N_1}(\nabla_{X_1} X_2, Z_2) \\ = \;& -g_{N_1}([X_1, Z_2], X_2) + (\sin 2\theta_1) Z_2[\theta_1] g_{N_1}(X_1, X_2) + g_{N_1}(\mathcal{A}_{Z_2} \omega\phi X_1, X_2) - \\ & g_{N_1}(\mathcal{A}_{Z_2} \omega X_1, \phi X_2) - g_{N_1}(\mathcal{H}\nabla_{Z_2} \omega X_1, \omega X_2). \end{aligned}$$

□

Theorem 3.8. *Let F be a PQBS submersion from a Kähler manifold (N_1, g_{N_1}, J) onto a Riemannian manifold (N_2, g_{N_2}). Then, the slant distribution D_2 defines a totally geodesic foliation on N_1 if and only if*

$$\begin{aligned} & g_{N_1}(\mathcal{T}_{Y_1} \omega\phi Q Y_2, U_1) \\ = \;& g_{N_1}(\mathcal{T}_{Y_1} \omega\phi Y_2, JPU_1 + \phi R U_1) + g_{N_1}(\mathcal{H}_{Y_1} \omega Q Y_2, \omega R U_1), \end{aligned}$$

$$\begin{aligned} & g_{N_1}([Y_1, U_2], Y_2) \\ = \;& (\sin 2\theta_1) U_2[\theta_1] g_{N_1}(Y_1, Y_2) + g_{N_1}(\mathcal{A}_{U_2} \omega\phi Y_1, Y_2) - \\ & g_{N_1}(\mathcal{A}_{U_2} \omega Y_1, \phi Y_2) - g_{N_1}(\mathcal{H}\nabla_{U_2} \omega Y_1, \omega Y_2), \end{aligned}$$

for $Y_1, Y_2 \in \Gamma(D_2), U_1 \in \Gamma(D \oplus D_1)$ and $U_2 \in \Gamma(\ker F_*)^{\perp}$.

Theorem 3.9. *Let $\digamma$ be a PQBS submersion from a Kähler manifold (N_1, g_{N_1}, J) onto a Riemannian manifold (N_2, g_{N_2}). Then, $\digamma$ is a totally geodesic map if and only if*

$$\begin{aligned} & -g_{N_1}([X_1, U_1], X_2) \\ = \ & g_{N_1}(\mathcal{V}\nabla_{U_1} PX_1, JX_2) + (\cos^2\theta_1) g_{N_1}(\mathcal{V}\nabla_{U_1} QX_1, X_2) - \\ & (\sin 2\theta_1) U_1[\theta_1] g_{N_1}(QX_1, X_2) - g_{N_1}(\mathcal{A}_{U_1}\omega\phi QX_1, X_2) + \\ & \cos^2\theta_2 g_{N_1}(\nabla_{U_1} RX_1, X_2) - g_{N_1}(\nabla_{U_1}\omega\phi RX_1, X_2) - \\ & (\sin 2\theta_2) U_1[\theta_2] g_{N_1}(RX_1, X_2) + g_{N_1}(\mathcal{H}\nabla_{U_1}\omega RX_1, CX_2) + \\ & g_{N_1}(\mathcal{A}_{U_1}\omega RX_1, BX_2), \end{aligned}$$

$$\begin{aligned} & g_{N_1}(\mathcal{A}_{U_1} PX_1 + \cos^2\theta_1 \mathcal{A}_{U_1} QX_1 + \cos^2\theta_2 \nabla_{U_1} RX_1, U_2) \\ = \ & g_{N_1}(\mathcal{H}\nabla_{U_1}\omega\phi PX_1 + \mathcal{H}\nabla_{U_1}\omega\phi QX_1 + \mathcal{H}\nabla_{U_1}\omega\phi RX_1, U_2) - \\ & g_{N_1}(\mathcal{A}_{U_1}\omega X_1, BU_2) - g_{N_1}(\mathcal{H}\nabla_{U_1}\omega X_1, CU_2), \end{aligned}$$

for $X_1, X_2 \in \Gamma(\ker \digamma_*)$ and $U_1, U_2 \in \Gamma(\ker \digamma_*)^\perp$.

Proof. Since $\digamma$ is a Riemannian submersion, we have

$$(\nabla \digamma_*)(U_1, U_2) = 0,$$

for $U_1, U_2 \in \Gamma(\ker \digamma_*)^\perp$.

For $X_1, X_2 \in \Gamma(\ker \digamma_*)$ and $U_1, U_2 \in \Gamma(\ker \digamma_*)^\perp$, using equations (2.2), (2.10), (2.8), (2.9), (3.2), (3.3) and Lemmas 3.2 and 3.3, we have

$$\begin{aligned} & g_{N_2}((\nabla \digamma_*)(X_1, X_2), \digamma_*(U_1)) \\ = \ & -g_{N_1}(\nabla_{X_1} X_2, U_1) \\ = \ & g_{N_1}([X_1, U_1], X_2) + g_{N_1}(\nabla_{X_1} U_1, X_2), \\ = \ & g_{N_1}([X_1, U_1], X_2) + g_{N_1}(\nabla_{U_1} PX_1, JX_2) + (\cos^2\theta_1) g_{N_1}(\nabla_{U_1} QX_1, X_2) - \\ & (\sin 2\theta_1) U_1[\theta_1] g_{N_1}(QX_1, X_2) - g_{N_1}(\nabla_{U_1}\omega\phi QX_1, X_2) + \\ & g_{N_1}(\nabla_{U_1}\omega QX_1, JX_2) + \cos^2\theta_2 g_{N_1}(\nabla_{U_1} RX_1, X_2) - \\ & g_{N_1}(\nabla_{U_1}\omega\phi RX_1, X_2) - (\sin 2\theta_2) U_1[\theta_2] g_{N_1}(RX_1, X_2) + \\ & g_{N_1}(\nabla_{U_1}\omega RX_1, JX_2), \\ = \ & g_{N_1}([X_1, U_1], X_2) + g_{N_1}(\mathcal{V}\nabla_{U_1} PX_1, JX_2) + (\cos^2\theta_1) g_{N_1}(\mathcal{V}\nabla_{U_1} QX_1, X_2) - \\ & (\sin 2\theta_1) U_1[\theta_1] g_{N_1}(QX_1, X_2) - g_{N_1}(\mathcal{A}_{U_1}\omega\phi QX_1, X_2) + \\ & \cos^2\theta_2 g_{N_1}(\nabla_{U_1} RX_1, X_2) - g_{N_1}(\nabla_{U_1}\omega\phi RX_1, X_2) - \\ & (\sin 2\theta_2) U_1[\theta_2] g_{N_1}(RX_1, X_2) + \\ & g_{N_1}(\mathcal{H}\nabla_{U_1}\omega RX_1, CX_2) + g_{N_1}(\mathcal{A}_{U_1}\omega RX_1, BX_2). \end{aligned}$$

Next, using equations (2.2), (2.10), (3.2), (3.3), (3.7) and Lemma 3.2 and 3.3, we have

$$\begin{aligned}
& g_{N_2}((\nabla F_*)(U_1, X_1), F_*(U_2)) \\
= \; & -g_{N_1}(\nabla_{U_1} X_1, U_2), \\
= \; & -g_{N_1}(\nabla_{U_1} JX_1, JU_2), \\
= \; & - - g_{N_1}(\nabla_{U_1} JPX_1 + JQX_1 + JRX_1, U_2), \\
= \; & -g_{N_1}(\nabla_{U_1} PX_1 + \cos^2\theta_1 \nabla_{U_1} QX_1 + \cos^2\theta_2 \nabla_{U_1} RX_1, U_2) + \\
& g_{N_1}(\nabla_{U_1} \omega\phi PX_1 + \nabla_{U_1} \omega\phi QX_1 + \nabla_{U_1} \omega\phi RX_1, U_2) - \\
& g_{N_1}(\nabla_{U_1} \omega X_1, JU_2) \\
= \; & -g_{N_1}(\mathcal{A}_{U_1} PX_1 + \cos^2\theta_1 \mathcal{A}_{U_1} QX_1 + \cos^2\theta_2 \nabla_{U_1} RX_1, U_2) + \\
& g_{N_1}(\mathcal{H}\nabla_{U_1} \omega\phi PX_1 + \mathcal{H}\nabla_{U_1} \omega\phi QX_1 + \mathcal{H}\nabla_{U_1} \omega\phi RX_1, U_2) - \\
& g_{N_1}(\mathcal{A}_{U_1} \omega X_1, BU_2) - g_{N_1}(\mathcal{H}\nabla_{U_1} \omega X_1, CU_2).
\end{aligned}$$

□

4. Example

Note that given an Euclidean space R^{2n} with coordinates $(y_1, y_2, ..., y_{2n-1}, y_{2n})$ we can canonically choose an almost complex structure J on R^{2n} as follows:

$$\begin{aligned}
& J(a_1 \frac{\partial}{\partial y_1} + a_2 \frac{\partial}{\partial y_2} + ... + a_{2n-1} \frac{\partial}{\partial y_{2n-1}} + a_{2n} \frac{\partial}{\partial y_{2n}}) \\
= \; & -a_2 \frac{\partial}{\partial y_1} + a_1 \frac{\partial}{\partial y_2} + ... - a_{2n} \frac{\partial}{\partial y_{2n-1}} + a_{2n-1} \frac{\partial}{\partial y_{2n}},
\end{aligned}$$

where $a_1, a_2, ..., a_{2n}$ are C^∞ functions defined on R^{2n}. Throughout this section, we will use this notation.

Example 4.1. Define a map $F : R^{10} \to R^4$ by

$$F(y_1, y_1, ..., y_{10}) = (\sin y_3 - \cos y_5, y_6, \sin y_7 - \cos y_9, y_{10}),$$

which is a PQBS submersion such that

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial y_1}, X_2 = \frac{\partial}{\partial y_2}, X_3 = \cos y_5 \frac{\partial}{\partial y_3} + \sin y_3 \frac{\partial}{\partial y_5}, \\
X_4 &= \frac{\partial}{\partial y_4}, X_5 = \cos y_9 \frac{\partial}{\partial y_7} + \sin y_7 \frac{\partial}{\partial y_9}, X_6 = \frac{\partial}{\partial y_8},
\end{aligned}$$

$$\ker F_* = D \oplus D_1 \oplus D_2,$$

where

$$\begin{aligned}
D &= < X_1 = \frac{\partial}{\partial y_1}, X_2 = \frac{\partial}{\partial y_2} >, \\
D_1 &= < X_3 = \cos y_5 \frac{\partial}{\partial y_3} + \sin y_3 \frac{\partial}{\partial y_5}, X_4 = \frac{\partial}{\partial y_4} >, \\
D_2 &= < X_5 = \cos y_9 \frac{\partial}{\partial y_7} + \sin y_7 \frac{\partial}{\partial y_9}, X_6 = \frac{\partial}{\partial y_8} >,
\end{aligned}$$

$$
\begin{aligned}
&(\ker F_*)^{\perp} \\
= \quad & < \sin y_3 \frac{\partial}{\partial y_3} - \cos y_5 \frac{\partial}{\partial y_5}, \frac{\partial}{\partial y_6}, \sin y_7 \frac{\partial}{\partial y_7} - \cos y_9 \frac{\partial}{\partial y_9} > .
\end{aligned}
$$

Thus F is a PQBS submersion with slant functions y_5 and y_5.

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