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EXAMINING SINGULAR DISSIPATIVE QUANTUM STURM–LIOUVILLE OPERATORS IN LIMIT-CIRCLE CASE WHENEVER $q > 1$

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Abstract. In this article, we examine dissipative singular quantum (q) Sturm–Liouville operators ($q > 1$) acting in a suitable Hilbert space, where the extensions of a minimal symmetric operator in limit-circle case (with deficiency indices $(2, 2)$) are presented. We create a self-adjoint dilation of the dissipative operator along with its incoming and outgoing spectral representations. These constructions enable us to find the scattering matrix of the dilation using the Weyl–Titchmarsh function associated with a self-adjoint q-Sturm–Liouville operator. Additionally, we establish a functional model for the dissipative operator and derive its characteristic function using the scattering matrix of the dilation (or the Weyl–Titchmarsh function). We prove theorems related to the completeness of the system of eigenfunctions and associated functions (root functions) for both dissipative and accumulative q-Sturm–Liouville operators.
Keywords: q-Sturm–Liouville equation, dissipative operator, self-adjoint dilation, Weyl–Titchmarsh function, characteristic function, completeness of the root functions.

1. Introduction

Quantum calculus is analogous to traditional infinitesimal calculus, but does not rely on the concept of limits. It has a lot of applications in several mathematical

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areas such as basic hypergeometric functions, combinatorics, fractal geometry, number theory, orthogonal polynomials, the calculus of variation, statistical physics and theory of relativity ([1, 26, 34] and the references therein). The concept of the q-derivative was introduced by Jackson in 1910 [20, 21]. Since then, the development of quantum calculus gained momentum, and most of the published work has been interested in problems of quantum calculus. Some q-analogues of definitions and theorems from ordinary calculus have been introduced by Kac and Cheung, including the q-derivative, q-integration, q-exponential function, q-trigonometric function, q-Taylor formula, q-Beta and Gamma functions, Euler–Maclaurin formula, and others [22]. Motivated by the formal research conducted by Exton [17], Annaby and Mansour [10] managed an in-depth investigation into q-calculus. Their results are expanded to encompass various versions of boundary value problems associated with q-Sturm–Liouville operators [11, 13, 16, 9, 14, 23, 24, 6, 8, 19, 4, 5, 7, 12] and have received a lot of attention in recent years.

In [11, 12], the authors developed the q-Titchmarsh–Weyl theory for singular q-Sturm–Liouville problems. Additionally, they introduced the concepts of singularities in q-limit-point and q-limit-circle cases. The q-Titchmarsh–Weyl theory is an extension of the classical Titchmarsh–Weyl theory to the context of quantum calculus. q-Titchmarsh–Weyl theory similarly focuses on the spectral properties of certain self-adjoint operators and explores the behavior of these operators, including their eigenvalues, eigenvectors, and associated spectral measures for q-Sturm–Liouville problems. It is known that dissipative operators comprise an important class of non-self-adjoint operators. We say that $\mathbf{A}$ (with dense domain $D(\mathbf{A})$) acting on a Hilbert space $\mathbf{H}$ is *dissipative* if $\operatorname{Im}(\mathbf{A}f, f) \geq 0$ for all $f \in D(\mathbf{A})$ and $\mathbf{A}$ is *accumulative* if $\operatorname{Im}(\mathbf{A}f, f) \leq 0$ for all $f \in D(\mathbf{A})$ [2, 3, 25, 30, 33, 32].

The spectral analysis of dissipative operators relies on the theory of self-adjoint dilations and the utilization of functional models. The dilation theory given by Sz. Nagy–Foiaş [30] and the scattering theory given by Lax–Phillips [27] are fundamental for constructing these functional models. Therefore, it is imperative to incorporate the characteristic function as a crucial concept, as it plays a central role in these theories for obtaining the spectral properties of dissipative operators. In the papers [4, 5], the author examined dissipative singular q-Sturm–Liouville problems for $0 < q < 1$ in the limit-circle and limit-point cases by constructing self-adjoint dilations for each dissipative operators and by using the scattering functions of these dilations, the author gave their characteristic functions. In contrast to the aforementioned studies on dissipative q-Sturm–Liouville operators with $0 < q < 1$ [2, 3, 19, 4, 5, 8, 16], we focus on spectral problems associated with dissipative singular q-Sturm–Liouville operators in limit-circle case for $q > 1$ in this study. The paper is structured as follows. Section 2 comprises some fundamental definitions and a lemma to follow the paper. Section 3 deals with the construction of the dissipative singular q-Sturm–Liouville operators in a suitable Hilbert space, where the extensions of minimal symmetric operator are explored in Weyl's limit-circle case at singular end point ∞. We establish a self-adjoint dilation of the dissipative operator in this section. In Section 4, we construct incoming and outgoing spectral representations of the dilation which provide us to determine the scattering matrix of the dilation using the scheme of Lax and Phillips [27]. This determination is expressed in terms of the Weyl–Titchmarsh function associated with a self-adjoint q-Sturm–Liouville operator. We establish a functional model for the dissipative operator by using the incoming spectral representation and define its characteristic function in relation to the Weyl–Titchmarsh function of a self-adjoint q-Sturm–Liouville operator (or in terms of the scattering matrix of the self-adjoint dilation) in this section. In conclusion, we also provide theorems on completeness of the system of eigenfunctions and associated functions (or root functions) of the dissipative and accumulative q-Sturm–Liouville operators in section 4 which rely on the results derived for the characteristic function.

2. Preliminaries

In this section, we introduce some of the q-notations and we set forth some fundamental definitions and equations needed for our subsequent discussion. We also introduce the main problem in this section. We assume that the reader of this paper is familiar with basic concepts of q-calculus, and for a review of the topic, we refer to the standard notations given in [10, 13, 28]. For $q \in \mathbb{R} := (-\infty, \infty)$ a set $C \subseteq \mathbb{R}$ is called a q-geometric set if, for every $t \in C$, $qt \in C$. If $C \subseteq \mathbb{R}$ is a q-geometric set, then it contains all geometric sequences $\{q^n t\}$ $(n = 0, 1, 2...)$, $t \in C$. Let y be a real or complex-valued function defined on a q-geometric set C. The q-difference operator is defined by

$$D_q y(t) := \frac{y(t) - y(qt)}{t(1-q)},\ t \in C\backslash\{0\}. \tag{2.1}$$

If $0 \in C$, then the q-derivative of a function y at zero is defined as $(0 < q < 1)$

$$D_q y(0) := \lim_{n\to\infty} \frac{y(q^n t) - y(0)}{q^n t},\ t \in C,$$

if the limit exists and does not depend on t. It is important for us to give the definition of $D_{q^{-1}}$ in the same manner for introducing the formulation of the extension problems. It is given by

$$D_{q^{-1}} y(t) := \begin{cases} \frac{y(t)-y(q^{-1}t)}{t(1-q^{-1})},\ t \in C\backslash\{0\}, \\ \\ D_q y(0),\ t = 0, \end{cases}$$

if $D_q y(0)$ exists. It is well known that the following equations that we will use in next sections can be found directly from the definition:

$$D_{q^{-1}} y(t) = (D_q y)(q^{-1}t),\ D_q^2 y(q^{-1}t) = qD_q[D_q y(q^{-1}t)] = D_{q^{-1}} D_q y(t).$$

As a right inverse of the q-difference operator, Jackson's q-integration is presented [20] by

$$\int_0^t y(s) d_q s := t(1-q) \sum_{n=0}^{\infty} q^n y(q^n t),\ t \in C,$$

if the series converges and in general, we have

$$\int_a^b y(s)d_qs := \int_0^b y(s)d_qs - \int_0^a y(s)d_qs,\ a,b \in C.$$

There is no unique canonical choice for the q-integration over $[0,\infty)$. Although Matsuo defined q-integration on the interval $[0,\infty)$ by

$$\int_0^{b\infty} y(s)d_qs := b(1-q)\sum_{n=-\infty}^{\infty} q^n y(bq^n),\ b>0,$$

if the series converges [29], Hanh in [18] defined it for a function y on the same interval as

$$\int_0^{\infty} y(s)d_qs := (1-q)\sum_{n=-\infty}^{\infty} q^n y(q^n),$$

if the series converges. We say that y is q-integrable on a q-geometric set C if and only if $\int_0^{b\infty} y(s)d_qs$ exists for all $t \in C$. Let C^* be a q-geometric set containing zero. A function y defined on C^* is called q-regular at zero if

$$\lim_{n\to\infty} y(q^n t) = y(0)$$

satisfies for all $t \in C^*$. Functions that are q-regular at zero generate an important class of functions because they consist of continuous functions. Because of this, we are interested in these kinds of functions in this paper. If y and z are q-regular at zero, there is a rule of q-integration by parts given as

$$\int_0^a z(t)D_qy(t)d_qt = (yz)(a) - (yz)(0) - \int_0^a D_qz(t)y(qt)d_qt$$

and the q-product rule is given by

$$D_q[y(t)z(t)] = y(qt)D_qz(t) + z(t)D_qy(t)$$

in [10].

Let's denote a singular q-Sturm–Liouville expression as A

$$(Af)(t) = \frac{1}{v(t)}\left(-\frac{1}{q}D_{q^{-1}}\left[w(t)D_qf(t)\right] + u(t)f(t)\right),\ t \in \mathbb{R}_{1+} := [1,\infty), \tag{2.2}$$

where v, w and u are real-valued functions defined on $\mathbb{R}_{1+}$ such that $w(t) \neq 0$, $v(t) > 0$ for all $t \in \mathbb{R}_{1+}$, $q > 1$ as well as D_q is the q-difference operator defined in (2.1). Throughout the paper, we will consider the operators related to (2.2) by introducing the Hilbert space $\mathcal{L}^2_{v,q}(\mathbb{R}_{1+})$ which consists of all complex-valued functions satisfying

$$\int_1^{\infty} v(t)\left|f(t)\right|^2 d_qt < \infty,\ f: \mathbb{R}_{1+} \to \mathbb{C}$$

and with the inner product

$$(f,g)=\int_1^\infty v(t)f(t)\overline{g(t)}d_qt,\ \ f,g:\mathbb{R}_{1+}\to\mathbb{C},$$

where $v(t)>0$ for all $t\in\mathbb{R}_{1+}$.

Let $\mathfrak{D}_{\max}$ denote a linear set of all functions $f\in\mathcal{L}^2_{v,q}(\mathbb{R}_{1+})$ such that $Af\in\mathcal{L}^2_{v,q}(\mathbb{R}_{1+})$. The *maximal operator* $\mathcal{Q}_{\max}$ on $\mathfrak{D}_{\max}$ is defined by $\mathcal{Q}_{\max}\,f=Af$. For each $f,g\in\mathfrak{D}_{\max}$, the *q-Wronski determinant* (or *q-Wronskian*) is defined by

$$\mathcal{W}_q[f,g](t)=f(t)D_qg(t)-D_qf(t)g(t),\ \ t\in\mathbb{R}_{1+}.$$

To get the results that we aimed for, we must present an essential definition which is called q-Green's formula (or q-Lagrange's identity) [10, 13, 11] given by

$$\int_1^t(Af)(s)\overline{g(s)}d_qs-\int_1^t f(s)\overline{(Ag)(s)}d_qs=[f,g]\,(t)-[f,g]\,(1),\ \ t\in\mathbb{R}_{1+},\tag{2.3}$$

for arbitrary $f,g\in\mathfrak{D}_{\max}$, here $[f,g](t)$ denotes the q-Lagrange bracket and written by

$$[f,g]\,(t):=w(t)[f(t)\overline{D_{q^{-1}}g(t)}-D_{q^{-1}}f(t)\overline{g(t)}],\ \ t\in\mathbb{R}_{1+}.$$

It is readily apparent from (2.3) that

$$[f,g]\,(\infty):=\lim_{t\to\infty}[f,g]\,(t)$$

exists and is finite for all $f,g\in\mathfrak{D}_{\max}$. Let us assume that $\mathfrak{D}_{\min}$ is the linear dense set in $\mathcal{L}^2_{v,q}(\mathbb{R}_{1+})$ of all vectors $f\in\mathfrak{D}_{\max}$ satisfying the conditions

$$f\,(1)=(wD_{q^{-1}}f)\,(1)=0,\ \ [f,g]\,(\infty)=0,\tag{2.4}$$

for arbitrary $g\in\mathfrak{D}_{\max}$. Please note that the restriction of the operator $\mathcal{Q}_{\max}$ to $\mathfrak{D}_{\min}$ is referred to as the *minimal operator* and is represented by $\mathcal{Q}_{\min}$. It is a closed symmetric operator with deficiency indices $(2,2)$ or $(1,1)$ and is symmetric from (2.4). Furthermore, the equality $\mathcal{Q}_{\max}=\mathcal{Q}^*_{\min}$ holds [5, 10, 11, 15, 12, 31]. We assume that Weyl's limit-circle case satisfies the expression A, that is the symmetric operator $\mathcal{Q}_{\min}$ has deficiency indices $(2,2)$ in this study ([5, 10, 11, 15, 12, 31]). Let us refer to τ and ϕ as the real-valued solutions of the equation $Af=0$ with the following initial conditions for $t\in\mathbb{R}_{1+}$.

$$\tau(1)=1,\ \ (wD_{q^{-1}}\tau)\,(1)=0,\ \ \phi(1)=0,\ \ (wD_{q^{-1}}\phi)\,(1)=1.\tag{2.5}$$

The Wronskian of the two solutions of the equation $Af=0$ is independent of t, and the two solutions of this equation are linearly independent if and only if their Wronskian is non-zero [10, 12, 13]. From conditions (2.5) and the constancy of the Wronskian, we can deduce that

$$\mathcal{W}_q[\tau,\phi](t)=\mathcal{W}_q[\tau,\phi](t)=1,\ \ t\in\mathbb{R}_{1+}.$$

It gives that τ and ϕ form a fundamental system of solutions of the equation $Af=0$. Given that $\mathcal{Q}_{\min}$ has deficiency indices $(2,2)$, both τ and ϕ are in $\mathcal{L}^2_{v,q}(\mathbb{R}_{1+})$, and moreover, they are in $\mathfrak{D}_{\max}$.

Lemma 2.1. *For any arbitrary functions $f, g \in \mathfrak{D}_{\max}$, the following equality (the Plücker identity) holds true*

$$[f,g](t) = [f,\tau](t)[\overline{g},\phi](t) - [f,\phi](t)[\overline{g},\tau](t),\ t \in \mathbb{R}_{1+} \cup \{\infty\}. \tag{2.6}$$

Proof. We know that the functions τ and ϕ are real-valued functions and $[\tau,\phi](t) = 1$ $(t \in \mathbb{R}_{1+} \cup \{\infty\})$, by considering these properties, we obtain

$$\begin{aligned}
&[f,\tau](t)[\overline{g},\phi](t) - [f,\phi](t)[\overline{g},\tau](t) \\
&= r(q^{-1}t)(fD_{q^{-1}}\tau - D_{q^{-1}}f\tau)(t)r(q^{-1}t)(\overline{g}D_{q^{-1}}\phi - \overline{D_{q^{-1}}g}\phi)(t) \\
&-r(q^{-1}t)(fD_{q^{-1}}\phi - D_{q^{-1}}f\phi)(t)r(t)(\overline{g}D_{q^{-1}}\tau - \overline{D_{q^{-1}}g}\tau)(t) \\
&= (r(q^{-1}t))^2(fD_{q^{-1}}\tau\overline{g}D_{q^{-1}}\phi - fD_{q^{-1}}\tau\overline{D_{q^{-1}}g}\phi - D_{q^{-1}}f\tau\overline{g}D_{q^{-1}}\phi \\
&+D_{q^{-1}}f\tau\overline{D_{q^{-1}}g}\phi - fD_{q^{-1}}\phi\overline{g}D_{q^{-1}}\tau + fD_{q^{-1}}\phi\overline{D_{q^{-1}}g}\tau \\
&+D_{q^{-1}}f\phi\overline{g}D_{q^{-1}}\tau - D_{q^{-1}}f\phi\overline{D_{q^{-1}}g}\tau)(t) \\
&= r(q^{-1}t)(-f\overline{D_{q^{-1}}g} + D_{q^{-1}}f\overline{g})(t)r(q^{-1}t)(D_{q^{-1}}\tau\phi - \tau D_{q^{-1}}\phi)(t) = [f,g](t).
\end{aligned}$$

It gives the proof of Lemma 2.1. □

Now, we will give brief information about Lax–Phillips method [27]. Because it is necessary for us to establish functional model and to examine the scattering properties of the dilation in the last section. The Lax–Phillips method is an important result in functional analysis for the study of existence and uniqueness of solutions in functional spaces. Let $\Theta(\lambda)$ be an arbitrary non-constant inner function ([2, 3, 25, 30, 33]) defined on the upper half-plane (we recall that a function Θ analytic in the upper half-plane $\mathbb{C}_+$ is called *inner function* on $\mathbb{C}_+$ if $|\Theta(\lambda)| \leq 1$ for $\lambda \in \mathbb{C}_+$, and $|\Theta(\lambda)| = 1$ for almost all $\lambda \in \mathbb{R}$). The symbols $\mathcal{H}^2_\pm$ refer to the Hardy classes [30, 33] in $\mathcal{L}^2(\mathbb{R})$ consisting of the functions analytically extendable to the upper and lower half-planes, respectively. Here $\mathcal{L}^2(\mathbb{R})$ is the Hilbert space consisting of all complex-valued functions f such that

$$\int_{-\infty}^{\infty} |f(t)|^2 \, dt < \infty.$$

Let us consider the nontrivial subspace $\mathcal{N} = \mathcal{H}^2_+ \ominus \Theta\mathcal{H}^2_+$. Then $\mathcal{N} \neq 0$ is a subspace of the Hilbert space $\mathcal{H}^2_+$. We consider the semigroup of the operators $\mathcal{X}(s)$ $(s \geq 0)$, $\mathcal{X}(s)\tau = \mathcal{P}\left[e^{i\lambda s}\tau\right]$, $\tau := \tau(\lambda) \in \mathcal{N}$, where $\mathcal{P}$ is the orthogonal projection from $\mathcal{H}^2_+$ onto $\mathcal{N}$, acts in the subspace $\mathcal{N}$. The generator of the semigroup $\{\mathcal{X}(s)\}$ is denoted by $\mathbf{Q}$:

$$\mathbf{Q}\tau = \lim_{s \to +0}[(is)^{-1}(\mathcal{X}(s)\tau - \tau)],$$

which is a dissipative operator acting in $\mathcal{N}$ with domain $D(\mathbf{Q})$ which consists of all functions $\tau \in \mathcal{N}$ such that the limit exists. The operator $\mathbf{Q}$ is called a *model dissipative operator*. This model dissipative operator, which is associated with the names of Lax and Phillips [27], is a special case of a more general model dissipative operator constructed by Sz.-Nagy and Foiaş [30]. The basic assertion is that Θ is the *characteristic function* of the operator $\mathbf{Q}$ [2, 3, 25, 30, 33, 32].

3. Dissipative q-Sturm–Liouville operator and its self-adjoint dilation

In this section, we deal with a q-dissipative operator and obtain self-adjoint dilation of the dissipative operator. Let's examine the operator $Q_{\alpha\beta}$, defined over $D(Q_{\alpha\beta})$, comprising vectors $y \in \mathfrak{D}_{\max}$ which satisfy the boundary conditions

$$(wD_{q^{-1}})y\,(1) - \alpha y\,(1) = 0,\ \ \alpha \in \mathbb{C}, \tag{3.1}$$

$$[y,\tau](\infty) - \beta[y,\phi](\infty) = 0,\ \ \operatorname{Im}\beta = 0 \text{ or } \beta = \infty. \tag{3.2}$$

It is clear that condition (3.2) becomes $[y,\phi](\infty) = 0$ whenever $\beta = \infty$.

Theorem 3.1. *The operator $Q_{\alpha\beta}$ is dissipative in $L^2_{v,q}(\mathbb{R}_{1+})$ whenever $\operatorname{Im}\alpha \geq 0$, $\operatorname{Im}\beta = 0$ or $\beta = \infty$.*

Proof. Assume that $y \in D(Q_{\alpha\beta})$ is arbitrary, then we write

$$(Q_{\alpha\beta}y, y) - (y, Q_{\alpha\beta}y) = [y,y](\infty) - [y,y](1). \tag{3.3}$$

It follows from Lemma (2.1) and condition (3.2) that

$$[y,y](\infty) = 0. \tag{3.4}$$

On the other hand, we get from the condition (3.1) that

$$[y,y](0) = -2i\operatorname{Im}\alpha\,|y(1)|^2\,. \tag{3.5}$$

For the next step, we obtain

$$\operatorname{Im}(Q_{\alpha\beta}y, y) = \operatorname{Im}\alpha\,|y\,(1)|^2 \geq 0 \text{ for } \operatorname{Im}\alpha \geq 0, \tag{3.6}$$

by considering (3.4) and (3.5) in (3.3). It completes the proof by the definition of dissipative operator. Note that, we come through from (3.6) that all eigenvalues of dissipative operator $Q_{\alpha\beta}$ lie in the closed upper half plane. □

Theorem 3.2. *The dissipative operator $Q_{\alpha\beta}$ does not have any real eigenvalue whenever $\operatorname{Im}\alpha > 0$, $\operatorname{Im}\beta = 0$ or $\beta = \infty$.*

Proof. Assume that λ_1 is a real eigenvalue of the operator $Q_{\alpha\beta}$ and $y_1(t) := y(t,\lambda_1)$ is the eigenfunction of the $Q_{\alpha\beta}$ corresponding to the eigenvalue λ_1. Since $(Q_{\alpha\beta}y_1, y_1) = \lambda_1\,\|y_1\|^2$, we write

$$\operatorname{Im}\alpha\,|y_1\,(1)|^2 = \operatorname{Im}\lambda_1\,\|y_1\|^2 = 0$$

and $y_1\,(1) = 0$ by using (3.6). It gives $(wD_{q^{-1}})y_1\,(1) = 0$ from the boundary condition (3.1) and it follows from the uniqueness theorem of the Cauchy problem for the equation $Ay = \lambda y$, $t \in \mathbb{R}_{1+}$, that $y_1(t) \equiv 0$. It completes the proof. □

It is pleasant that the equality (3.6) also presents that the operator $Q_{\alpha\beta}$ becomes an accumulative (self-adjoint) operator in $\mathcal{L}^2_{v,q}(\mathbb{R}_{1+})$ if $\operatorname{Im}\alpha \leq 0, \operatorname{Im}\beta = 0$ or $\beta = \infty, (\operatorname{Im}\alpha = 0$ or $\alpha = \infty, \operatorname{Im}\beta = 0$ or $\beta = \infty)$. In the case where $\alpha = \infty$, the condition (3.1) should be altered to $y(1) = 0$. As a result of this, we present the following theorem which is analogous to the Theorem 3.2.

Corollary 3.1. *If* $\operatorname{Im}\alpha < 0$, $\operatorname{Im}\beta = 0$ *or* $\beta = \infty$, *then the accumulative operator* $Q_{\alpha\beta}$ *has no real eigenvalues.*

For the next step of this section, we investigate the dissipative operators $Q_{\alpha\beta}$ $(\operatorname{Im}\alpha > 0, \operatorname{Im}\beta = 0$ or $\beta = \infty)$ generated by the expression (2.2) and the boundary conditions (3.1) and (3.2). We consider the Hilbert spaces denoted as $\mathcal{L}^2(\mathbb{R}_{1-})$ $(\mathbb{R}_{1-} := (-\infty, 1])$ and $\mathcal{L}^2(\mathbb{R}_{1+})$ $(\mathbb{R}_{1+} := [1, \infty])$ comprising all functions ρ_- and ρ_+, respectively, satisfying

$$\int_{-\infty}^{1} |\rho_-(t)|^2 \, dt < \infty, \quad \int_{1}^{\infty} |\rho_+(t)|^2 \, dt < \infty$$

with the inner product

$$(\rho_-, \gamma_-)_{\mathcal{L}^2(\mathbb{R}_{1-})} = \int_{-\infty}^{1} \rho_-(t)\overline{\gamma_-(t)}dt, \; (\rho_+, \gamma_+)_{\mathcal{L}^2(\mathbb{R}_{1+})} = \int_{1}^{\infty} \rho_+(t)\overline{\gamma_+(t)}dt.$$

We create another Hilbert space by adding the above two Hilbert spaces to the our main Hilbert space $H := \mathcal{L}^2_{v,q}(\mathbb{R}_{1+})$, then we find an orthogonal sum of Hilbert space as $\mathcal{H} = \mathcal{L}^2(\mathbb{R}_{1-}) \oplus H \oplus \mathcal{L}^2(\mathbb{R}_{1+})$, and we call it as the *main Hilbert space of the dilation.* In the space $\mathcal{H}$, we call the operator $\mathcal{B}_{\alpha\beta}$ generated by the expression

$$\mathcal{B}\langle \rho_-, y, \rho_+ \rangle = \langle i\frac{d\rho_-}{d\xi}, Ay, i\frac{d\rho_+}{d\varsigma} \rangle \tag{3.7}$$

on the set $\mathfrak{D}(\mathcal{B}_{\alpha\beta})$ of vectors $\langle \rho_-, y, \rho_+ \rangle$ satisfying the conditions $y \in \mathfrak{D}_{\max}$, $\rho_\mp \in \mathcal{W}^1_2(\mathbb{R}_{1\pm})$ and

$$(wD_{q^{-1}}y)(1) - \alpha y(1) = \delta\rho_-(1), \; (wD_{q^{-1}}y)(1) - \overline{\alpha} y(1) = \delta\rho_+(1), \tag{3.8}$$

$$[y, \tau](\infty) - \beta[y, \phi](\infty) = 0, \tag{3.9}$$

where $\delta^2 := 2\operatorname{Im}\alpha$, $\delta > 0$, and $\mathcal{W}^1_2(\mathbb{R}_{1\pm})$ is the Sobolev space consisting of all functions $f \in \mathcal{L}^2(\mathbb{R}_{1\pm})$ such that f are locally absolutely continuous functions on $\mathbb{R}_\pm$ and $f' \in \mathcal{L}^2(\mathbb{R}_{1\pm})$. Based on this information, we present the following theorem.

Theorem 3.3. *The operator* $\mathcal{B}_{\alpha\beta}$ *is self-adjoint in* $\mathcal{H}$.

Proof. Let assume that $f, h \in \mathfrak{D}(\mathcal{B}_{\alpha\beta}), f = \langle \rho_-, y, \rho_+ \rangle$ and $h = \langle \gamma_-, z, \gamma_+ \rangle$. Then, integrating by parts and using (2.3), we obtain that

$$(\mathcal{B}_{\alpha\beta}f, h)_{\mathcal{H}} = \int_{-\infty}^{1} i\rho'_-\overline{\gamma}_- d\xi + (Ay, z)_H + \int_{1}^{\infty} i\rho'_+\overline{\gamma}_+ d\varsigma \tag{3.10}$$

$$= i\rho_-(1)\,\overline{\gamma}_-(1) - i\rho_+(1)\,\overline{\gamma}_+(1) + [y,z](\infty) - [y,z](1) + (f,\mathcal{B}_{\alpha\beta}h)_{\mathcal{H}}.$$

If we consider Lemma 2.1 by using the conditions (3.8), (3.9) for the components of the vectors f and h, we find

$$i\rho_-(1)\,\overline{\gamma_-(1)} - i\rho_+(1)\,\overline{\gamma_+(1)} + [y,z](\infty) - [y,z](1) = 0.$$

It gives that $(\mathcal{B}_{\alpha\beta}f,h)_{\mathcal{H}} = (f,\mathcal{B}_{\alpha\beta}h)_{\mathcal{H}}$, that is $\mathcal{B}_{\alpha\beta}$ is symmetric and it means that if we show that $\mathcal{B}^*_{\alpha\beta} \subseteq \mathcal{B}_{\alpha\beta}$, we get the selfadjointness of $\mathcal{B}_{\alpha\beta}$. Let us take $h = \langle\gamma_-, z, \gamma_+\rangle \in \mathfrak{D}(\mathcal{B}^*_{\alpha\beta})$. From this, we write $\mathcal{B}^*_{\alpha\beta}h = h^*$ whenever $h^* = \langle\gamma^*_-, z^*, \gamma^*_+\rangle \in \mathcal{H}$. It follows from that

$$(\mathcal{B}_{\alpha\beta}f,h)_{\mathcal{H}} = (f,h^*)_{\mathcal{H}},\ \forall f \in \mathfrak{D}(\mathcal{B}_{\alpha\beta}). \tag{3.11}$$

By considering the definition of the operator $\mathcal{B}$ defined in (3.7), it is easy for us to show $\gamma_\mp \in \mathcal{W}^1_2(\mathbb{R}_\pm)$, $z \in \mathfrak{D}_{\max}$ and $h^* = \mathcal{B}h$ with appropriate selection of components for $f \in (\mathcal{B}_{\alpha\beta})$ in (3.11). After that the equation (3.11) becomes $(\mathcal{B}f,h)_{\mathcal{H}} = (f,\mathcal{B}h)_{\mathcal{H}},\ \forall f \in \mathfrak{D}(\mathcal{B}_{\alpha\beta})$ and so that the sum of the integral terms in the bilinear form $(\mathcal{B}f,h)_{\mathcal{H}}$ is equal to zero, it implies

$$i\rho_-(1)\,\overline{\gamma_-(1)} - i\rho_+(1)\,\overline{\gamma_+(1)} + [y,z](\infty) - [y,z](1) = 0 \tag{3.12}$$

for all $f = \langle\rho_-, y, \rho_+\rangle \in \mathfrak{D}(\mathcal{B}_{\alpha\beta})$. On the other hand, we obtain

$$y(1) = -\frac{i}{\delta}(\rho_+(1) - \rho_-(1)),$$

$$(wD_{q^{-1}}y)(1) = \delta\rho_-(1) - \frac{i\alpha}{\delta}(\rho_+(1) - \rho_-(1)).$$

by choosing the $y(1)$ and $(wD_{q^{-1}}y)(1)$ in the boundary conditions (3.8). Then if we take in consideration these equations in (3.12), we write

$$i\rho_-(1)\,\overline{\gamma_-(1)} - i\rho_+(1)\,\overline{\gamma_+(1)} = [y,z](1) - [y,z](\infty)$$

$$= -\frac{i}{\delta}(\rho_+(1) - \rho_-(1))(\overline{wz'})(1) - \delta[\rho_-(1) - \frac{i\alpha}{\delta^2}(\rho_+(1) - \rho_-(1))]\overline{z}(1)$$

$$-[y,\tau](\infty)[\overline{z},\tau](\infty) + [y,\phi](\infty)[\overline{z},\phi](\infty) = -\frac{i}{\delta}(\rho_+(1) - \rho_-(1))(\overline{wz'})(1)$$

$$-\delta[\rho_-(1) - \frac{i\alpha}{\delta^2}(\rho_+(1) - \rho_-(1))]\overline{z}(1) - ([\overline{z},\tau](\infty) - \beta[\overline{z},\phi](\infty))[y,\phi](\infty).$$

Since the values $\rho_\pm(1)$ can be arbitrary complex numbers, by comparing the coefficient of $\rho_\pm(1)$ on the left and right sides of the last equality, we find that $h = \langle\gamma_-, z, \gamma_+\rangle$ satisfies the following boundary conditions

$$(wD_{q^{-1}})z(1) - \alpha z(1) = \delta\gamma_-(1),$$

$$(wD_{q^{-1}})z(1) - \overline{\alpha}z(1) = \delta\gamma_+(1)$$

and

$$[z,\tau](\infty) - \beta[z,\phi](\infty) = 0.$$

It gives the inclusion $\mathcal{B}^*_{\alpha\beta} \subseteq \mathcal{B}_{\alpha\beta}$, and hence $\mathcal{B}_{\alpha\beta} = \mathcal{B}^*_{\alpha\beta}$. □

To give the relation between the operators $\mathcal{B}_{\alpha\beta}$ and $Q_{\alpha\beta}$, it is necessary to emphasize the following brief information. The self-adjoint operator $\mathcal{B}_{\alpha\beta}$ generates a unitary group $\mathcal{Y}(s) = \exp[i\mathcal{B}_{\alpha\beta}s]$ in $\mathcal{H}$ for $s \in \mathbb{R}$. Assume that $\mathcal{P}_1 : \mathcal{H} \to \mathcal{L}^2_{v,q}(\mathbb{R}_{1+})$ and $\mathcal{P}_2 : \mathcal{L}^2_{v,q}(\mathbb{R}_{1+}) \to \mathcal{H}$ are two mappings with $\mathcal{P}_1 : \langle \rho_-, y, \rho_+ \rangle \to y$ and $\mathcal{P}_2 : y \to \langle 0, y, 0 \rangle$, respectively. Furthermore, if we assume that $\mathcal{X}(s) = \mathcal{P}_1\mathcal{Y}(s)\mathcal{P}_2$ for $(s \geq 0)$, the family $\{\mathcal{X}(s) = \mathcal{P}_1\mathcal{Y}(s)\mathcal{P}_2\}$ $(s \geq 0)$ of operators is a strongly continuous semigroup of non-unitary contractions on $\mathcal{L}^2_{v,q}(\mathbb{R}_{1+})$. Let us consider the generator of this semigroup given by $M_{\alpha\beta}$ with $M_{\alpha\beta} = \lim_{s\to 1^+}(is)^{-1}(\mathcal{X}(s)y - y)$. The domain of $M_{\alpha\beta}$ consists of all the vectors for which the limit exists. The operator $M_{\alpha\beta}$ is dissipative and the operator $B_{\alpha\beta}$ is called the self-adjoint dilation of $M_{\beta\gamma}$ [31, 32].

Theorem 3.4. *The operator $\mathcal{B}_{\alpha\beta}$ is a self-adjoint dilation of the dissipative operator $Q_{\alpha\beta}$.*

Proof. In order to obtain the result, we need to show $Q_{\alpha\beta} = M_{\alpha\beta}$ and to get this equation we must prove

$$\mathcal{P}_1(\mathcal{B}_{\alpha\beta} - \lambda I)^{-1}\mathcal{P}_2 y = (Q_{\alpha\beta} - \lambda I)^{-1}y, \ y \in H, \ \operatorname{Im}\lambda < 0. \tag{3.13}$$

With this aim, let us assume $(\mathcal{B}_{\alpha\beta} - \lambda I)^{-1}\mathcal{P}_2 y = h = \langle \gamma_-, z, \gamma_+ \rangle$. Then it gives $(\mathcal{B}_{\alpha\beta} - \lambda I)h = \mathcal{P}_2 y$, the equation (3.12) is also equivalent to the equation $Az - \lambda z = y$, and

$$\gamma_-(\xi) = \gamma_-(1)e^{-i\lambda\xi}, \ \gamma_+(\varsigma) = \gamma_+(1)e^{-i\lambda\varsigma}.$$

Since $h \in \mathfrak{D}(\mathcal{B}_{\alpha\beta})$, and hence $\gamma_- \in \mathcal{L}^2(\mathbb{R}_{1-})$. It follows from that $\gamma_-(1) = 0$ and z satisfies the boundary conditions

$$(wD_{q^{-1}}y)(1) - \alpha y(1) = 0, \ [z, \tau]_\infty - [z, \phi]_1 = 0.$$

Therefore, $z \in \mathfrak{D}(Q_{\alpha\beta})$, and since a point λ with $\operatorname{Im}\lambda < 0$ cannot be an eigenvalue of a dissipative operator, we write $z = (Q_{\alpha\beta} - \lambda I)^{-1}y$. Then, we rewrite $\gamma_+(1)$ as $\gamma_+(1) = \delta^{-1}((wD_{q^{-1}})z\,(1) - \overline{\alpha}z(1))$. From this equality, the assumption $(\mathcal{B}_{\alpha\beta} - \lambda I)^{-1}\mathcal{P}_2 y = h = \langle \gamma_-, z, \gamma_+ \rangle$ becomes

$$(\mathcal{B}_{\alpha\beta} - \lambda I)^{-1}\mathcal{P}_2 y = \langle 1, (Q_{\alpha\beta} - \lambda I)^{-1}y, \delta^{-1}[(wD_{q^{-1}})z\,(1) - \overline{\alpha}z(1)]e^{-i\lambda\varsigma} \rangle,$$

for $y \in H$ and $\operatorname{Im}\lambda < 0$. Then, we obtain (3.13) by applying the mapping P_1 to the last equality and by considering (3.13) for $\operatorname{Im}\lambda < 0$, we find

$$(Q_{\alpha\beta} - \lambda I)^{-1} = P_1(\mathcal{B}_{\alpha\beta} - \lambda I)^{-1}\mathcal{P}_2 = -iP_1\int_1^\infty \mathcal{Y}(s)e^{-i\lambda s}ds\mathcal{P}_2$$

$$= -i\int_1^\infty \mathcal{X}(s)e^{-i\lambda s}ds = (M_{\alpha\beta} - \lambda I)^{-1} \ .$$

It follows from that $Q_{\alpha\beta} = M_{\alpha\beta}$ and it completes the proof. □

4. Scattering theory of the dilation, functional model of the dissipative operator and completeness theorems for the dissipative and accumulative operators

In this section, we determine the scattering function of the dilation in terms of the Weyl–Titchmarsh function of the self-adjoint operator according to Lax–Phillips method [27]. Furthermore, we formulate a functional model by using incoming spectral representation and we define its characteristic function using either the Weyl–Titchmarsh function of a self-adjoint q-Sturm–Liouville operator or the scattering matrix of the self-adjoint dilation. Finally, we prove the completeness theorem of the system of eigenfunctions and associated functions (or root functions) of the dissipative and accumulative q-Sturm–Liouville operators. Let us begin with the properties of 'incoming' and 'outgoing' subspaces. This implies that the unitary group $\{\mathcal{Y}(s)\}$ ($s \in \mathbb{R}$) has a crucial characteristic which enables us to apply the Lax–Phillips scheme [27] to it. Let us take $\mathcal{D}^- := \left\langle \mathcal{L}^2(\mathbb{R}_{1-}), 0, 0 \right\rangle$ and $\mathcal{D}^+ := \left\langle 0, 0, \mathcal{L}^2(\mathbb{R}_{1+}) \right\rangle$. The following lemma gives their properties.

Lemma 4.1. *$\mathcal{D}^- := \left\langle \mathcal{L}^2(\mathbb{R}_{1-}), 0, 0 \right\rangle$ and $\mathcal{D}^+ := \left\langle 0, 0, \mathcal{L}^2(\mathbb{R}_{1+}) \right\rangle$ satisfy the following properties:*

(i) $\mathcal{Y}(s)\mathcal{D}^- \subset \mathcal{D}^-$, $s \leq 0$, *and* $\mathcal{Y}(s)\mathcal{D}^+ \subset \mathcal{D}^+$, $s \geq 0$

(ii) $\bigcap\limits_{s\leq 0} \mathcal{Y}(s)\mathcal{D}^- = \bigcap\limits_{s\geq 0} \mathcal{Y}(s)\mathcal{D}^+ = \{0\}$

(iii) $\overline{\bigcup\limits_{s\geq 0} \mathcal{Y}(s)\mathcal{D}^-} = \overline{\bigcup\limits_{s\leq 0} \mathcal{Y}(s)\mathcal{D}^+} = \mathcal{H}$

(iv) $\mathcal{D}^- \perp \mathcal{D}^+$.

Proof. The proof of item (i) is obvious from the definitions. We will demonstrate property (ii) exclusively for $\mathcal{D}^+$ since the proof for $\mathcal{D}^-$ follows a similar approach. Let us define $\mathcal{R}_\lambda = (\mathcal{B}_{\alpha\beta} - \lambda I)^{-1}$, for all λ with $\operatorname{Im}\lambda < 0$, firstly. Then, for any $f = \langle 0, 0, \rho_+ \rangle \in \mathcal{D}^+$, we obtain

$$\mathcal{R}_\lambda f = \langle 0, 0, -ie^{-i\lambda\xi} \int_1^\xi e^{i\lambda s} \rho_+(s) ds \rangle.$$

It implies $\mathcal{R}_\lambda f \in \mathcal{D}^+$. Hence, if $h \perp \mathcal{D}^+$, then it gives

$$(\mathcal{R}_\lambda f, h)_\mathcal{H} = -i \int_1^\infty e^{-i\lambda s} (\mathcal{Y}(s) f, h)_\mathcal{H} ds = 0, \ \ \operatorname{Im}\lambda < 0$$

and it means that for all $s \geq 0$, $(\mathcal{Y}(s)f, h)_\mathcal{H} = 0$. Hence, $\mathcal{Y}(s)\mathcal{D}^+ \subset \mathcal{D}^+$ for $s \geq 0$, and it completes the property (i). To prove property (ii), we write mappings $\mathcal{P}_1^+ : \mathcal{H} \to \mathcal{L}^2(\mathbb{R}_{1+})$ and $\mathcal{P}_2^+ : \mathcal{L}^2(\mathbb{R}_{1+}) \to \mathcal{D}^+$ with the rule $\mathcal{P}_1^+ : \langle \rho_-, y, \rho_+ \rangle \to \rho_+$

and $\mathcal{P}_2^+ : \rho \to \langle 0, 0, \rho\rangle$, respectively. Note that the semigroup of isometries $\mathcal{Y}^+(s) = \mathcal{P}_1^+ \mathcal{Y}(s) \mathcal{P}_2^+, \quad s \geq 0$, is a one-sided shift in $\mathcal{L}^2(\mathbb{R}_{1+})$. Moreover, the generator of the semigroup of the one-sided shift $\mathcal{U}(s)$ in $\mathcal{L}^2(\mathbb{R}_{1+})$ is the differential operator $i\frac{d}{d\xi}$ with the boundary condition $\rho(1) = 0$. On the other hand, the generator B of the semigroup of isometries $\mathcal{Y}^+(s), s \geq 0$, is the operator $B\rho = \mathcal{P}_1^+ \mathcal{B}_{\alpha\beta} \mathcal{P}_2^+ \rho = \mathcal{P}_1^+ \mathcal{B}_{\alpha\beta} \langle 0, 0, \rho\rangle = \mathcal{P}_1^+ \langle 0, 0, i\frac{d\rho}{d\xi}\rangle = i\frac{d\rho}{d\xi}$, where $\rho \in \mathcal{W}_2^1(\mathbb{R}_{1+})$ and $\rho(1) = 0$. Since a semigroup is uniquely determined by its generator, we obtain $\mathcal{Y}^+(s) = \mathcal{U}(s)$, and gives,

$$\bigcap_{s\geq 0} \mathcal{Y}(s)\mathcal{D}^+ = \langle 0, 0, \bigcap_{s\geq 0} \mathcal{U}(s)\mathcal{L}^2(\mathbb{R}_{1+})\rangle = \{0\}.$$

It completes the proof of (ii). It is known that the scattering matrix is defined by using the spectral representation theory with respect to the Lax-Phillips scattering theory. Because of this, we will go on by their constructions and we also give the proof of (iii) of the incoming and outgoing subspaces, but to do this proof, we need auxiliary lemmas. Firstly, let us recall the definition of the *completely non-self-adjoint operator* and then we give these auxiliary lemmas to prove property (iii). □

Definition 4.1. The linear operator T with the domain $D(T)$ acting in a Hilbert space $\mathbf{H}$ is called completely non-self-adjoint or simple if there is no invariant subspace $D(T) \supseteq N$ $(N \neq \{0\})$ of the operator T, *where the restriction of* T to N is self-adjoint [2, 25, 33].

Lemma 4.2. *The dissipative operator $Q_{\alpha\beta}$ is completely non-self-adjoint, that is, simple.*

Proof. Let us suppose that $H^\sim \subset H$ be a non-trivial subspace where $Q_{\alpha\beta}$ has a self-adjoint operator part $Q^\sim_{\alpha\beta}$ with domain

$$\mathfrak{D}(Q^\sim_{\alpha\beta}) = H^\sim \cap \mathfrak{D}(Q_{\alpha\beta}).$$

If $f \in \mathfrak{D}(Q^\sim_{\alpha\beta})$, then $f \in \mathfrak{D}(Q^{\sim *}_{\alpha\beta})$, as well as $(wD_{q^{-1}}y)(1) - \alpha y(1) = 0$, $(wD_{q^{-1}}y)(1) - \overline{\alpha} y(1) = 0$ and $[y, \tau](\infty) - \beta[y, \phi](\infty) = 0$. From this, for the eigenfunctions $y(t, \lambda)$ of the operator $Q_{\alpha\beta}$ that lies in $H^\sim$ and is an eigenvector of $Q^\sim_{\alpha\beta}$, we have $y(1, \lambda) = 0$, $(wD_{q^{-1}})(1, \lambda) = 0$, and then by the uniqueness theorem of the Cauchy problem for the equation $Ay = \lambda y$, $t \in \mathbb{R}_{1+}$, we have $y(t, \lambda) \equiv 0$. Since all solutions of $Ay = \lambda y\,(t \in \mathbb{R}_{1+})$ belong to $\mathcal{L}^2_{w,q}(\mathbb{R}_{1+})$, it can be concluded that the resolvent $\mathcal{R}_\lambda(Q_{\alpha\beta})$ of the operator $Q_{\alpha\beta}$ is a Hilbert-Schmidt operator, and hence the spectrum of $Q_{\alpha\beta}$ is purely discrete. Hence, by the theorem on expansion in eigenvectors of the self-adjoint operator $Q^\sim_{\alpha\beta}$, we write $H^\sim = \{0\}$, it gives that $Q_{\alpha\beta}$ is simple. The lemma is proved. □

Lemma 4.3. *Assume*

$$\mathcal{K}^- = \overline{\bigcup_{s\geq 0} \mathcal{Y}(s)\mathcal{D}^-},\ \mathcal{K}^+ = \overline{\bigcup_{s\leq 0} \mathcal{Y}(s)\mathcal{D}^+}.$$

Then, the equality $\mathcal{K}^- + \mathcal{K}^+ = \mathcal{H}$ *is true.*

Proof. Considering property (i) of Lemma 4.1 for the subspace $\mathcal{D}^+$, it is straightforward to demonstrate that the subspace $\mathcal{H}^\sim = \mathcal{H} \ominus (H + \mathcal{K}^+)$ remains invariant under the action of the group $\{\mathcal{Y}(s)\}$ and takes the form $\mathcal{H}^\sim = \langle 0, H^\sim, 0\rangle$, where $H^\sim$ is a subspace within H. Therefore, if the subspace $\mathcal{H}^\sim$ (and hence, also $H^\sim$) were non-trivial, then the unitary group $\{\mathcal{Y}^\sim(s)\}$, restricted to this subspace, would be a unitary part of the group $\{\mathcal{Y}(s)\}$, and hence the restriction $Q^\sim_{\alpha\beta}$ of $Q_{\alpha\beta}$ to $H^\sim$ would be a self-adjoint operator in $H^\sim$. But, this is a contradiction from the previous lemma, i.e, from the simplicity of the operator $Q_{\alpha\beta}$. As a result of this, we have $\mathcal{H}^\sim = \{0\}$. It completes the proof. □

To present other two lemmas, let us use some notations. We will denote the self-adjoint operator generated from the expression A and the boundary conditions $y(1) = 0$, $[y,\tau](\infty) - \beta[y,\phi](\infty) = 0$ for $y \in \mathfrak{D}_{\max}$ by $Q_{\infty\beta}$ and let $\psi(t,\lambda)$ and $\upsilon(t,\lambda)$ be the solutions of the equation $A(y) = \lambda y$ ($t \in \mathbb{R}_{1+}$) satisfying the conditions $\psi(1,\lambda) = 0$, $(wD_{q^{-1}}\psi)(1,\lambda) = 1$ as well as $\upsilon(1,\lambda) = 1$, $(wD_{q^{-1}}\upsilon)(1,\lambda) = 0$. Moreover, the *Weyl–Titchmarch function* which is represented by $m_{\infty\beta}(\lambda)$ of the self-adjoint operator $Q_{\infty\gamma}$ is determined by the condition $[\upsilon + m_{\infty\beta}\psi, \tau](\infty) - \beta[\upsilon + m_{\infty\beta}\psi, \phi](\infty) = 0$. It follows from that

$$m_{\infty\beta}(\lambda) = -\frac{[\upsilon,\tau](\infty) - \beta[\upsilon,\phi](\infty)}{[\psi,\tau](\infty) - \beta[\psi,\phi](\infty)}. \tag{4.1}$$

It can be said from (4.1) that $m_{\infty\beta}(\lambda)$ is a meromorphic function on the complex plane $\mathbb{C}$ with a countable number of poles lying along the real axis, which align precisely with the eigenvalues of the operator $Q_{\infty\beta}$ [11, 12]. Furthermore, the function $m_{\infty\beta}$ has the following properties:

$$\operatorname{Im}\lambda \operatorname{Im}(m_{\infty\beta})(\lambda) > 0$$

for $\operatorname{Im}\lambda \neq 0$ and $\overline{m_{\infty\beta}(\lambda)} = m_{\infty\beta}(\bar{\lambda})$ for $\lambda \in \mathbb{C}$, except the reel poles of $m_{\infty\beta}(\lambda)$. On the other hand, let us introduce the following notations: $a(t,\lambda) := \upsilon(t,\lambda) + m_{\infty\beta}(\lambda)\psi(t,\lambda)$,

$$\Theta_{\alpha\beta}(\lambda) := \frac{m_{\infty\beta}(\lambda) - \alpha}{m_{\infty\beta}(\lambda) - \overline{\alpha}}. \tag{4.2}$$

If we introduce the vectors

$$\Upsilon^-_\lambda(t,\xi,\varsigma) = \langle e^{-i\lambda\xi}, (m_{\infty\beta}(\lambda) - \alpha)^{-1}\delta a(t,\lambda), \overline{\times}_{\alpha\beta}(\lambda) e^{-i\lambda\varsigma}\rangle,$$

these vectors do not belong to the space $\mathcal{H}$ for real λ. But, $\Upsilon^-_\lambda(t,\xi,\varsigma)$ satisfies the equation $\mathcal{A}\Upsilon = \lambda\Upsilon$ and the corresponding boundary conditions for the operator $\mathcal{B}_{\alpha\beta}$. Using these vectors, we can define transformation $\Psi_- : f \to \tilde{f}_-(\lambda)$ by $(\Psi_- f)(\lambda) := \tilde{f}_-(\lambda) := \frac{1}{\sqrt{2\pi}}(f, \Upsilon^-_\lambda)_{\mathcal{H}}$ on the vector $f = \langle \rho_-, y, \rho_+\rangle$, where ρ_-, ρ_+ and y are smooth, compactly supported functions.

Lemma 4.4. *The isometric transformation Ψ_- maps $\mathcal{K}^-$ onto $\mathcal{L}^2(\mathbb{R})$. For all vectors $f, h \in \mathcal{K}^-$, the Parseval equality and the inversion formula are valid as*

$$(f,h)_{\mathcal{H}} = (\tilde{f}_-, \tilde{h}_-)_{\mathcal{L}^2} = \int_{-\infty}^{\infty} \tilde{f}_-(\lambda)\overline{\tilde{h}_-(\lambda)}d\lambda, \;\; f = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{f}_-(\lambda)\Upsilon_\lambda^- d\lambda,$$

where $\tilde{f}_-(\lambda) = (\Psi_- f)(\lambda)$ and $\tilde{h}_-(\lambda) = (\Psi_- h)(\lambda)$.

Proof. For $f, h \in \mathcal{D}^-$, $f = \langle \rho_-, 0, 0\rangle$, $h = \langle \rho_-, 0, 0\rangle$, write

$$\tilde{f}_-(\lambda) := \frac{1}{\sqrt{2\pi}}(f, \Upsilon_\lambda^-)_{\mathcal{H}} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{1} \rho_-(\xi)e^{i\lambda\xi}d\xi \in H_-^2,$$

and, from Parseval equality for Fourier integrals, it becomes

$$(f,h)_{\mathcal{H}} = \int_{-\infty}^{1} \rho_-(\xi)\overline{\gamma_-}(\xi)d\xi = \int_{-\infty}^{\infty} \tilde{f}_-(\lambda)\overline{\tilde{h}_-(\lambda)}d\lambda = (\Psi_- f, \Psi_- g)_{\mathcal{L}^2}.$$

Now, our aim is to extend the Parseval equality to the whole of $\mathcal{K}^-$. For this purpose, we consider in $\mathcal{K}^-$ the dense set $\mathcal{H}_-^\sim$ of vectors obtained from the smooth, compactly supported functions in $\mathcal{D}^-$: $f \in \mathcal{H}_-^\sim$ if $f = \mathcal{Y}(l)f_0$, $f_0 = \langle \rho_-, 0, 0\rangle$, $\rho_- \in C_0^\infty(\mathbb{R}_{1-})$, where $l = l_f$ is a non-negative number (depending on f). In this case, if $f, h \in \mathcal{K}^-$, then for $l > l_f$ and $l > l_h$, we obtain $\mathcal{Y}(-l)f, \mathcal{Y}(-l)h \in \mathcal{D}^-$ and moreover, the first components of these vectors belong to $C_0^\infty(\mathbb{R}_{1-})$. Therefore, since the operators $\mathcal{Y}(s)$ $(s \in \mathbb{R})$ are unitary, the equality

$$\Psi_-\mathcal{Y}(-l)f = (\mathcal{Y}(-l)f, U_\lambda^-)_{\mathcal{H}} = e^{-i\lambda l}(f, U_\lambda^-)_{\mathcal{H}} = e^{-i\lambda l}\Phi_- f,$$

implies that

$$(f,h)_{\mathcal{H}} = (\mathcal{Y}(-l)f, \mathcal{Y}(-l)h)_{\mathcal{H}} = (\Psi_-\mathcal{Y}(-l)f, \Psi_-\mathcal{Y}(-l)h)_{L^2}$$

$$= (e^{-i\lambda l}\Psi_- f, e^{-i\lambda l}\Psi_- h)_{\mathcal{L}^2} = (\Psi_- f, \Psi_- h)_{\mathcal{L}^2}. \tag{4.3}$$

If we take closure in (4.3), we get the Parseval equality for the whole space $\mathcal{K}^-$. The inversion formula follows from the Parseval equality if all integrals are taken as limits in the mean of integrals over finite intervals. Finally, we find

$$\Psi_-\mathcal{K}^- = \overline{\bigcup_{s\geq 0} \Psi_-\mathcal{Y}(s)\mathcal{D}^-} = \overline{\bigcup_{s\geq 0} e^{-i\lambda s}\mathcal{H}_-^2} = \mathcal{L}^2(\mathbb{R}),$$

it means that Ψ_- maps $\mathcal{K}^-$ onto the whole of $\mathcal{L}^2(\mathbb{R})$. The lemma is proved. $\square$

Similarly, let us set

$$\Upsilon_\lambda^+(t,\xi,\varsigma) = \langle \Theta_{\alpha\beta}(\lambda)e^{-i\lambda\xi}, (m_{\infty\beta}(\lambda)-\overline{\alpha})^{-1}\delta a(t,\lambda), e^{-i\lambda\varsigma}\rangle.$$

Note that the vectors $\Upsilon_\lambda^+(t,\xi,\varsigma)$ for real λ do not belong to the space $\mathcal{H}$. However, $\Upsilon_\lambda^+(t,\xi,\varsigma)$ satisfies the equation $\mathcal{B}\Upsilon = \lambda\Upsilon$ $(\lambda \in \mathbb{R})$ and the boundary conditions (3.8) and (3.9). Using $\Upsilon_\lambda^+(t,\xi,\varsigma)$, we define the transformation $\Psi_+ : f \to \tilde{f}_+(\lambda)$ on vectors $f = \langle \rho_-, y, \rho_+\rangle$, where ρ_-, ρ_+, and y are smooth, compactly supported functions by setting

$$(\Psi_+ f)(\lambda) := \tilde{f}_+(\lambda) := \frac{1}{\sqrt{2\pi}}(f, \Upsilon_\lambda^+)_{\mathcal{H}}.$$

The proof of the next result is analogous to that of Lemma 4.4.

Lemma 4.5. *The transformation Ψ_+ isometrically maps $\mathcal{K}^+$ onto $\mathcal{L}^2(\mathbb{R})$. For all vectors $f, h \in \mathcal{K}^+$, the Parseval equality and the inversion formula hold:*

$$(f,h)_{\mathcal{H}} = (\tilde{f}_+, \tilde{h}_+)_{\mathcal{L}^2} = \int_{-\infty}^{\infty} \tilde{f}_+(\lambda)\overline{\tilde{h}_+(\lambda)}d\lambda,\ f = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{f}_+(\lambda)U_\lambda^+ d\lambda,$$

where $\tilde{f}_+(\lambda) = (\Psi_+ f)(\lambda)$ and $\tilde{h}_+(\lambda) = (\Psi_+ h)(\lambda)$.

For $\lambda \in \mathbb{R}$, the function $\Theta_{\alpha\beta}(\lambda)$ satisfies $|\Theta_{\alpha\beta}(\lambda)| = 1$ from (4.2). Hence, it follows from the explicit formula for the vectors Υ_λ^+ and Υ_λ^- that for $\lambda \in \mathbb{R}$

$$\Upsilon_\lambda^- = \overline{\Theta}_{\alpha\beta}(\lambda)\Upsilon_\lambda^+. \tag{4.4}$$

It follows from Lemmas 4.4 and 4.5 that $\mathcal{K}^- = \mathcal{K}^+$. Together with Lemma 4.3, this gives $\mathcal{H} = \mathcal{K}^- = \mathcal{K}^+$, and property (**iii**) above has been established for the incoming and outgoing subspaces, that is, it completes the proof of item (**iii**) of Lemma 4.1.

Because of these results, we can say that the transformation Ψ_- isometrically maps onto $\mathcal{L}^2(\mathbb{R})$ with the subspace $\mathcal{D}^-$ mapped onto $\mathcal{H}_-^2$ and the operators $\mathcal{Y}(s)$ are transformed into the operators of multiplication by $e^{i\lambda s}$. In other words, Ψ_- is the incoming spectral representation for the group $\{\mathcal{Y}(s)\}$. Similarly, Ψ_+ is the outgoing spectral representation for $\{\mathcal{Y}(s)\}$. It follows from (4.4) that the passage from the Ψ_+-representation of a vector $f \in \mathcal{H}$ to its Ψ_--representation is realized by multiplication of the function $\Theta_{\alpha\beta}(\lambda) : \tilde{f}_-(\lambda) = \Theta_{\alpha\beta}(\lambda)\tilde{f}_+(\lambda)$. With respect to [27], the scattering function (matrix) of the group $\{\mathcal{Y}(s)\}$ according to the subspaces $\mathcal{D}^-$ and $\mathcal{D}^+$, is the coefficient by which the Ψ_--representation of a vector $f \in \mathcal{H}$ must be multiplied in order to get the corresponding Ψ_+-representation: $\tilde{f}_+(\lambda) = \overline{\Theta}_{\alpha\beta}(\lambda)\tilde{f}_-(\lambda)$ and thus we directly give the following theorem.

Theorem 4.1. *The function $\overline{\Theta}_{\alpha\beta}(\lambda)$ is the scattering matrix of the group $\{\mathcal{Y}(s)\}$ (of the self-adjoint operator $\mathcal{B}_{\alpha\beta}$). Let suppose that $\mathbf{N} = \langle 0, H, 0\rangle$, it brings $\mathcal{H} = \mathcal{D}^- \oplus \mathbf{N} \oplus \mathcal{D}^+$. It follows from the explicit form of the unitary transformation Ψ_-*

$$\mathcal{H} \to \mathcal{L}^2(\mathbb{R}),\ f \to \tilde{f}_-(\lambda) = (\Psi_- f)(\lambda),\ \mathcal{D}^- \to \mathcal{H}_-^2,\ \mathcal{D}^+ \to \Theta_{\alpha\beta}\mathcal{H}_+^2, \tag{4.5}$$

$$\mathbf{N} \to \mathcal{H}_+^2 \ominus \Theta_{\alpha\beta}\mathcal{H}_+^2,\ \mathcal{Y}(s)f \to (\Psi_-\mathcal{Y}(s)\Psi_-^{-1}\tilde{f}_-)(\lambda) = e^{i\lambda s}\tilde{f}_-(\lambda). \tag{4.6}$$

The formulae (4.5) and (4.6) show that our operator $Q_{\alpha\beta}$ is a unitary equivalent to the model dissipative operator with the characteristic function $\Theta_{\alpha\beta}(\lambda)$. Since the characteristic functions of unitary equivalent dissipative operators are the same (see [33, 32, 30]), we present the next theorem.

Theorem 4.2. *The characteristic function of the dissipative operator $Q_{\alpha\beta}$ aligns with the function $\Theta_{\alpha\beta}(\lambda)$ defined in (4.2).*

The characteristic function is highly beneficial in determining whether all eigenfunctions and associated functions of a dissipative operator $Q_{\alpha\beta}$ span the whole space or not. This analysis can be conducted by verifying the absence of the singular factor $s(\lambda)$ in the factorization $\Theta_{\alpha\beta}(\lambda) = s(\lambda)B(\lambda)$, where $B(\lambda)$ represents a Blaschke product [5, 3, 33, 32, 30].

Theorem 4.3. *For all values of α with $\operatorname{Im}\alpha > 0$, except possibly for a single value $\beta = \beta_0$, and for fixed β with $\operatorname{Im}\beta = 0$ or $\beta = 0$, the characteristic function $\Theta_{\alpha\beta}$ of the dissipative operator $Q_{\alpha\beta}$ is a Blaschke product. The spectrum of $Q_{\alpha\beta}$ is purely discrete and lies in the open upper half-plane. The operator $Q_{\alpha\beta}$ $(\alpha \neq \alpha_0)$ has a countable number of isolated eigenvalues with finite multiplicities and limit points at infinity. The system of all eigenfunctions and associated functions (or root functions) of the dissipative operator $Q_{\alpha\beta}$ $(\alpha \neq \alpha_0)$ is complete in the space $\mathcal{L}^2_{v,q}(\mathbb{R}_{1+})$.*

Proof. It can be easily seen from (4.2) that $\Theta_{\alpha\beta}$ is an inner function in the upper half-plane and it is meromorphic in the whole λ-plane. We have the factorization

$$\Theta_{\alpha\beta}(\lambda) = e^{i\lambda d} B_{\alpha\beta}(\lambda), \tag{4.7}$$

where $B_{\alpha\beta}(\lambda)$ is the Blaschke product and $d = d(\alpha) \geq 0$. Because of this from (4.7), we write

$$|\Theta_{\alpha\beta}(\lambda)| = \left|e^{i\lambda d}\right| |B_{\alpha\beta}(\lambda)| \leq e^{-d(\alpha)\operatorname{Im}\lambda}, \;\; \operatorname{Im}\lambda \geq 0. \tag{4.8}$$

On the other hand, if we express $m_{\infty\beta}(\lambda)$ in terms of $\Theta_{\alpha\beta}(\lambda)$, we obtain

$$m_{\infty\gamma}(\lambda) = \frac{\overline{\beta}\Theta_{\beta\gamma}(\lambda) - \beta}{\Theta_{\beta\gamma}(\lambda) - 1} \tag{4.9}$$

from (4.4). If $d(\alpha) > 0$ for a given value α $(\operatorname{Im}\alpha > 0)$, then (4.8) gives us that

$$\lim_{s\to+\infty} \Theta_{\alpha\beta}(is) = 0,$$

and then (4.9) leads to

$$\lim_{s\to+\infty} m_{\infty\beta}(is) = -\alpha.$$

Since $m_{\infty\beta}(\lambda)$ is independent of α, $d(\alpha)$ can be non-zero at not more than a single point $\alpha = \alpha_0$ and, further $d(\alpha)$ can be non-zero at not more then a single point $\alpha = \alpha_0$ (and, further, $\alpha_0 = -\lim_{s\to+\infty} m_{\infty\beta}(is)$). Therefore, the proof is completed. □

It is well-known that a linear operator $\mathbf{Q}$ acting in the Hilbert space $\mathbf{H}$ is accumulative if and only if $-\mathbf{Q}$ is dissipative. So that all results concerning dissipative operators can also be valid for accumulative operators. Then, the Theorem 4.3 yields the following result.

Corollary 4.1. *For all values of α with $\operatorname{Im}\alpha < 0$, except possibly for a single value $\alpha = \alpha_1$, and for fixed β with $\operatorname{Im}\beta = 0$ or $\beta = 0$, the characteristic function $\Theta_{\alpha\beta}$ of the accumulative operator $Q_{\alpha\beta}$ is a Blaschke product. The spectrum of $Q_{\alpha\beta}$ is purely discrete and lies in the open lower half-plane. The operator $Q_{\alpha\beta}$ $(\beta \neq \beta_1)$ has a countable number of isolated eigenvalues with finite multiplicities and limit points at infinity. The system of all eigenvectors and associated functions (or root functions) of the accumulative operator $Q_{\alpha\beta}$ $(\alpha \neq \alpha_1)$ is complete in the space $\mathcal{L}^2_{v,q}(\mathbb{R}_{1+})$.*

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