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A UNIQUE EQUIVALENCE CLASS OF NEAR-POINTS OF COINCIDENCE FOR GENERALIZED CONTRACTIONS IN INTERVAL SPACES

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Abstract. In this article, we discuss some theorems of near-points of coincidence for generalized contractive mappings in metric and normed interval spaces that are also weakly compatible self-mappings. In these theorems, we investigate the existence and uniqueness of common near-fixed points. We also provide examples to demonstrate the validity of our work.
Keywords: Metric interval spaces, Normed interval spaces, Near-fixed points, Near-points of coincidence, Null set, Triangle inequality, Contractive mappings.

1. Introduction

Since the development of Banach's contraction principle, it has been widely used in various areas of mathematics and has been extended to different fields. It has been the main tool for identifying fixed points for many years [3]. For the first time, Alber et al. presented the weak contraction principle in Hilbert spaces as a generalization of Banach's contraction principle [1]. Rhoades then extended this principle to metric spaces and obtained good results [10]. For the first time, multivalued mappings and contractions were merged by Nadler [7]. Fréchet presented the concept of a metric space in 1906 [4]. Khan applied the metric space to domain theory problems and computer science [6]. With many efforts, the notion of the metric space has been

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generalized in various ways. In 2018, Wu introduced the metric interval space (MIS) and the normed interval space (NIS) for the collection of closed and bounded intervals $\mathcal{I}$ in $\mathbb{R}$ and defined the notion of near-fixed points for these spaces [14]. In 2020, Ullah and et al. [13] presented the near-coincidence point theorems in metric interval spaces and hyperspaces via a simulation function. In 2021, Sarwar et al. [11] introduced the concept of cone interval b-metric space over Banach algebras. In addition, Sarwar et al. [12] studied some near-fixed point results in MIS and NIS by using α-admissibility and the concept of simulation functions. In 2022, Joshi and Tomar [5] introduced a b-interval metric space to generalize MIS and studied the topological properties in b-interval metric space. The interval space $\mathcal{I}$ cannot be regarded as a vector space because there is no additive inverse member for any non-degenerated closed interval in $\mathcal{I}$. Therefore, the customary normed space $(\mathcal{I}, \|\cdot\|)$ cannot be used for the interval space. Hence, the concept of the NIS has been expressed with the help of the null set. Banach's contraction principle states that a fixed point can be defined for the mapping T provided that $(\mathcal{I}, d)$ is a customary metric space. Nevertheless, for some distance functions such as $d([k,l], [u,v]) = |(k+l)-(u+v)|$, the space $(\mathcal{I}, d)$ does not satisfy the conditions of a metric space. Therefore, the MIS has been introduced based on the null set. The MIS and the NIS proposed by Wu are important because they provide a suitable framework for studying contractive mappings on the interval space [14]. There is no additive inverse member for any non-degenerated closed interval in $\mathcal{I}$. Therefore, $\mathcal{I}$ cannot be a linear space. The algebraic structure of $\mathcal{I}$ is named a quasilinear space (QLS) [9, 16]. The QLS was first stated by Aseev in 1986 [2].

One of the advantages of using bounded and closed intervals is that they can be used to describe uncertain data. In some situations, data collection cannot be done accurately. For example, measuring the level of liquids and the stock price cannot be precisely calculated in short time intervals. In these cases, the level of liquids and the stock price in a short time can be supposed in a bounded closed interval, which indicates uncertainty. Therefore, the interval analysis can be helpful in engineering, economics and social sciences problems to cope with uncertainty [15].

The objective of this article is to establish several common near-fixed points theorems for weakly compatible contractive mappings in the MIS and the NIS and investigate the existence and uniqueness of near-points of coincidence for these mappings. Although several contractive mappings have been presented in conventional metric spaces in some articles [8], these contractive mappings have not yet been studied in interval spaces and are investigated for the first time in this research. In addition, the second theorem of this article has not yet been stated in any space. Therefore, we provide new definitions of different types of contractive mappings in interval spaces and prove some theorems about the existence and uniqueness of near-fixed points for these mappings in the MIS and the NIS. We also provide some examples.

In this paper, in Section 2., we define the interval space and the null set. In Section 3., we introduce the MIS and the NIS and their properties. In Section 4., we define different types of contraction mappings in the MIS and the NIS and prove the near-fixed point theorems for these mappings. Finally, in Section 5., we give a general conclusion for this article.

2. Preliminaries

Suppose $\mathcal{I}$ is the collection of all closed and bounded intervals $[k, l]$ in $\mathbb{R}$, such that $k \leq l$. The addition and scalar multiplication operations on $\mathcal{I}$ are defined as follows:

$$[k, l] \oplus [u, v] = [k + u, l + v] \quad \texttt{and} \quad a\,[k, l] = \begin{cases} [ak, al] & \texttt{if } a \geq 0 \\ [al, ak] & \texttt{if } a < 0. \end{cases}$$

Note that $\mathcal{I}$ is not a customary vector space considering the mentioned two operations, because there is no additive inverse member for any non-degenerated closed interval. Clearly, we have $[0, 0] \in \mathcal{I}$ as a zero member. However, the following subtraction does not give a zero element for some $[k, l] \in \mathcal{I}$,

$$[k, l] \ominus [k, l] = [k, l] \oplus [-l, -k] = [k - l, l - k]\,.$$

The null set is defined by:

$$\Omega = \big\{\, [k, l] \ominus [k, l] : \; [k, l] \in \mathcal{I} \big\}.$$

It is clear that

$$\Omega = \big\{\, [-a, a] : \;\; a \geq 0 \big\} = \big\{ a\,[-1, 1] : \;\; a \geq 0 \big\}.$$

For more details, see [14].

Because some members of $\mathcal{I}$ have not an additive inverse member, $\mathcal{I}$ cannot be a linear space. The algebraic structure of $\mathcal{I}$ is named a quasilinear space (QLS) [9]. The QLS was first stated by Aseev in 1986 [2]. He defined the QLS as follows.

Definition 2.1. [2] A set X is called a quasilinear space (QLS) if a partial order "$\leq$", an algebraic addition operation, and a multiplication operation on real numbers are expressed in it such that the following conditions hold for any elements $x, y, z, v \in X$ and any $\alpha, \beta \in \mathbb{R}$:

(1) $x \leq x$,

(2) $x \leq z$ if $x \leq y$ and $y \leq z$,

(3) $x = y$ if $x \leq y$ and $y \leq x$,

(4) $x + y = y + x$,

(5) $x + (y + z) = (x + y) + z$,

(6) there exists an element $\theta \in X$ such that $x + \theta = x$,

(7) $\alpha \cdot (\beta \cdot x) = (\alpha \cdot \beta) \cdot x$,

(8) $\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$,

(9) $1 \cdot x = x$,

(10) $0 \cdot x = \theta$,

(11) $(\alpha + \beta) \cdot x \leq \alpha \cdot x + \beta \cdot x$,

(12) $x + z \leq y + v$ if $x \leq y$ and $z \leq v$,

(13) $\alpha \cdot x \leq \alpha \cdot y$ if $x \leq y$.

An important example of quasilinear spaces is $\mathcal{I}$, the collection of all closed real intervals, with the inclusion relation " $\subseteq$ ", and with the addition and real-scalar multiplication operations defined as follows:

$$A + B = \{a + b : a \in A, b \in B\} \quad \texttt{and} \quad \lambda \cdot A = \{\lambda a : a \in A\}.$$

This set is denoted by $\Omega_C(\mathbb{R})$. For more details, see [9, 16].

Definition 2.2. [2] Let X be a QLS. A function $\|\cdot\|_X : X \to \mathbb{R}$ is named a norm if the following conditions hold:

(1) $\|x\|_X > 0$ if $x \neq 0$,

(2) $\|x + y\|_X \leq \|x\|_X + \|y\|_X$,

(3) $\|\alpha \cdot x\|_X = |\alpha| \|x\|_X$,

(4) if $x \leq y$, then $\|x\|_X \leq \|y\|_X$,

(5) if for any $\varepsilon > 0$ there exists an element $x_\varepsilon \in X$ such that $x \leq y + x_\varepsilon$ and $\|x_\varepsilon\|_X \leq \varepsilon$ then $x \leq y$.

A quasilinear space X with a norm on it is named a normed quasilinear space (NQLS).

Let X be an NQLS. Hausdorff metric or norm metric on X is defined by the equality

$$h_X(x, y) = \inf\{r \geq 0 : x \leq y + a_1^r, y \leq x + a_2^r, \|a_i^r\| \leq r\}.$$

Remark 2.1. [14] In any interval space, the following facts are valid:

- The distributive law is generally incorrect, i.e.,

$$(\alpha + \beta)\,[k, l] \neq \alpha\,[k, l] \oplus \beta\,[k, l] \quad \texttt{for any} \quad [k, l] \in \mathcal{I} \quad \texttt{and} \quad \alpha, \beta \in \mathbb{R}.$$

  For example, for $\alpha = -5$, $\beta = 3$ and $[k, l] = [-1, 3]$, the equality is incorrect.

- The distributive law is true provided that the scalars are both positive or both negative, i.e.,

$(\alpha + \beta)[k,l] = \alpha[k,l] \oplus \beta[k,l]$ **for any** $[k,l] \in \mathcal{I}$ **and** $\alpha, \beta > 0$ **or** $\alpha, \beta < 0$.

- For any $[k,l], [u,v], [m,n] \in \mathcal{I}$, we have

$$[m,n] \ominus ([k,l] \oplus [u,v]) = [m,n] \oplus (-[k,l]) \oplus (-[u,v]) = [m,n] \ominus [k,l] \ominus [u,v]. \tag{2.1}$$

- We define $[k,l] \overset{\Omega}{=} [u,v]$ if and only if

$$\exists \omega_1, \omega_2 \in \Omega \ \ \texttt{such that} \ \ [k,l] \oplus \omega_1 = [u,v] \oplus \omega_2. \tag{2.2}$$

For example, $[-1,3] \overset{\Omega}{=} [-2,4]$ by choosing $\omega_1 = [-2,2]$ and $\omega_2 = [-1,1]$.
Clearly, $[k,l] = [u,v]$ implies $[k,l] \overset{\Omega}{=} [u,v]$ by choosing $\omega_1 = \omega_2 = [0,0]$. However, the inverse is not usually true. A class based on the relation $\overset{\Omega}{=}$ for any $[k,l] \in \mathcal{I}$, is defined by:

$$\Big\langle [k,l] \Big\rangle = \Big\{ [u,v] \in \mathcal{I} : [k,l] \overset{\Omega}{=} [u,v] \Big\}. \tag{2.3}$$

$\Big\langle \mathcal{I} \Big\rangle$ represents the family of all classes $\Big\langle [k,l] \Big\rangle$ for $[k,l] \in \mathcal{I}$.

Proposition 2.1. *[14] The relation $\overset{\Omega}{=}$ is a reflexive, symmetric and transitive relation, and therefore, $\overset{\Omega}{=}$ will be an equivalence relation.*

This proposition shows that the classes (2.3) provide the equivalence classes. Moreover, $[u,v] \in \langle [k,l] \rangle$ implies that $\langle [k,l] \rangle = \langle [u,v] \rangle$. Note that the quotient collection $\mathcal{I}$ will not necessarily be a customary vector space(see [14]).

3. Metric and normed interval spaces

Definition 3.1. [14] Suppose that $\mathcal{I}$ is the collection of all bounded and closed intervals in $\mathbb{R}$ and $d : \mathcal{I} \times \mathcal{I} \to \mathbb{R}^+$ is a mapping. A pair $(\mathcal{I}, d)$ is named a metric interval space (MIS) if the following three conditions hold for d:
(i) $d([k,l], [u,v]) = 0$ if and only if $[k,l] \overset{\Omega}{=} [u,v]$ for all $[k,l], [u,v] \in \mathcal{I}$;
(ii) $d([k,l], [u,v]) = d([u,v], [k,l])$ for all $[k,l], [u,v] \in \mathcal{I}$;
(iii) $d([k,l], [u,v]) \leq d([k,l], [m,n]) + d([m,n], [u,v])$ for all $[k,l], [u,v], [m,n] \in \mathcal{I}$.
We have a pseudo-metric interval space $(\mathcal{I}, d)$ if conditions (ii) and (iii) hold for d. We say that the null equality hold for d if it satisfies the following equality for any $\omega_1, \omega_2 \in \Omega$ and $[k,l], [u,v] \in \mathcal{I}$:

$$d([k,l] \oplus \omega_1, [u,v] \oplus \omega_2) = d([k,l], [u,v]),$$

and specifically

- $d([k,l] \oplus \omega_1, [u,v]) = d([k,l], [u,v])$,
- $d([k,l], [u,v] \oplus \omega_2) = d([k,l], [u,v])$.

Definition 3.2. [14] The sequence $\{ [k_n, l_n] \}_{n=1}^{\infty}$ is named a convergent sequence in pseudo-metric interval space $(\mathcal{I}, d)$ if and only if $\lim_{n\to\infty} d([k_n, l_n], [k,l]) = 0$ for some $[k,l] \in \mathcal{I}$. In fact, the member $[k,l]$ is the limit of sequence $\{ [k_n, l_n] \}_{n=1}^{\infty}$.

Proposition 3.1. *[14] If the sequence $\{ [k_n, l_n] \}_{n=1}^{\infty}$ exists in $\mathcal{I}$ such that the null equality holds for d and $\lim_{n\to\infty} d([k_n, l_n], [k,l]) = 0$, then $\lim_{n\to\infty} d([k_n, l_n], [u,v]) = 0$ for any $[u,v] \in \langle [k,l] \rangle$.*

Definition 3.3. [14] Suppose that the sequence $\{ [k_n, l_n] \}_{n=1}^{\infty}$ exists in $\mathcal{I}$ such that $\lim_{n\to\infty} d([k_n, l_n], [k,l]) = 0$ for some $[k,l] \in \mathcal{I}$. Then the class $\langle [k,l] \rangle$ is named the class limit of $\{ [k_n, l_n] \}_{n=1}^{\infty}$. In addition, the uniqueness of the class limit in an MIS is easily obtained (see[14]).

Definition 3.4. [14] Let $\{ [k_n, l_n] \}_{n=1}^{\infty}$ be a sequence in the MIS $(\mathcal{I}, d)$ such that

$$\forall \varepsilon > 0, \ \exists N \in \mathbb{N} \ \texttt{such that} \ d([k_n, l_n], [k_m, l_m]) < \varepsilon, \ \ \forall n, m > N.$$

Then $\{ [k_n, l_n] \}_{n=1}^{\infty}$ is called a Cauchy sequence in $\mathcal{I}$. Moreover, a subset $\mathcal{W}$ of $\mathcal{I}$ in the MIS $(\mathcal{I}, d)$ is complete if and only if every Cauchy sequence $\{ [k_n, l_n] \}_{n=1}^{\infty}$ in $\mathcal{W}$ converges to an element $[k,l] \in \mathcal{W}$.

Example 3.1. [14] Assume that $d : \mathcal{I} \times \mathcal{I} \to \mathbb{R}^+$ is defined by $d([k,l], [u,v]) = |(k+l) - (u+v)|$. Then, it is obvious that $(\mathcal{I}, d)$ is a complete metric interval space (CMIS) and the null equality holds for d.

Definition 3.5. A subset $\mathcal{Y}$ of $\mathcal{I}$ in a complete metric interval space $(\mathcal{I}, d)$ is closed if the limit of every sequence $\{ [k_n, l_n] \}_{n=1}^{\infty}$ in $\mathcal{Y}$ is an element of $\mathcal{Y}$.

Definition 3.6. [14] For a mapping $\| \cdot \| : \mathcal{I} \to \mathbb{R}^+$ that $\mathbb{R}^+$ is nonnegative real numbers, we present the following features:
(i) $\|\lambda [k,l] \| = |\lambda| \cdot \| [k,l] \|$, $\forall [k,l] \in \mathcal{I}$ and $\lambda \in \mathbb{F}$;
(i°) $\|\lambda [k,l] \| = |\lambda| \cdot \| [k,l] \|$, $\forall [k,l] \in \mathcal{I}$ and $\lambda \in \mathbb{F}$ with $\lambda \neq 0$;
(ii) $\| [k,l] \oplus [u,v] \| \leq \| [k,l] \| + \| [u,v] \|$, $\forall [k,l], [u,v] \in \mathcal{I}$;
(iii) $\| [k,l] \| = 0$ implies $[k,l] \in \Omega$.

- It is said that $(\mathcal{I}, \| \cdot \|)$ is a pseudo-seminormed interval space if it satisfies Cases (i°) and (ii).
- It is said that $(\mathcal{I}, \| \cdot \|)$ is a normed interval space (NIS) if it satisfies Cases (i), (ii) and (iii).

- It is said that the null condition holds for $\|\cdot\|$ if item (iii) is changed to $\|[k,l]\| = 0 \Leftrightarrow [k,l] \in \Omega$.
- It is said that the null super-inequality holds for $\|\cdot\|$ if $\|[k,l] \oplus \omega\| \geq \|[k,l]\|$ for all $[k,l] \in \mathcal{I}$ and $\omega \in \Omega$.
- It is said that the null equality holds for $\|\cdot\|$ if $\|[k,l] \oplus \omega\| = \|[k,l]\|$ for all $[k,l] \in \mathcal{I}$ and $\omega \in \Omega$.

A complete normed interval space is named a Banach interval space (BIS).

Example 3.2. [14] Assume that the mapping $\|\cdot\| : \mathcal{I} \to \mathbb{R}^+$ is defined by $\|[k,l]\| = |k+l|$. Then, it is obvious that $(\mathcal{I}, \|\cdot\|)$ is a BIS and the null equality holds for $\|\cdot\|$.

For more details about normed interval spaces, see [14].

4. Near-fixed point results

Definition 4.1. [14] A point $[k,l] \in \mathcal{I}$ is named a near-fixed point for a self-mapping $T : \mathcal{I} \to \mathcal{I}$ if $T[k,l] \stackrel{\Omega}{=} [k,l]$.

According to the definition, we have $T[k,l] \stackrel{\Omega}{=} [k,l]$ if and only if there exist $[-b_1, b_1], [-b_2, b_2] \in \Omega$ where $b_1, b_2 \in \mathbb{R}^+$ so that at least one of the following equalities holds:

- $T[k,l] \oplus [-b_1, b_1] = [k,l]$;
- $T[k,l] = [k,l] \oplus [-b_1, b_1]$;
- $T[k,l] \oplus [-b_1, b_1] = [k,l] \oplus [-b_2, b_2]$.

Definition 4.2. [11] A point $[k,l] \in \mathcal{I}$ is a common near-fixed point for functions $T, S : \mathcal{I} \to \mathcal{I}$ if $T[k,l] \stackrel{\Omega}{=} S[k,l] \stackrel{\Omega}{=} [k,l]$.

Example 4.1. Assume that $T, S : \mathcal{I} \to \mathcal{I}$ are defined by

$$T[k,l] = [k-3, l+3] \quad \text{and} \quad S[k,l] = [k-4, l+4].$$

We indicate that $[k,l]$ is a common near-fixed point of T and S. For $\omega_1 = [0,0]$ and $\omega_2 = [-3,3] \in \Omega$, we have $T[k,l] \stackrel{\Omega}{=} [k,l]$, i.e.,

$$\begin{aligned} [k-3, l+3] \stackrel{\Omega}{=} [k,l] \quad &\Longleftrightarrow \quad [k-3, l+3] \oplus [0,0] = [k,l] \oplus [-3,3] \\ &\Longleftrightarrow \quad [k-3, l+3] = [k-3, l+3]. \end{aligned}$$

Similarly, for $\omega_1 = [0,0]$ and $\omega_2 = [-4,4] \in \Omega$, we have $S[k,l] \stackrel{\Omega}{=} [k,l]$. According to Proposition 2.1, because $\stackrel{\Omega}{=}$ is an equivalence relation, then $T[k,l] \stackrel{\Omega}{=} S[k,l]$. Hence $T[k,l] \stackrel{\Omega}{=} S[k,l] \stackrel{\Omega}{=} [k,l]$.

Definition 4.3. [11] Suppose that $T_1, T_2 : \mathcal{I} \to \mathcal{I}$ are two mappings on $\mathcal{I}$. If $[u,v] \overset{\Omega}{=} T_1[k,l] \overset{\Omega}{=} T_2[k,l]$ for $[k,l], [u,v] \in \mathcal{I}$, then $[u,v]$ is named a near-point of coincidence of T_1 and T_2, and $[k,l]$ is named a near-coincidence point of T_1 and T_2.

Definition 4.4. [11] Suppose that $T_1, T_2 : \mathcal{I} \to \mathcal{I}$ are two mappings on $\mathcal{I}$. If $T_1T_2[k,l] \overset{\Omega}{=} T_2T_1[k,l]$, whenever $T_1[k,l] \overset{\Omega}{=} T_2[k,l]$ for $[k,l] \in \mathcal{I}$, then the pair $\{T_1, T_2\}$ is called weakly compatible.

Lemma 4.1. *Assume that* $\lim_{n\to\infty} d([k_n, l_n], [k,l]) = 0$ *in the MIS* $(\mathcal{I}, d)$. *Then for each* $[u,v] \in \mathcal{I}$, *we have* $\lim_{n\to\infty} d([k_n, l_n], [u,v]) = d([k,l], [u,v])$.

Proof. Due to the triangle inequality, we obtain

$$d([k_n, l_n], [u,v]) \leq d([k_n, l_n], [k,l]) + d([k,l], [u,v])$$

and

$$d([k,l], [u,v]) \leq d([k,l], [k_n, l_n]) + d([k_n, l_n], [u,v]).$$

So we obtain

$$0 \leq |d([k_n, l_n], [u,v]) - d([k,l], [u,v])| \leq d([k_n, l_n], [k,l]).$$

Taking $n \to \infty$ and using the assumption of Lemma, we have $\lim_{n\to\infty} d([k_n, l_n], [u,v]) = d([k,l], [u,v])$. $\square$

Lemma 4.2. *[11] Suppose that* S *and* T *are weakly compatible self-mappings of* $\mathcal{I}$. *If* $\langle [u,v] \rangle$ *is a unique class of near-points of coincidence for* S *and* T, *then* $\langle [u,v] \rangle$ *is a unique class of common near-fixed points for* S *and* T.

Theorem 4.1. *Suppose that* $(\mathcal{I}, d)$ *is a CMIS and the null equality holds for* d. *Assume that* $F, G, T, S : (\mathcal{I}, d) \to (\mathcal{I}, d)$ *are self-mappings of* $\mathcal{I}$ *such that* $F\mathcal{I} \subseteq T\mathcal{I}$, $G\mathcal{I} \subseteq S\mathcal{I}$ *and*

$$d(F[k,l], G[u,v]) \leq \phi\big(M([k,l], [u,v])\big), \tag{4.1}$$

for all $[k,l], [u,v] \in \mathcal{I}$, *where a function* $\phi : [0,\infty) \to [0,\infty)$ *is continuous and nondecreasing with* $\phi(t) < t$ *for each* $t > 0$ *and*

$$\begin{aligned} M([k,l],[u,v]) &= \max\big\{d(S[k,l], T[u,v]), d(F[k,l], S[k,l]), d(G[u,v], T[u,v]), \\ &\quad \frac{d(F[k,l], T[u,v]) + d(G[u,v], S[k,l])}{2}\big\}. \end{aligned}$$

If one of the sets $F\mathcal{I}$, $G\mathcal{I}$, $S\mathcal{I}$ *and* $T\mathcal{I}$ *is a closed subset of* $(\mathcal{I}, d)$, *then*
(i) F *and* S *have a unique equivalence class of near-points of coincidence.*
(ii) G *and* T *have a unique equivalence class of near-points of coincidence.*
In addition, if the pairs $\{F, S\}$ *and* $\{G, T\}$ *are weakly compatible self-mappings, then a unique class of common near-fixed points exists for* F, G, T *and* S.

Proof. Let $[k_0, l_0] \in \mathcal{I}$ be an arbitrary element. Since $F\mathcal{I} \subseteq T\mathcal{I}$, there exists $[k_1, l_1] \in \mathcal{I}$ such that $T[k_1, l_1] = F[k_0, l_0]$. Since $G\mathcal{I} \subseteq S\mathcal{I}$, there exists $[k_2, l_2] \in \mathcal{I}$ such that $S[k_2, l_2] = G[k_1, l_1]$. By continuing this process, we define sequences $\{[k_n, l_n]\}_{n=1}^{\infty}$ and $\{[u_n, v_n]\}_{n=1}^{\infty}$ in $\mathcal{I}$ such that

$$\begin{aligned} &[u_{2n}, v_{2n}] = T[k_{2n+1}, l_{2n+1}] = F[k_{2n}, l_{2n}], \\ &[u_{2n+1}, v_{2n+1}] = S[k_{2n+2}, l_{2n+2}] = G[k_{2n+1}, l_{2n+1}] \ \forall n \in \mathbb{N}. \end{aligned} \tag{4.2}$$

we also have

$$[u_{2n-1}, v_{2n-1}] = S[k_{2n}, l_{2n}] = G[k_{2n-1}, l_{2n-1}] \ \forall n \in \mathbb{N}. \tag{4.3}$$

Now we show that $\{[u_n, v_n]\}_{n=1}^{\infty}$ in $\mathcal{I}$ is a Cauchy sequence in $(\mathcal{I}, d)$. According to the triangle inequality, we get

$$\begin{aligned} d([u_{2q-1}, v_{2q-1}], [u_{2q+1}, v_{2q+1}]) &\leq d([u_{2q-1}, v_{2q-1}], [u_{2q}, v_{2q}]) \\ &\quad + d([u_{2q}, v_{2q}], [u_{2q+1}, v_{2q+1}]) \\ &\leq 2\max\big\{d([u_{2q-1}, v_{2q-1}], [u_{2q}, v_{2q}]), \\ &\quad d([u_{2q}, v_{2q}], [u_{2q+1}, v_{2q+1}])\big\}, \end{aligned} \tag{4.4}$$

Thus from (4.2), (4.3) and (4.4), we obtain

$$\begin{aligned} &M([k_{2q}, l_{2q}], [k_{2q+1}, l_{2q+1}]) = \max\big\{d(S[k_{2q}, l_{2q}], T[k_{2q+1}, l_{2q+1}]), \\ &\quad d(F[k_{2q}, l_{2q}], S[k_{2q}, l_{2q}]), d(G[k_{2q+1}, l_{2q+1}], T[k_{2q+1}, l_{2q+1}]), \\ &\quad \frac{d(S[k_{2q}, l_{2q}], G[k_{2q+1}, l_{2q+1}]) + d(F[k_{2q}, l_{2q}], T[k_{2q+1}, l_{2q+1}])}{2}\big\} \\ = &\max\big\{d([u_{2q-1}, v_{2q-1}], [u_{2q}, v_{2q}]), d([u_{2q}, v_{2q}], [u_{2q-1}, v_{2q-1}]), \\ &\quad d([u_{2q+1}, v_{2q+1}], [u_{2q}, v_{2q}]), \\ &\quad \frac{d([u_{2q-1}, v_{2q-1}], [u_{2q+1}, v_{2q+1}]) + d([u_{2q}, v_{2q}], [u_{2q}, v_{2q}])}{2}\big\} \\ \leq &\max\big\{d([u_{2q-1}, v_{2q-1}], [u_{2q}, v_{2q}]), d([u_{2q}, v_{2q}], [u_{2q+1}, v_{2q+1}])\big\}. \end{aligned}$$

Since ϕ is a non-decreasing function, we obtain

$$\begin{aligned} \phi\big(M([k_{2q}, l_{2q}], [k_{2q+1}, l_{2q+1}])\big) &\leq \phi\big(\max\big\{d([u_{2q-1}, v_{2q-1}], [u_{2q}, v_{2q}]), \\ &\quad d([u_{2q}, v_{2q}], [u_{2q+1}, v_{2q+1}])\big\}\big). \end{aligned}$$

Therefore, putting $[k, l] = [k_{2q}, l_{2q}]$ and $[u, v] = [k_{2q+1}, l_{2q+1}]$ in (4.1) and using the above inequality, we get

$$\begin{aligned} d([u_{2q}, v_{2q}], [u_{2q+1}, v_{2q+1}]) &\leq \phi\big(\max\big\{d([u_{2q-1}, v_{2q-1}], [u_{2q}, v_{2q}]), \\ &\quad d([u_{2q}, v_{2q}], [u_{2q+1}, v_{2q+1}])\big\}\big). \end{aligned} \tag{4.5}$$

Similarly, we have

$$\begin{aligned} d([u_{2q+1}, v_{2q+1}], [u_{2q+2}, v_{2q+2}]) &\leq \phi\big(\max\big\{d([u_{2q}, v_{2q}], [u_{2q+1}, v_{2q+1}]), \\ &\quad d([u_{2q+1}, v_{2q+1}], [u_{2q+2}, v_{2q+2}])\big\}\big). \end{aligned} \tag{4.6}$$

Therefore, from (4.5) and (4.6) for all $n \geq 1$, we have

$$d([u_n, v_n], [u_{n+1}, v_{n+1}]) \leq \phi\big(\max\{d([u_{n-1}, v_{n-1}], [u_n, v_n]), d([u_n, v_n], [u_{n+1}, v_{n+1}])\}\big). \tag{4.7}$$

Assume that there exists $q \in \mathbb{N}$ such that $d([u_{2q-1}, v_{2q-1}], [u_{2q}, v_{2q}]) = 0$. Then $[u_{2q-1}, v_{2q-1}] \stackrel{\Omega}{=} [u_{2q}, v_{2q}]$ and from (4.5), we have

$$d([u_{2q}, v_{2q}], [u_{2q+1}, v_{2q+1}]) \leq \phi\big(d([u_{2q}, v_{2q}], [u_{2q+1}, v_{2q+1}])\big).$$

According to $\phi(t) < t$ for each $t > 0$, the above inequality results that

$$\begin{aligned} d([u_{2q}, v_{2q}], [u_{2q+1}, v_{2q+1}]) &\leq \phi\big(d([u_{2q}, v_{2q}], [u_{2q+1}, v_{2q+1}])\big) \\ &< d([u_{2q}, v_{2q}], [u_{2q+1}, v_{2q+1}]). \end{aligned}$$

which implies that $d([u_{2q}, v_{2q}], [u_{2q+1}, v_{2q+1}]) = 0$. Hence $[u_{2q}, v_{2q}] \stackrel{\Omega}{=} [u_{2q+1}, v_{2q+1}]$. Thus from (4.6), we obtain

$$d([u_{2q+1}, v_{2q+1}], [u_{2q+2}, v_{2q+2}]) \leq \phi\big(d([u_{2q+1}, v_{2q+1}], [u_{2q+2}, v_{2q+2}])\big),$$

which follows that $[u_{2q+1}, v_{2q+1}] \stackrel{\Omega}{=} [u_{2q+2}, v_{2q+2}]$. Therefore, we have

$$[u_{2q-1}, v_{2q-1}] \stackrel{\Omega}{=} [u_{2q}, v_{2q}] \stackrel{\Omega}{=} [u_{2q+1}, v_{2q+1}] \stackrel{\Omega}{=} [u_{2q+2}, v_{2q+2}] \stackrel{\Omega}{=} \cdots$$

If we suppose that there exists $q \in \mathbb{N}$ such that $d([u_{2q}, v_{2q}], [u_{2q+1}, v_{2q+1}]) = 0$, then the similar result holds. Now we assume that

$$d([u_n, v_n], [u_{n+1}, v_{n+1}]) > 0, \ \forall n \in \mathbb{N}. \tag{4.8}$$

Since $\phi(t) < t$ for each $t > 0$, (4.7) implies that

$$d([u_n, v_n], [u_{n+1}, v_{n+1}]) < \max\{d([u_{n-1}, v_{n-1}], [u_n, v_n]), d([u_n, v_n], [u_{n+1}, v_{n+1}])\}.$$

Then we have $d([u_n, v_n], [u_{n+1}, v_{n+1}]) < d([u_{n-1}, v_{n-1}], [u_n, v_n])$. Therefore

$$\max\{d([u_{n-1}, v_{n-1}], [u_n, v_n]), d([u_n, v_n], [u_{n+1}, v_{n+1}])\} = d([u_{n-1}, v_{n-1}], [u_n, v_n]).$$

Hence from (4.7), we obtain

$$d([u_n, v_n], [u_{n+1}, v_{n+1}]) \leq \phi\big(d([u_{n-1}, v_{n-1}], [u_n, v_n])\big) \quad \forall n \geq 1. \tag{4.9}$$

By repeating the above inequality n times, we have

$$d([u_n, v_n], [u_{n+1}, v_{n+1}]) \leq \phi^n\big(d([u_0, v_0], [u_1, v_1])\big). \tag{4.10}$$

This shows that

$$\lim_{n\to\infty} d([u_n, v_n], [u_{n+1}, v_{n+1}]) = 0. \tag{4.11}$$

Now using (4.10) and the triangle inequality, we have

$$\begin{aligned} d([u_n,v_n],[u_{n+m},v_{n+m}]) &\leq d([u_n,v_n],[u_{n+1},v_{n+1}]) \\ &\quad + \cdots + d([u_{n+m-1},v_{n+m-1}],[u_{n+m},v_{n+m}]) \\ &\leq \phi^n\big(d([u_0,v_0],[u_1,v_1])\big) \\ &\quad + \cdots + \phi^{n+m-1}\big(d([u_0,v_0],[u_1,v_1])\big). \end{aligned}$$

Thus $d([u_n,v_n],[u_{n+m},v_{n+m}]) \to 0$, as $n \to \infty$. Therefore, $\{[u_n,v_n]\}_{n=1}^{\infty}$ is a Cauchy sequence in the MIS $(\mathcal{I},d)$. Due to the completeness of the MIS $(\mathcal{I},d)$, there is $[u,v] \in \mathcal{I}$ such that

$$d([u_n,v_n],[u,v]) \to 0. \tag{4.12}$$

Since the null equality holds for d, according to Proposition 3.1 and (4.12), we have

$$\lim_{n\to\infty} d([u_n,v_n],[\tilde{u},\tilde{v}]) = 0, \quad \texttt{for any } [\tilde{u},\tilde{v}] \in \big\langle [u,v] \big\rangle. \tag{4.13}$$

Thus, (4.13) implies that

$$\lim_{n\to\infty} d([u_{2n},v_{2n}],[\tilde{u},\tilde{v}]) = \lim_{n\to\infty} d([u_{2n-1},v_{2n-1}],[\tilde{u},\tilde{v}]) = 0. \tag{4.14}$$

So from (4.2), (4.3) and (4.14), we obtain

$$\begin{aligned} &\lim_{n\to\infty} d(T[k_{2n+1},l_{2n+1}],[\tilde{u},\tilde{v}]) = \lim_{n\to\infty} d(F[k_{2n},l_{2n}],[\tilde{u},\tilde{v}]) = 0, \\ &\lim_{n\to\infty} d(S[k_{2n},l_{2n}],[\tilde{u},\tilde{v}]) = \lim_{n\to\infty} d(G[k_{2n-1},l_{2n-1}],[\tilde{u},\tilde{v}]) = 0. \end{aligned} \tag{4.15}$$

Now without loss of generality, we suppose that $S\mathcal{I}$ is a closed subset of the MIS $(\mathcal{I},d)$. Then duo to (4.15), there is $[a,b] \in \mathcal{I}$ such that $[\tilde{u},\tilde{v}] \overset{\Omega}{=} S[a,b]$, i.e.,

$$[\tilde{u},\tilde{v}] \oplus \omega_1 = S[a,b] \oplus \omega_2, \quad \texttt{for some } \omega_1,\omega_2 \in \Omega. \tag{4.16}$$

Now we show that $d(F[a,b],[\tilde{u},\tilde{v}]) = 0$. Assume that $d(F[a,b],[\tilde{u},\tilde{v}]) > 0$. From (4.1), (4.2) and the triangle inequality, we get

$$\begin{aligned} d(F[a,b],[\tilde{u},\tilde{v}]) &\leq d(F[a,b],G[k_{2n+1},l_{2n+1}]) + d(G[k_{2n+1},l_{2n+1}],[\tilde{u},\tilde{v}]) \\ &\leq \phi\big(M([a,b],[k_{2n+1},l_{2n+1}])\big) + d([u_{2n+1},v_{2n+1}],[\tilde{u},\tilde{v}]). \end{aligned} \tag{4.17}$$

Using (4.2) and (4.16) and and making use of the triangle inequality and the null equality, we obtain

$$\begin{aligned} M([a,b],[k_{2n+1},l_{2n+1}]) &= \max\big\{d(S[a,b],T[k_{2n+1},l_{2n+1}]), \\ &\quad d(F[a,b],S[a,b]), d(G[k_{2n+1},l_{2n+1}],T[k_{2n+1},l_{2n+1}]), \\ &\quad \frac{d(S[a,b],G[k_{2n+1},l_{2n+1}]) + d(F[a,b],T[k_{2n+1},l_{2n+1}])}{2}\big\} \\ &= \max\big\{d(S[a,b]\oplus\omega_2,T[k_{2n+1},l_{2n+1}]), \end{aligned}$$

$$
\begin{aligned}
& d(F[a,b], S[a,b] \oplus \omega_2), d(G[k_{2n+1}, l_{2n+1}], T[k_{2n+1}, l_{2n+1}]), \\
& \frac{d(S[a,b] \oplus \omega_2, G[k_{2n+1}, l_{2n+1}]) + d(F[a,b], T[k_{2n+1}, l_{2n+1}])}{2}\Big\} \\
= {} & \max\Big\{ d([\tilde{u},\tilde{v}] \oplus \omega_1, [u_{2n}, v_{2n}]), d(F[a,b], [\tilde{u},\tilde{v}] \oplus \omega_1), \\
& d([u_{2n+1}, v_{2n+1}], [u_{2n}, v_{2n}]), \\
& \frac{d([\tilde{u},\tilde{v}] \oplus \omega_1, [u_{2n+1}, v_{2n+1}]) + d(F[a,b], [u_{2n}, v_{2n}])}{2}\Big\} \\
\leq {} & \max\Big\{ d([\tilde{u},\tilde{v}], [u_{2n}, v_{2n}]), d(F[a,b], [\tilde{u},\tilde{v}]), d([u_{2n+1}, v_{2n+1}], [u_{2n}, v_{2n}]), \\
& \frac{d([\tilde{u},\tilde{v}], [u_{2n+1}, v_{2n+1}]) + d(F[a,b], [\tilde{u},\tilde{v}]) + d([\tilde{u},\tilde{v}], [u_{2n}, v_{2n}])}{2}\Big\}, \qquad (4.18)
\end{aligned}
$$

On the other hand, we have

$$d(F[a,b], [\tilde{u},\tilde{v}]) \leq M([a,b], [k_{2n+1}, l_{2n+1}]). \qquad (4.19)$$

Therefore, by letting $n \to \infty$ in (4.18) and (4.19) and using of (4.11) and (4.14), we obtain

$$\lim_{n\to\infty} M([a,b], [k_{2n+1}, l_{2n+1}]) = d(F[a,b], [\tilde{u},\tilde{v}]). \qquad (4.20)$$

Thus taking $n \to \infty$ in (4.17) and using of (4.14), (4.20) and the continuity of ϕ, we have

$$
\begin{aligned}
d(F[a,b], [\tilde{u},\tilde{v}]) &\leq \lim_{n\to\infty} \phi\big(M([a,b], [k_{2n+1}, l_{2n+1}])\big) + \lim_{n\to\infty} d([u_{2n+1}, v_{2n+1}], [\tilde{u},\tilde{v}]) \\
&= \phi\big(d(F[a,b], [\tilde{u},\tilde{v}])\big).
\end{aligned}
$$

Since $\phi(t) < t$ for each $t > 0$, we have

$$d(F[a,b], [\tilde{u},\tilde{v}]) \leq \phi\big(d(F[a,b], [\tilde{u},\tilde{v}])\big) < d(F[a,b], [\tilde{u},\tilde{v}]),$$

which is a contradiction. Thus $d(F[a,b], [\tilde{u},\tilde{v}]) = 0$ and $F[a,b] \stackrel{\Omega}{=} [\tilde{u},\tilde{v}]$, i.e.,

$$F[a,b] \oplus \omega_3 = [\tilde{u},\tilde{v}] \oplus \omega_4, \quad \texttt{for some } \omega_3, \omega_4 \in \Omega. \qquad (4.21)$$

Due to $[\tilde{u},\tilde{v}] \stackrel{\Omega}{=} S[a,b]$, Proposition 2.1 and (4.21), we have

$$[\tilde{u},\tilde{v}] \stackrel{\Omega}{=} F[a,b] \stackrel{\Omega}{=} S[a,b], \quad \texttt{for all } [\tilde{u},\tilde{v}] \in \big\langle [u,v] \big\rangle. \qquad (4.22)$$

Thus, any $[\tilde{u},\tilde{v}] \in \langle [u,v] \rangle$ is a near-point of coincidence for F and S. Now we show that $\langle [u,v] \rangle$ is a unique class of near-points of coincidence for F and S. Assume that $\langle [\bar{u},\bar{v}] \rangle$ is another class of near-points of coincidence for F and S such that $\langle [\bar{u},\bar{v}] \rangle \neq \langle [u,v] \rangle$. Then, we have $[\bar{u},\bar{v}] \notin \langle [u,v] \rangle$, $[\bar{u},\bar{v}] \stackrel{\Omega}{=} F[\bar{a},\bar{b}] \stackrel{\Omega}{=} S[\bar{a},\bar{b}]$ and $d([\tilde{u},\tilde{v}], [\bar{u},\bar{v}]) > 0$ for all $[\tilde{u},\tilde{v}] \in \langle [u,v] \rangle$. In other words, there exist $\varpi_1, \varpi_2, \varpi_3 \in \Omega$ such that

$$[\bar{u},\bar{v}] \oplus \varpi_1 = F[\bar{a},\bar{b}] \oplus \varpi_2 = S[\bar{a},\bar{b}] \oplus \varpi_3. \qquad (4.23)$$

From (4.1), (4.2), (4.23) and the triangle inequality, we have

$$\begin{aligned} d([\bar{u},\bar{v}],[\tilde{u},\tilde{v}]) &= d([\bar{u},\bar{v}]\oplus\varpi_1,[\tilde{u},\tilde{v}]) = d(F[\bar{a},\bar{b}]\oplus\varpi_2,[\tilde{u},\tilde{v}]) = d(F[\bar{a},\bar{b}],[\tilde{u},\tilde{v}]) \\ &\leq d(F[\bar{a},\bar{b}],G[k_{2n+1},l_{2n+1}]) + d(G[k_{2n+1},l_{2n+1}],[\tilde{u},\tilde{v}]) \\ &\leq \phi\big(M([\bar{a},\bar{b}],[k_{2n+1},l_{2n+1}])\big) + d(G[k_{2n+1},l_{2n+1}],[\tilde{u},\tilde{v}]) \\ &= \phi\big(M([\bar{a},\bar{b}],[k_{2n+1},l_{2n+1}])\big) + d([u_{2n+1},v_{2n+1}],[\tilde{u},\tilde{v}]). \end{aligned} \tag{4.24}$$

From (4.2), (4.23) and the null equality, we obtain

$$\begin{aligned} &M([\bar{a},\bar{b}],[k_{2n+1},l_{2n+1}]) = \max\Big\{d(S[\bar{a},\bar{b}],T[k_{2n+1},l_{2n+1}]), \\ &\quad d(F[\bar{a},\bar{b}],S[\bar{a},\bar{b}]), d(G[k_{2n+1},l_{2n+1}],T[k_{2n+1},l_{2n+1}]), \\ &\quad \frac{d(S[\bar{a},\bar{b}],G[k_{2n+1},l_{2n+1}]) + d(F[\bar{a},\bar{b}],T[k_{2n+1},l_{2n+1}])}{2}\Big\} \\ &= \max\Big\{d(S[\bar{a},\bar{b}]\oplus\varpi_3,T[k_{2n+1},l_{2n+1}]), \\ &\quad d(F[\bar{a},\bar{b}]\oplus\varpi_2,S[\bar{a},\bar{b}]\oplus\varpi_3), d(G[k_{2n+1},l_{2n+1}],T[k_{2n+1},l_{2n+1}]), \\ &\quad \frac{d(S[\bar{a},\bar{b}]\oplus\varpi_3,G[k_{2n+1},l_{2n+1}]) + d(F[\bar{a},\bar{b}]\oplus\varpi_2,T[k_{2n+1},l_{2n+1}])}{2}\Big\} \\ &= \max\Big\{d([\bar{u},\bar{v}]\oplus\varpi_1,[u_{2n},v_{2n}]), d([\bar{u},\bar{v}]\oplus\varpi_1,[\bar{u},\bar{v}]\oplus\varpi_1), \\ &\quad d([u_{2n+1},v_{2n+1}],[u_{2n},v_{2n}]), \\ &\quad \frac{d([\bar{u},\bar{v}]\oplus\varpi_1,[u_{2n+1},v_{2n+1}]) + d([\bar{u},\bar{v}]\oplus\varpi_1,[u_{2n},v_{2n}])}{2}\Big\} \\ &= \max\Big\{d([\bar{u},\bar{v}],[u_{2n},v_{2n}]), d([\bar{u},\bar{v}],[\bar{u},\bar{v}]), d([u_{2n+1},v_{2n+1}],[u_{2n},v_{2n}]), \\ &\quad \frac{d([\bar{u},\bar{v}],[u_{2n+1},v_{2n+1}]) + d([\bar{u},\bar{v}],[u_{2n},v_{2n}])}{2}\Big\}, \end{aligned} \tag{4.25}$$

Notice that according to Lemma 4.1 and (4.14), we have $\lim_{n\to\infty} d([\bar{u},\bar{v}],[u_{2n},v_{2n}]) = d([\bar{u},\bar{v}],[\tilde{u},\tilde{v}])$.
Therefore, taking $n \to \infty$ in (4.24), making use of (4.11), (4.14) and (4.25) and using the continuity of ϕ and Lemma 4.1, we obtain

$$\begin{aligned} d([\bar{u},\bar{v}],[\tilde{u},\tilde{v}]) &\leq \lim_{n\to\infty} \phi\big(M([\bar{a},\bar{b}],[k_{2n+1},l_{2n+1}])\big) + \lim_{n\to\infty} d([u_{2n+1},v_{2n+1}],[\tilde{u},\tilde{v}]) \\ &= \phi\big(d([\bar{u},\bar{v}],[\tilde{u},\tilde{v}])\big). \end{aligned}$$

Since $\phi(t) < t$ for each $t > 0$, we have

$$d([\bar{u},\bar{v}],[\tilde{u},\tilde{v}]) \leq \phi\big(d([\bar{u},\bar{v}],[\tilde{u},\tilde{v}])\big) < d([\bar{u},\bar{v}],[\tilde{u},\tilde{v}]),$$

which is a contradiction. So $d([\bar{u},\bar{v}],[\tilde{u},\tilde{v}]) = 0$, i.e., $[\bar{u},\bar{v}] \stackrel{\Omega}{=} [\tilde{u},\tilde{v}]$ for all $[\tilde{u},\tilde{v}] \in \langle [u,v] \rangle$ and $\langle [\bar{u},\bar{v}] \rangle = \langle [u,v] \rangle$. Thus F and S have a unique class of near-points of coincidence. Therefore, we proved (i).
According to $F\mathcal{I} \subseteq T\mathcal{I}$ and (4.21), there is $[c,d] \in \mathcal{I}$ such that $[\tilde{u},\tilde{v}] \stackrel{\Omega}{=} T[c,d]$, i.e.,

$$T[c,d] \oplus \omega_5 = [\tilde{u},\tilde{v}] \oplus \omega_6, \quad \texttt{for some } \omega_5,\omega_6 \in \Omega. \tag{4.26}$$

Now we show that $d(G[c,d],[\tilde{u},\tilde{v}]) = 0$ for all $[\tilde{u},\tilde{v}] \in \langle [u,v] \rangle$. Assume that $d(G[c,d],[\tilde{u},\tilde{v}]) > 0$. From (4.1) and (4.21), we obtain

$$\begin{aligned} d([\tilde{u},\tilde{v}],G[c,d]) &= d([\tilde{u},\tilde{v}]\oplus\omega_4, G[c,d]) = d(F[a,b]\oplus\omega_3, G[c,d]) \\ &= d(F[a,b],G[c,d]) \le \phi\big(M([a,b],[c,d])\big). \end{aligned} \tag{4.27}$$

Using (4.16), (4.21) and (4.26) and making use of the null equality, we have

$$\begin{aligned} M([a,b],[c,d]) &= \max\big\{d(S[a,b],T[c,d]), d(F[a,b],S[a,b]), d(G[c,d],T[c,d]), \\ &\quad \frac{d(S[a,b],G[c,d]) + d(F[a,b],T[c,d])}{2}\big\} \\ &= \max\big\{d(S[a,b]\oplus\omega_2, T[c,d]\oplus\omega_5), \\ &\quad d(F[a,b]\oplus\omega_3, S[a,b]\oplus\omega_2), d(G[c,d],T[c,d]\oplus\omega_5), \\ &\quad \frac{d(S[a,b]\oplus\omega_2, G[c,d]) + d(F[a,b]\oplus\omega_3, T[c,d]\oplus\omega_5)}{2}\big\} \\ &= \max\big\{d([\tilde{u},\tilde{v}]\oplus\omega_1, [\tilde{u},\tilde{v}]\oplus\omega_6), d([\tilde{u},\tilde{v}]\oplus\omega_4, [\tilde{u},\tilde{v}]\oplus\omega_1), \\ &\quad d(G[c,d],[\tilde{u},\tilde{v}]\oplus\omega_6), \frac{d([\tilde{u},\tilde{v}]\oplus\omega_1, G[c,d]) + d([\tilde{u},\tilde{v}]\oplus\omega_4, [\tilde{u},\tilde{v}]\oplus\omega_6)}{2}\big\} \\ &= \max\big\{d([\tilde{u},\tilde{v}],[\tilde{u},\tilde{v}]), d([\tilde{u},\tilde{v}],[\tilde{u},\tilde{v}]), d(G[c,d],[\tilde{u},\tilde{v}]), \\ &\quad \frac{d([\tilde{u},\tilde{v}],G[c,d]) + d([\tilde{u},\tilde{v}],[\tilde{u},\tilde{v}])}{2}\big\} = d([\tilde{u},\tilde{v}],G[c,d]). \end{aligned} \tag{4.28}$$

Thus from (4.27), (4.28) and the property of ϕ, we obtian

$$d([\tilde{u},\tilde{v}],G[c,d]) \le \phi\big(d([\tilde{u},\tilde{v}],G[c,d])\big) < d([\tilde{u},\tilde{v}],G[c,d]),$$

which is a contradiction. Therefore, we have $d(G[c,d],[\tilde{u},\tilde{v}]) = 0$ and $G[c,d] \stackrel{\Omega}{=} [\tilde{u},\tilde{v}]$, i.e.,

$$G[c,d]\oplus\omega_7 = [\tilde{u},\tilde{v}]\oplus\omega_8, \quad \texttt{for some } \omega_7,\omega_8\in\Omega. \tag{4.29}$$

Due to (4.26), Proposition 2.1 and (4.29), we have

$$[\tilde{u},\tilde{v}] \stackrel{\Omega}{=} G[c,d] \stackrel{\Omega}{=} T[c,d], \quad \texttt{for all } [\tilde{u},\tilde{v}] \in \langle [u,v] \rangle. \tag{4.30}$$

Thus, any $[\tilde{u},\tilde{v}] \in \langle [u,v] \rangle$ is a near-point of coincidence for G and T. Therefore, from Proposition 2.1, (4.22) and (4.30), we have

$$[\tilde{u},\tilde{v}] \stackrel{\Omega}{=} G[c,d] \stackrel{\Omega}{=} T[c,d] \stackrel{\Omega}{=} F[a,b] \stackrel{\Omega}{=} S[a,b], \quad \texttt{for all } [\tilde{u},\tilde{v}] \in \langle [u,v] \rangle. \tag{4.31}$$

Now we show that $\langle [u,v] \rangle$ is a unique class of near-points of coincidence for G and T. Suppose that $\langle [\bar{u},\bar{v}] \rangle$ is another class of near-points of coincidence for G and T such that $\langle [\bar{u},\bar{v}] \rangle \neq \langle [u,v] \rangle$. Thus, we have $[\bar{u},\bar{v}] \notin \langle [u,v] \rangle$, $[\bar{u},\bar{v}] \stackrel{\Omega}{=} G\left[\bar{c},\bar{d}\right] \stackrel{\Omega}{=} T\left[\bar{c},\bar{d}\right]$ and $d([\tilde{u},\tilde{v}],[\bar{u},\bar{v}]) > 0$ for all $[\tilde{u},\tilde{v}] \in \langle [u,v] \rangle$. In other words, there exist $\varpi_4,\varpi_5,\varpi_6\in\Omega$ such that

$$[\bar{u},\bar{v}]\oplus\varpi_4 = G\left[\bar{c},\bar{d}\right]\oplus\varpi_5 = T\left[\bar{c},\bar{d}\right]\oplus\varpi_6. \tag{4.32}$$

Using (4.1), (4.2) and (4.32) and making use of the triangle inequality and the null equality, we obtain

$$\begin{aligned} d([u_{2n}, v_{2n}], [\bar{u}, \bar{v}]) &= d([u_{2n}, v_{2n}], [\bar{u}, \bar{v}] \oplus \varpi_4) = d(F[k_{2n}, l_{2n}], G[\bar{c}, \bar{d}] \oplus \varpi_5) \\ &= d(F[k_{2n}, l_{2n}], G[\bar{c}, \bar{d}]) \leq \phi\big(M([k_{2n}, l_{2n}], [\bar{c}, \bar{d}])\big). \end{aligned} \tag{4.33}$$

From (4.2), (4.3), (4.32) and the null equality, we have

$$\begin{aligned} &M([k_{2n}, l_{2n}], [\bar{c}, \bar{d}]) = \max\big\{d(S[k_{2n}, l_{2n}], T[\bar{c}, \bar{d}]), \\ &\quad d(F[k_{2n}, l_{2n}], S[k_{2n}, l_{2n}]), d(G[\bar{c}, \bar{d}], T[\bar{c}, \bar{d}]), \\ &\quad \frac{d(S[k_{2n}, l_{2n}], G[\bar{c}, \bar{d}]) + d(F[k_{2n}, l_{2n}], T[\bar{c}, \bar{d}])}{2}\big\} \\ &= \max\big\{d(S[k_{2n}, l_{2n}], T[\bar{c}, \bar{d}] \oplus \varpi_6), \\ &\quad d(F[k_{2n}, l_{2n}], S[k_{2n}, l_{2n}]), d(G[\bar{c}, \bar{d}] \oplus \varpi_5, T[\bar{c}, \bar{d}] \oplus \varpi_6), \\ &\quad \frac{d(S[k_{2n}, l_{2n}], G[\bar{c}, \bar{d}] \oplus \varpi_5) + d(F[k_{2n}, l_{2n}], T[\bar{c}, \bar{d}] \oplus \varpi_6)}{2}\big\} \\ &= \max\big\{d([u_{2n-1}, v_{2n-1}], [\bar{u}, \bar{v}] \oplus \varpi_4), \\ &\quad d([u_{2n}, v_{2n}], [u_{2n-1}, v_{2n-1}]), d([\bar{u}, \bar{v}] \oplus \varpi_4, [\bar{u}, \bar{v}] \oplus \varpi_4), \\ &\quad \frac{d([u_{2n-1}, v_{2n-1}], [\bar{u}, \bar{v}] \oplus \varpi_4) + d([u_{2n}, v_{2n}], [\bar{u}, \bar{v}] \oplus \varpi_4)}{2}\big\} \\ &= \max\big\{d([u_{2n-1}, v_{2n-1}], [\bar{u}, \bar{v}]), d([u_{2n}, v_{2n}], [u_{2n-1}, v_{2n-1}]), d([\bar{u}, \bar{v}], [\bar{u}, \bar{v}]), \\ &\quad \frac{d([u_{2n-1}, v_{2n-1}], [\bar{u}, \bar{v}]) + d([u_{2n}, v_{2n}], [\bar{u}, \bar{v}])}{2}\big\}, \end{aligned} \tag{4.34}$$

Notice that according to Lemma 4.1 and (4.14), we have $\lim_{n\to\infty} d([\bar{u}, \bar{v}], [u_{2n}, v_{2n}]) = d([\bar{u}, \bar{v}], [\tilde{u}, \tilde{v}])$.
Then taking $n \to \infty$ in (4.33), making use of (4.11), (4.14) and (4.34) and using the continuity of ϕ and Lemma 4.1, we have

$$d([\bar{u}, \bar{v}], [\tilde{u}, \tilde{v}]) \leq \phi\big(d([\bar{u}, \bar{v}], [\tilde{u}, \tilde{v}])\big).$$

Since $\phi(t) < t$ for each $t > 0$, we have

$$d([\bar{u}, \bar{v}], [\tilde{u}, \tilde{v}]) \leq \phi\big(d([\bar{u}, \bar{v}], [\tilde{u}, \tilde{v}])\big) < d([\bar{u}, \bar{v}], [\tilde{u}, \tilde{v}]),$$

which is a contradiction. So $d([\bar{u}, \bar{v}], [\tilde{u}, \tilde{v}]) = 0$ for all $[\tilde{u}, \tilde{v}] \in \langle [u, v] \rangle$. Hence, $\langle [\bar{u}, \bar{v}] \rangle = \langle [u, v] \rangle$. Thus G and T have a unique class of near-points of coincidence. Therefore, we proved (ii).
Since F and S are weakly compatible self-mappings and $\langle [u, v] \rangle$ is a unique class of near-points of coincidence for F and S, then according to Lemma 4.2, there is a unique class of common near-fixed points $\langle [u, v] \rangle$ for F and S, i.e.,

$$[\tilde{u}, \tilde{v}] \overset{\Omega}{=} F[\tilde{u}, \tilde{v}] \overset{\Omega}{=} S[\tilde{u}, \tilde{v}], \quad \texttt{for all } [\tilde{u}, \tilde{v}] \in \langle [u, v] \rangle. \tag{4.35}$$

On the other hand, since G and T are weakly compatible self-mappings and $\langle\, [u,v]\, \rangle$ is a unique class of near-points of coincidence for G and T, then according to Lemma 4.2, there is a unique class of common near-fixed points $\langle\, [u,v]\, \rangle$ for F and S, i.e.,

$$[\tilde{u},\tilde{v}] \stackrel{\Omega}{=} G\,[\tilde{u},\tilde{v}] \stackrel{\Omega}{=} T\,[\tilde{u},\tilde{v}]\,, \quad \texttt{for all } [\tilde{u},\tilde{v}] \in \langle\, [u,v]\, \rangle. \tag{4.36}$$

Therefore due to (4.35), (4.36) and Proposition 2.1, there is a unique class of common near-fixed points $\langle\, [u,v]\, \rangle$ for F, T, G and S. $\square$

Example 4.2. Let $T, S : \mathcal{I} \to \mathcal{I}$ and $F, G : \mathcal{I} \to \mathcal{I}$ in the MIS be defined by $T\,[k,l] = S\,[k,l] = [k,l]$ and $F\,[k,l] = G\,[k,l] = \left[\frac{k}{3}, \frac{l}{3}\right]$, respectively. Then we have $F\mathcal{I} \subseteq T\mathcal{I} = \mathcal{I}$ and $G\mathcal{I} \subseteq S\mathcal{I} = \mathcal{I}$. Define $d : \mathcal{I} \times \mathcal{I} \to \mathbb{R}^+$ by $d([k,l]\,, [u,v]) = |(k+l) - (u+v)|$ for all $[k,l]\,, [u,v] \in \mathcal{I}$. Suppose that $\phi(t) = \frac{t}{2}$. Therefore by (4.1), we have

$$d\Big(\left[\frac{k}{3},\frac{l}{3}\right], \left[\frac{u}{3},\frac{v}{3}\right]\Big) \le \phi\Big(\max\Big\{d([k,l]\,,[u,v]), d\Big(\left[\frac{k}{3},\frac{l}{3}\right], [k,l]\Big), d\Big(\left[\frac{u}{3},\frac{v}{3}\right], [u,v]\Big), \frac{d\Big([k,l]\,, \left[\frac{u}{3},\frac{v}{3}\right]\Big) + d\Big(\left[\frac{k}{3},\frac{l}{3}\right], [u,v]\Big)}{2}\Big\}\Big)$$

$$\Longrightarrow \frac{1}{3}\Big|(k+l) - (u+v)\Big| \le \frac{1}{2}\Big(\max\Big\{\Big|(k+l) - (u+v)\Big|, \frac{2}{3}\Big|k+l\Big|, \frac{2}{3}\Big|u+v\Big|, \frac{\Big|(k+l) - \frac{1}{3}(u+v)\Big| + \Big|\frac{1}{3}(k+l) - (u+v)\Big|}{2}\Big\}\Big).$$

Thus, this example satisfies all conditions Theorem 4.1. Note that any $[-k,k] \in \Omega$ is a coincidence near-point for S and F. Therefore, S and F have a unique equivalence class of near-points of coincidence $\Big\langle\, [-k,k]\, \Big\rangle$, where $k \in \mathbb{R}$. Notice that according to definition of the equivalence classes, we have $\Big\langle\, [-k,k]\, \Big\rangle = \Big\langle\, [-1,1]\, \Big\rangle = \Big\langle\, [-2,2]\, \Big\rangle$. Moreover, the pairs $\{F,S\}$ and $\{G,T\}$ are weakly compatible. Thus, any $[-k,k] \in \Omega$ is a common near-fixed point for T, G, F and S. Hence, T, G, F and S have a unique equivalence class of common near-fixed points $\Big\langle\, [-k,k]\, \Big\rangle$ in $\mathcal{I}$.

Corollary 4.1. *Suppose that $(\mathcal{I}, \|\cdot\|)$ is a BIS and the null equality holds for $\|\cdot\|$. Assume that $F, G, T, S : (\mathcal{I}, \|\cdot\|) \to (\mathcal{I}, \|\cdot\|)$ are self-mappings of $\mathcal{I}$ such that $F\mathcal{I} \subseteq T\mathcal{I}$, $G\mathcal{I} \subseteq S\mathcal{I}$ and*

$$\|F\,[k,l] \ominus G\,[u,v]\,\| \le \phi\big(M([k,l]\,,[u,v])\big),$$

for all $[k,l]\,,[u,v] \in \mathcal{I}$, where a function $\phi : [0,\infty) \to [0,\infty)$ is continuous and nondecreasing with $\phi(t) < t$ for each $t > 0$ and

$$M([k,l]\,,[u,v]) = \max\Big\{\|S\,[k,l] \ominus T\,[u,v]\,\|, \|F\,[k,l] \ominus S\,[k,l]\,\|, \|G\,[u,v] \ominus T\,[u,v]\,\|, \frac{\|F\,[k,l] \ominus T\,[u,v]\,\| + \|G\,[u,v] \ominus S\,[k,l]\,\|}{2}\Big\}.$$

If one of the sets $F\mathcal{I}$, $G\mathcal{I}$, $S\mathcal{I}$ and $T\mathcal{I}$ is a closed subset of $(\mathcal{I}, \|\cdot\|)$, then
(i) *F and S have a unique equivalence class of near-points of coincidence.*
(ii) *G and T have a unique equivalence class of near-points of coincidence.*
In addition, if the pairs $\{F, S\}$ and $\{G, T\}$ are weakly compatible self-mappings, then a unique class of common near-fixed points exists for F, G, T and S.

Corollary 4.2. *Let (X, h_X) be a complete quasilinear space, U be an open subset of X and V be a closed subset of X, with $U \subset V$. Let $H : V \times [0,1] \to X$ be an operator such that the following conditions hold:*
(i) *$x \neq H(x,t)$ for each $x \in V \backslash U$ and each $t \in [0,1]$,*
(ii) *there exists $\varphi \in \phi$ (where $\phi : [0,\infty) \to [0,\infty)$ is a continuous and non-decreasing function with $\phi(t) < t$ for each $t > 0$) such that for each $t \in [0,1]$ and each $x, y \in V$, we have $h_X(H(x,t), H(y,t)) \leq \varphi(h_X(x,y))$,*
(iii) *there exists a continuous function $\mu : [0,1] \to \mathbb{R}$ such that for all $p, q \in [0,1]$ and each $x \in V$, we have $h_X(H(x,p), H(x,q)) \leq |\mu(p) - \mu(q)|$,*
(iv) *$\psi : [0,+\infty) \to [0,+\infty)$ is stictly non-decreasing* (`here`; $\psi(x) = x - \varphi(x)$)*.*
Then $H(\cdot, 0)$ has a fixed point if and only if $H(\cdot, 1)$ has a fixed point.

Proof. See the proof of Theorem 3.1 in [8]. □

Theorem 4.2. *Suppose that $(\mathcal{I}, d)$ is a CMIS and the null equality holds for d. Assume that $F, G, T, S : (\mathcal{I}, d) \to (\mathcal{I}, d)$ are self-mappings of $\mathcal{I}$ such that $F\mathcal{I} \subseteq T\mathcal{I}$, $G\mathcal{I} \subseteq S\mathcal{I}$ and*

$$d(F[k,l], G[u,v]) \leq \phi\big(M([k,l],[u,v])\big), \tag{4.37}$$

for all $[k,l], [u,v] \in \mathcal{I}$, where a function $\phi : [0,\infty) \to [0,\infty)$ is continuous and nondecreasing with $\phi(t) < t$ for each $t > 0$ and

$$\begin{aligned} M([k,l],[u,v]) \;=\; & \alpha d(S[k,l], T[u,v]) + \beta\big[d(F[k,l], S[k,l]) + d(G[u,v], T[u,v])\big] \\ & + \gamma\big[d(F[k,l], T[u,v]) + d(G[u,v], S[k,l])\big], \end{aligned}$$

with $\alpha, \beta, \gamma \geq 0$, $\alpha + 2\beta + 2\gamma \leq 1$. If one of the sets $F\mathcal{I}$, $G\mathcal{I}$, $S\mathcal{I}$ and $T\mathcal{I}$ is a closed subset of $(\mathcal{I}, d)$, then
(i) *F and S have a unique equivalence class of near-points of coincidence.*
(ii) *G and T have a unique equivalence class of near-points of coincidence.*
In addition, if the pairs $\{F, S\}$ and $\{G, T\}$ are weakly compatible self-mappings, then a unique equivalence class of common near-fixed points exists for F, G, T and S.

Proof. Consider an arbitrary $[k_0, l_0] \in \mathcal{I}$. Since $F\mathcal{I} \subseteq T\mathcal{I}$, there exists $[k_1, l_1] \in \mathcal{I}$ such that $T[k_1, l_1] = F[k_0, l_0]$. Since $G\mathcal{I} \subseteq S\mathcal{I}$, there exists $[k_2, l_2] \in \mathcal{I}$ such that $S[k_2, l_2] = G[k_1, l_1]$. By continuing this process, we define sequences $\{[k_n, l_n]\}_{n=1}^{\infty}$ and $\{[u_n, v_n]\}_{n=1}^{\infty}$ in $\mathcal{I}$ such that

$$\begin{aligned} [u_{2n}, v_{2n}] &= T[k_{2n+1}, l_{2n+1}] = F[k_{2n}, l_{2n}], \\ [u_{2n+1}, v_{2n+1}] &= S[k_{2n+2}, l_{2n+2}] = G[k_{2n+1}, l_{2n+1}] \ \forall n \in \mathbb{N}. \end{aligned} \tag{4.38}$$

we also have

$$[u_{2n-1}, v_{2n-1}] = S\,[k_{2n}, l_{2n}] = G\,[k_{2n-1}, l_{2n-1}] \;\; \forall n \in \mathbb{N}. \tag{4.39}$$

According to (4.38), (4.39) and the triangle inequality, we have

$$\begin{aligned}
& M([k_{2q}, l_{2q}], [k_{2q+1}, l_{2q+1}]) = \alpha d(S\,[k_{2q}, l_{2q}], T\,[k_{2q+1}, l_{2q+1}]) \\
& \quad +\beta\big[d(F\,[k_{2q}, l_{2q}], S\,[k_{2q}, l_{2q}]) + d(G\,[k_{2q+1}, l_{2q+1}], T\,[k_{2q+1}, l_{2q+1}])\big] \\
& \quad +\gamma\big[d(S\,[k_{2q}, l_{2q}], G\,[k_{2q+1}, l_{2q+1}]) + d(F\,[k_{2q}, l_{2q}], T\,[k_{2q+1}, l_{2q+1}])\big] \\
= \; & \alpha d([u_{2q-1}, v_{2q-1}], [u_{2q}, v_{2q}]) \\
& \quad +\beta\big[d([u_{2q}, v_{2q}], [u_{2q-1}, v_{2q-1}]) + d([u_{2q+1}, v_{2q+1}], [u_{2q}, v_{2q}]\,\big] \\
& \quad +\gamma\big[d([u_{2q-1}, v_{2q-1}], [u_{2q+1}, v_{2q+1}]) + d([u_{2q}, v_{2q}], [u_{2q}, v_{2q}])\big] \\
\leq \; & \alpha d([u_{2q-1}, v_{2q-1}], [u_{2q}, v_{2q}]) + \beta\big[d([u_{2q}, v_{2q}], [u_{2q-1}, v_{2q-1}]) \\
& \quad +d([u_{2q+1}, v_{2q+1}], [u_{2q}, v_{2q}])\big] + \gamma\big[d([u_{2q-1}, v_{2q-1}], [u_{2q}, v_{2q}]) \\
& \quad +d([u_{2q}, v_{2q}], [u_{2q+1}, v_{2q+1}])\big].
\end{aligned}$$

Since ϕ is a non-decreasing function, we obtain

$$\begin{aligned}
& \phi\big(M([k_{2q}, l_{2q}], [k_{2q+1}, l_{2q+1}])\big) \leq \phi\big(\alpha d([u_{2q-1}, v_{2q-1}], [u_{2q}, v_{2q}]) \\
& \quad +\beta\big[d([u_{2q}, v_{2q}], [u_{2q-1}, v_{2q-1}]) + d([u_{2q+1}, v_{2q+1}], [u_{2q}, v_{2q}])\big] \\
& \quad +\gamma\big[d([u_{2q-1}, v_{2q-1}], [u_{2q}, v_{2q}]) + d([u_{2q}, v_{2q}], [u_{2q+1}, v_{2q+1}])\big]\big).
\end{aligned}$$

Putting $[k, l] = [k_{2q}, l_{2q}]$ and $[u, v] = [k_{2q+1}, l_{2q+1}]$ in (4.37) and using the above inequality, we get

$$\begin{aligned}
d([u_{2q}, v_{2q}], [u_{2q+1}, v_{2q+1}]) \;\leq\; & \phi\big((\alpha+\beta+\gamma) d([u_{2q-1}, v_{2q-1}], [u_{2q}, v_{2q}]) \\
& +(\beta+\gamma) d([u_{2q+1}, v_{2q+1}], [u_{2q}, v_{2q}])\big).
\end{aligned} \tag{4.40}$$

Similarly, we have

$$\begin{aligned}
d([u_{2q+1}, v_{2q+1}], [u_{2q+2}, v_{2q+2}]) \;\leq\; & \phi\big((\alpha+\beta+\gamma) d([u_{2q}, v_{2q}], [u_{2q+1}, v_{2q+1}]), \\
& +(\beta+\gamma) d([u_{2q+1}, v_{2q+1}], [u_{2q+2}, v_{2q+2}])\big).
\end{aligned} \tag{4.41}$$

Therefore, from (4.40) and (4.41), we obtain

$$\begin{aligned}
d([u_n, v_n], [u_{n+1}, v_{n+1}]) \;\leq\; & \phi\big((\alpha+\beta+\gamma) d([u_{n-1}, v_{n-1}], [u_n, v_n]), \\
& +(\beta+\gamma) d([u_n, v_n], [u_{n+1}, v_{n+1}])\big),
\end{aligned} \tag{4.42}$$

for all $n \geq 1$. Assume that there exists $q \in \mathbb{N}$ such that $d([u_{2q-1}, v_{2q-1}], [u_{2q}, v_{2q}]) = 0$. Then $[u_{2q-1}, v_{2q-1}] \overset{\Omega}{=} [u_{2q}, v_{2q}]$ and from (4.40), we have

$$\begin{aligned}
d([u_{2q}, v_{2q}], [u_{2q+1}, v_{2q+1}]) \;&\leq\; \phi\big((\beta+\gamma) d([u_{2q}, v_{2q}], [u_{2q+1}, v_{2q+1}])\big) \\
&\leq\; \phi\big(d([u_{2q}, v_{2q}], [u_{2q+1}, v_{2q+1}])\big) \; (\texttt{since } (\beta+\gamma) \leq 1).
\end{aligned}$$

According to $\phi(t) < t$ for each $t > 0$, the above inequality results that

$$\begin{aligned} d([u_{2q}, v_{2q}], [u_{2q+1}, v_{2q+1}]) &\leq \phi\big(d([u_{2q}, v_{2q}], [u_{2q+1}, v_{2q+1}])\big) \\ &< d([u_{2q}, v_{2q}], [u_{2q+1}, v_{2q+1}]), \end{aligned}$$

which implies that $d([u_{2q}, v_{2q}], [u_{2q+1}, v_{2q+1}]) = 0$. Hence, $[u_{2q}, v_{2q}] \stackrel{\Omega}{=} [u_{2q+1}, v_{2q+1}]$. Thus from (4.41) and $\beta + \gamma \leq 1$, we obtain

$$\begin{aligned} d([u_{2q+1}, v_{2q+1}], [u_{2q+2}, v_{2q+2}]) &\leq \phi\big((\beta+\gamma) d([u_{2q+1}, v_{2q+1}], [u_{2q+2}, v_{2q+2}])\big) \\ &\leq \phi\big(d([u_{2q+1}, v_{2q+1}], [u_{2q+2}, v_{2q+2}])\big) \\ &< d([u_{2q+1}, v_{2q+1}], [u_{2q+2}, v_{2q+2}]). \end{aligned}$$

which follows that $[u_{2q+1}, v_{2q+1}] \stackrel{\Omega}{=} [u_{2q+2}, v_{2q+2}]$. Thus we have

$$[u_{2q-1}, v_{2q-1}] \stackrel{\Omega}{=} [u_{2q}, v_{2q}] \stackrel{\Omega}{=} [u_{2q+1}, v_{2q+1}] \stackrel{\Omega}{=} [u_{2q+2}, v_{2q+2}] \stackrel{\Omega}{=} \cdots$$

Moreover, if we suppose that there exists $q \in \mathbb{N}$ such that $d([u_{2q}, v_{2q}], [u_{2q+1}, v_{2q+1}]) = 0$, then the similar result holds. Now we suppose that

$$d([u_n, v_n], [u_{n+1}, v_{n+1}]) > 0, \ \forall n \in \mathbb{N}.$$

Since $\phi(t) < t$ for each $t > 0$, (4.42) implies that

$$\begin{aligned} d([u_n, v_n], [u_{n+1}, v_{n+1}]) &< (\alpha+\beta+\gamma) d([u_{n-1}, v_{n-1}], [u_n, v_n]), \\ &\quad +(\beta+\gamma) d([u_n, v_n], [u_{n+1}, v_{n+1}]). \end{aligned}$$

So, we have

$$(1-\beta-\gamma) d([u_n, v_n], [u_{n+1}, v_{n+1}]) < (\alpha+\beta+\gamma) d([u_{n-1}, v_{n-1}], [u_n, v_n]).$$

Therefore, we obtain

$$\begin{aligned} d([u_n, v_n], [u_{n+1}, v_{n+1}]) &< \frac{(\alpha+\beta+\gamma)}{(1-\beta-\gamma)} d([u_{n-1}, v_{n-1}], [u_n, v_n]) \\ &\leq d([u_{n-1}, v_{n-1}], [u_n, v_n]) \ (\texttt{since } \frac{(\alpha+\beta+\gamma)}{(1-\beta-\gamma)} \leq 1). \end{aligned}$$

Hence $\big\{d([u_n, v_n], [u_{n+1}, v_{n+1}])\big\}$ is a nonnegative, monotone non-increasing and bounded below sequence. Therefore there is an $r \geq 0$ such that

$$\lim_{n\to\infty} d([u_n, v_n], [u_{n+1}, v_{n+1}]) = r. \tag{4.43}$$

Suppose that $r > 0$. Taking $n \to \infty$ in (4.42) and using (4.43) and the continuity of ϕ, we have

$$r \leq \phi\big((\alpha+2\beta+2\gamma) r\big). \tag{4.44}$$

According to the nondecreasing property of ϕ and $\alpha + 2\beta + 2\gamma \leq 1$, (4.44) implies that

$$r \leq \phi\big((\alpha + 2\beta + 2\gamma)r\big) \leq \phi(r) \Rightarrow r \leq \phi(r).$$

Since $\phi(t) < t$ for each $t > 0$, then we have

$$r \leq \phi(r) < r,$$

which implies that $r = 0$, i.e.,

$$\lim_{n\to\infty} d([u_n, v_n], [u_{n+1}, v_{n+1}]) = 0. \tag{4.45}$$

Now we indicate that $\{[u_n, v_n]\}_{n=1}^{\infty}$ is a Cauchy sequence in the MIS $(\mathcal{I}, d)$. Due to (4.45), it is adequate to to demonstrate that $\{[u_{2n}, v_{2n}]\}_{n=1}^{\infty}$ can be classified as a Cauchy sequence in $(\mathcal{I}, d)$. Assume that $\{[u_{2n}, v_{2n}]\}$ is not a Cauchy sequence. Then there is an $\varepsilon > 0$ that we can detect two sequences of positive integers $\{2m(h)\}$ and $\{2n(h)\}$ with $2n(h) > 2m(h) > h$ such that

$$d\big([u_{2m(h)}, v_{2m(h)}], [u_{2n(h)}, v_{2n(h)}]\big) \geq \varepsilon$$

and

$$d\big([u_{2m(h)}, v_{2m(h)}], [u_{2n(h)-2}, v_{2n(h)-2}]\big) < \varepsilon,$$

for all positive integers h. Therefore, we have

$$\begin{aligned}
\varepsilon &\leq d\big([u_{2m(h)}, v_{2m(h)}], [u_{2n(h)}, v_{2n(h)}]\big)\\
&\leq d\big([u_{2m(h)}, v_{2m(h)}], [u_{2n(h)-2}, v_{2n(h)-2}]\big)\\
&\quad + d\big([u_{2n(h)-2}, v_{2n(h)-2}], [u_{2n(h)-1}, v_{2n(h)-1}]\big)\\
&\quad + d\big([u_{2n(h)-1}, v_{2n(h)-1}], [u_{2n(h)}, v_{2n(h)}]\big)\\
&< \varepsilon + d\big([u_{2n(h)-2}, v_{2n(h)-2}], [u_{2n(h)-1}, v_{2n(h)-1}]\big)\\
&\quad + d\big([u_{2n(h)-1}, v_{2n(h)-1}], [u_{2n(h)}, v_{2n(h)}]\big)
\end{aligned}$$

Letting $h \to \infty$ in the above inequality and making use of (4.45), we have

$$\lim_{h\to\infty} d\big([u_{2m(h)}, v_{2m(h)}], [u_{2n(h)}, v_{2n(h)}]\big) = \varepsilon. \tag{4.46}$$

Again,

$$\begin{aligned}
d\big([u_{2m(h)}, v_{2m(h)}], [u_{2n(h)}, v_{2n(h)}]\big)&\\
&\leq d\big([u_{2m(h)}, v_{2m(h)}], [u_{2m(h)-1}, v_{2m(h)-1}]\big)\\
&\quad + d\big([u_{2m(h)-1}, v_{2m(h)-1}], [u_{2n(h)+1}, v_{2n(h)+1}]\big)\\
&\quad + d\big([u_{2n(h)+1}, v_{2n(h)+1}], [u_{2n(h)}, v_{2n(h)}]\big),
\end{aligned} \tag{4.47}$$

and

$$\begin{aligned} d\big(\big[u_{2m(h)-1}, v_{2m(h)-1}\big] , \big[u_{2n(h)+1}, v_{2n(h)+1}\big] \big) &\leq d\big(\big[u_{2m(h)-1}, v_{2m(h)-1}\big] , \big[u_{2m(h)}, v_{2m(h)}\big] \big) \\ &\quad + d\big(\big[u_{2m(h)}, v_{2m(h)}\big] , \big[u_{2n(h)}, v_{2n(h)}\big] \big) \\ &\quad + d\big(\big[u_{2n(h)}, v_{2n(h)}\big] , \big[u_{2n(h)+1}, v_{2n(h)+1}\big] \big). \end{aligned} \tag{4.48}$$

Taking $h \to \infty$ in the inequalities (4.47) and (4.48) and using (4.45) and (4.46), we get

$$\lim_{h\to\infty} d\big(\big[u_{2m(h)-1}, v_{2m(h)-1}\big] , \big[u_{2n(h)+1}, v_{2n(h)+1}\big] \big) = \varepsilon. \tag{4.49}$$

Moreover,

$$\begin{aligned} d\big(\big[u_{2m(h)-1}, v_{2m(h)-1}\big] , \big[u_{2n(h)}, v_{2n(h)}\big] \big) &\leq d\big(\big[u_{2m(h)-1}, v_{2m(h)-1}\big] , \big[u_{2n(h)+1}, v_{2n(h)+1}\big] \big) \\ &\quad + d\big(\big[u_{2n(h)+1}, v_{2n(h)+1}\big] , \big[u_{2n(h)}, v_{2m(h)}\big] \big), \end{aligned} \tag{4.50}$$

and

$$\begin{aligned} d\big(\big[u_{2n(h)+1}, v_{2n(h)+1}\big] , \big[u_{2m(h)-1}, v_{2m(h)-1}\big] \big) &\leq d\big(\big[u_{2n(h)+1}, v_{2n(h)+1}\big] , \big[u_{2n(h)}, v_{2n(h)}\big] \big) \\ &\quad + d\big(\big[u_{2n(h)}, v_{2n(h)}\big] , \big[u_{2m(h)-1}, v_{2m(h)-1}\big] \big). \end{aligned} \tag{4.51}$$

Furthermore,

$$\begin{aligned} d\big(\big[u_{2m(h)-1}, v_{2m(h)-1}\big] , \big[u_{2n(h)}, v_{2n(h)}\big] \big) &\leq d\big(\big[u_{2m(h)-1}, v_{2m(h)-1}\big] , \big[u_{2m(h)}, v_{2m(h)}\big] \big) \\ &\quad + d\big(\big[u_{2m(h)}, v_{2m(h)}\big] , \big[u_{2n(h)+1}, v_{2n(h)+1}\big] \big), \end{aligned} \tag{4.52}$$

and

$$\begin{aligned} d\big(\big[u_{2m(h)}, v_{2m(h)}\big] , \big[u_{2n(h)+1}, v_{2n(h)+1}\big] \big) &\leq d\big(\big[u_{2m(h)}, v_{2m(h)}\big] , \big[u_{2m(h)-1}, v_{2m(h)-1}\big] \big) \\ &\quad + d\big(\big[u_{2m(h)-1}, v_{2m(h)-1}\big] , \big[u_{2n(h)}, v_{2n(h)}\big] \big) \\ &\quad + d\big(\big[u_{2n(h)}, v_{2n(h)}\big] , \big[u_{2n(h)+1}, v_{2n(h)+1}\big] \big). \end{aligned} \tag{4.53}$$

Letting $h \to \infty$ in the inequalities (4.50)-(4.53) and using (4.45), (4.46) and (4.49), we obtain

$$\lim_{h\to\infty} d\big(\big[u_{2m(h)-1}, v_{2m(h)-1}\big] , \big[u_{2n(h)}, v_{2n(h)}\big] \big) = \varepsilon, \tag{4.54}$$

$$\lim_{h\to\infty} d\big(\big[u_{2m(h)}, v_{2m(h)}\big] , \big[u_{2n(h)+1}, v_{2n(h)+1}\big] \big) = \varepsilon. \tag{4.55}$$

By putting $[k,l] = \big[k_{2m(h)}, l_{2m(h)}\big]$ and $[u,v] = \big[k_{2n(h)+1}, l_{2n(h)+1}\big]$ in (4.37) and using (4.38), (4.39), we obtain

$$\begin{aligned}
& d\big(\big[u_{2m(h)}, v_{2m(h)}\big] , \big[u_{2n(h)+1}, v_{2n(h)+1}\big] \big)\big) \\
= \;& d\big(F\big[k_{2m(h)}, l_{2m(h)}\big] , G\big[k_{2n(h)+1}, l_{2n(h)+1}\big] \big) \\
\leq \;& \phi\Big(\alpha d\big(S\big[k_{2m(h)}, l_{2m(h)}\big] , T\big[k_{2n(h)+1}, l_{2n(h)+1}\big] \big) \\
& \quad +\beta\Big[d\big(F\big[k_{2m(h)}, l_{2m(h)}\big] , S\big[k_{2m(h)}, l_{2m(h)}\big] \big) \\
& \quad +d\big(G\big[k_{2n(h)+1}, l_{2n(h)+1}\big] , T\big[k_{2n(h)+1}, l_{2n(h)+1}\big] \big)\Big] \\
& \quad +\gamma\Big[d\big(F\big[k_{2m(h)}, l_{2m(h)}\big] , T\big[k_{2n(h)+1}, l_{2n(h)+1}\big] \big) \\
& \quad +d\big(G\big[k_{2n(h)+1}, l_{2n(h)+1}\big] , S\big[k_{2m(h)}, l_{2m(h)}\big] \big)\Big]\Big) \\
= \;& \phi\Big(\alpha d\big(\big[u_{2m(h)-1}, v_{2m(h)-1}\big] , \big[u_{2n(h)}, v_{2n(h)}\big] \big) \\
& \quad +\beta\Big[d\big(\big[u_{2m(h)}, v_{2m(h)}\big] , \big[u_{2m(h)-1}, v_{2m(h)-1}\big] \big) \\
& \quad +d\big(\big[u_{2n(h)+1}, v_{2n(h)+1}\big] , \big[u_{2n(h)}, v_{2n(h)}\big] \big)\Big] \\
& \quad +\gamma\Big[d\big(\big[u_{2m(h)}, v_{2m(h)}\big] , \big[u_{2n(h)}, v_{2n(h)}\big] \big) \\
& \quad +d\big(\big[u_{2n(h)+1}, v_{2n(h)+1}\big] , \big[u_{2m(h)-1}, v_{2m(h)-1}\big] \big)\Big]\Big).
\end{aligned}$$

Taking $h \to \infty$ in the above inequality, making use of (4.45)-(4.46), (4.49) and (4.54)-(4.55) and using the continuity of ϕ, we have

$$\varepsilon \leq \phi\big((\alpha + 2\gamma)\varepsilon\big).$$

According to the nondecreasing property of ϕ and $\alpha + 2\gamma \leq 1$, it results that

$$\varepsilon \leq \phi((\alpha + 2\gamma)\varepsilon) \leq \phi(\varepsilon),$$

Since $\varepsilon > 0$ and $\phi(t) < t$ for each $t > 0$, then we have

$$\varepsilon \leq \phi(\varepsilon) < \varepsilon,$$

which is a contradiction. Thus $\big\{ [u_{2n}, v_{2n}] \big\}_{n=1}^{\infty}$ is a Cauchy sequence in $(\mathcal{I}, d)$. Hence according to (4.45), $\big\{ [u_n, v_n] \big\}_{n=1}^{\infty}$ is also a Cauchy sequence in $(\mathcal{I}, d)$. So due to the completeness of the MIS $(\mathcal{I}, d)$, there is $[u, v] \in \mathcal{I}$ such that

$$d([u_n, v_n] , [u, v]) \to 0, \tag{4.56}$$

Since the null equality holds for d, according to Proposition 3.1 and (4.56), we have

$$\lim_{n\to\infty} d([u_n, v_n] , [\tilde{u}, \tilde{v}]) = 0, \quad \texttt{for any} \quad [\tilde{u}, \tilde{v}] \in \big\langle [u, v] \big\rangle. \tag{4.57}$$

Therefore, (4.57) implies that

$$\lim_{n\to\infty} d([u_{2n}, v_{2n}], [\tilde{u}, \tilde{v}]) = \lim_{n\to\infty} d([u_{2n-1}, v_{2n-1}], [\tilde{u}, \tilde{v}]) = 0. \tag{4.58}$$

Then from (4.38), (4.39) and (4.58), we obtain

$$\begin{aligned} &\lim_{n\to\infty} d(T[k_{2n+1}, l_{2n+1}], [\tilde{u}, \tilde{v}]) = \lim_{n\to\infty} d(F[k_{2n}, l_{2n}], [\tilde{u}, \tilde{v}]) = 0, \\ &\lim_{n\to\infty} d(S[k_{2n}, l_{2n}], [\tilde{u}, \tilde{v}]) = \lim_{n\to\infty} d(G[k_{2n-1}, l_{2n-1}], [\tilde{u}, \tilde{v}]) = 0. \end{aligned} \tag{4.59}$$

Now without loss of generality, we suppose that $S\mathcal{I}$ is a closed subset of the MIS $(\mathcal{I}, d)$. Then duo to (4.59), there is $[a,b] \in \mathcal{I}$ such that $[\tilde{u}, \tilde{v}] \stackrel{\Omega}{=} S[a,b]$, i.e.,

$$[\tilde{u}, \tilde{v}] \oplus \omega_1 = S[a,b] \oplus \omega_2, \quad \texttt{for some } \omega_1, \omega_2 \in \Omega. \tag{4.60}$$

Now we show that $d(F[a,b], [\tilde{u}, \tilde{v}]) = 0$ for all $[\tilde{u}, \tilde{v}] \in \langle [u,v] \rangle$. Suppose that $d(F[a,b], [\tilde{u}, \tilde{v}]) > 0$. Then from (4.37), (4.39) and the triangle inequality, we get

$$\begin{aligned} d(F[a,b], [\tilde{u}, \tilde{v}]) &\leq d(F[a,b], G[k_{2n+1}, l_{2n+1}]) + d(G[k_{2n+1}, l_{2n+1}], [\tilde{u}, \tilde{v}]) \\ &\leq \phi\big(M([a,b], [k_{2n+1}, l_{2n+1}])\big) + d([u_{2n+1}, v_{2n+1}], [\tilde{u}, \tilde{v}]). \end{aligned} \tag{4.61}$$

By using (4.38), (4.39) and (4.60) and making use of the triangle inequality and the null equality, we obtain

$$\begin{aligned} &M([a,b], [k_{2n+1}, l_{2n+1}]) = \alpha d(S[a,b], T[k_{2n+1}, l_{2n+1}]) \\ &\qquad + \beta\big[d(F[a,b], S[a,b]) + d(G[k_{2n+1}, l_{2n+1}], T[k_{2n+1}, l_{2n+1}])\big] \\ &\qquad + \gamma\big[d(S[a,b], G[k_{2n+1}, l_{2n+1}]) + d(F[a,b], T[k_{2n+1}, l_{2n+1}])\big] \\ &= \alpha d(S[a,b] \oplus \omega_2, T[k_{2n+1}, l_{2n+1}]) \\ &\qquad + \beta\big[d(F[a,b], S[a,b] \oplus \omega_2) + d(G[k_{2n+1}, l_{2n+1}], T[k_{2n+1}, l_{2n+1}])\big] \\ &\qquad + \gamma\big[d(S[a,b] \oplus \omega_2, G[k_{2n+1}, l_{2n+1}]) + d(F[a,b], T[k_{2n+1}, l_{2n+1}])\big] \\ &= \alpha d([\tilde{u}, \tilde{v}] \oplus \omega_1, [u_{2n}, v_{2n}]) + \beta\big[d(F[a,b], [\tilde{u}, \tilde{v}] \oplus \omega_1) \\ &\qquad + d([u_{2n+1}, v_{2n+1}], [u_{2n}, v_{2n}])\big] \\ &\qquad + \gamma\big[d([\tilde{u}, \tilde{v}] \oplus \omega_1, [u_{2n+1}, v_{2n+1}]) + d(F[a,b], [u_{2n}, v_{2n}])\big] \\ &\leq \alpha d([\tilde{u}, \tilde{v}], [u_{2n}, v_{2n}]) + \beta\big[d(F[a,b], [\tilde{u}, \tilde{v}]) + d([u_{2n+1}, v_{2n+1}], [u_{2n}, v_{2n}])\big] \\ &\qquad + \gamma\big[d([\tilde{u}, \tilde{v}], [u_{2n+1}, v_{2n+1}]) + d(F[a,b], [\tilde{u}, \tilde{v}]) + d([\tilde{u}, \tilde{v}], [u_{2n}, v_{2n}])\big]. \end{aligned} \tag{4.62}$$

Taking $n \to \infty$ in (4.61), making use of (4.45) and (4.58) and using (4.62) and the continuity of ϕ, we have

$$\begin{aligned} d(F[a,b][\tilde{u}, \tilde{v}]) &\leq \lim_{n\to\infty} \phi\big(M([a,b], [k_{2n+1}, l_{2n+1}])\big) + \lim_{n\to\infty} d([u_{2n+1}, v_{2n+1}], [\tilde{u}, \tilde{v}]) \\ &= \phi\big((\beta+\gamma) d(F[a,b], [\tilde{u}, \tilde{v}])\big) \\ &\leq \phi\big(d(F[a,b], [\tilde{u}, \tilde{v}])\big) \ \ (\texttt{since } \alpha + \beta \leq 1). \end{aligned}$$

Since $\phi(t) < t$ for each $t > 0$, we have

$$d(F[a,b],[\tilde{u},\tilde{v}]) \leq \phi(d(F[a,b],[\tilde{u},\tilde{v}])) < d(F[a,b],[\tilde{u},\tilde{v}]),$$

which is a contradiction. Thus $d(F[a,b],[\tilde{u},\tilde{v}]) = 0$ and $F[a,b] \stackrel{\Omega}{=} [\tilde{u},\tilde{v}]$, i.e.,

$$F[a,b] \oplus \omega_3 = [\tilde{u},\tilde{v}] \oplus \omega_4, \quad \texttt{for some } \omega_3, \omega_4 \in \Omega. \tag{4.63}$$

Due to $[\tilde{u},\tilde{v}] \stackrel{\Omega}{=} S[a,b]$, Proposition 2.1 and (4.63), we have

$$[\tilde{u},\tilde{v}] \stackrel{\Omega}{=} F[a,b] \stackrel{\Omega}{=} S[a,b], \quad \texttt{for all } [\tilde{u},\tilde{v}] \in \big\langle [u,v] \big\rangle. \tag{4.64}$$

Therefore, any $[\tilde{u},\tilde{v}] \in \big\langle [u,v] \big\rangle$ is a near-point of coincidence for F and S. Now we show that $\big\langle [u,v] \big\rangle$ is a unique class of near-points of coincidence for F and S. Assume that $\big\langle [\bar{u},\bar{v}] \big\rangle$ is another class of near-points of coincidence for F and S such that $\big\langle [\bar{u},\bar{v}] \big\rangle \neq \big\langle [u,v] \big\rangle$. Then, we have $[\bar{u},\bar{v}] \notin \big\langle [u,v] \big\rangle$, $[\bar{u},\bar{v}] \stackrel{\Omega}{=} F\big[\bar{a},\bar{b}\big] \stackrel{\Omega}{=} S\big[\bar{a},\bar{b}\big]$ and $d([\tilde{u},\tilde{v}],[\bar{u},\bar{v}]) > 0$ for all $[\tilde{u},\tilde{v}] \in \big\langle [u,v] \big\rangle$. In other words, there exist $\varpi_1, \varpi_2, \varpi_3 \in \Omega$ such that

$$[\bar{u},\bar{v}] \oplus \varpi_1 = F\big[\bar{a},\bar{b}\big] \oplus \varpi_2 = S\big[\bar{a},\bar{b}\big] \oplus \varpi_3. \tag{4.65}$$

Using (4.37), (4.39) and (4.65) and making use of the triangle inequality and the null equality, we obtain

$$\begin{aligned} d([\bar{u},\bar{v}],[\tilde{u},\tilde{v}]) &= d([\bar{u},\bar{v}] \oplus \varpi_1, [\tilde{u},\tilde{v}]) = d(F\big[\bar{a},\bar{b}\big] \oplus \varpi_2, [\tilde{u},\tilde{v}]) = d(F\big[\bar{a},\bar{b}\big],[\tilde{u},\tilde{v}]) \\ &\leq d(F\big[\bar{a},\bar{b}\big], G[k_{2n+1}, l_{2n+1}]) + d(G[k_{2n+1}, l_{2n+1}],[\tilde{u},\tilde{v}]) \\ &\leq \phi\big(M(\big[\bar{a},\bar{b}\big],[k_{2n+1}, l_{2n+1}])\big) + d([u_{2n+1}, v_{2n+1}],[\tilde{u},\tilde{v}]). \end{aligned} \tag{4.66}$$

By using (4.39), (4.65) and the null equality, we obtain

$$\begin{aligned} & M(\big[\bar{a},\bar{b}\big],[k_{2n+1}, l_{2n+1}]) = \alpha d(S\big[\bar{a},\bar{b}\big], T[k_{2n+1}, l_{2n+1}]) \\ & \qquad + \beta\big[d(F\big[\bar{a},\bar{b}\big], S\big[\bar{a},\bar{b}\big]) + d(G[k_{2n+1}, l_{2n+1}], T[k_{2n+1}, l_{2n+1}])\big] \\ & \qquad + \gamma\big[d(S\big[\bar{a},\bar{b}\big], G[k_{2n+1}, l_{2n+1}]) + d(F\big[\bar{a},\bar{b}\big], T[k_{2n+1}, l_{2n+1}])\big] \\ &= \alpha d(S\big[\bar{a},\bar{b}\big] \oplus \varpi_3, T[k_{2n+1}, l_{2n+1}]) \\ & \qquad + \beta\big[d(F\big[\bar{a},\bar{b}\big] \oplus \varpi_2, S\big[\bar{a},\bar{b}\big] \oplus \varpi_3) + d(G[k_{2n+1}, l_{2n+1}], T[k_{2n+1}, l_{2n+1}])\big] \\ & \qquad + \gamma\big[d(S\big[\bar{a},\bar{b}\big] \oplus \varpi_3, G[k_{2n+1}, l_{2n+1}]) + d(F\big[\bar{a},\bar{b}\big] \oplus \varpi_2, T[k_{2n+1}, l_{2n+1}])\big] \\ &= \alpha d([\bar{u},\bar{v}] \oplus \varpi_1, [u_{2n}, v_{2n}]) + \beta\big[d([\bar{u},\bar{v}] \oplus \varpi_1, [\bar{u},\bar{v}] \oplus \varpi_1) \\ & \qquad + d([u_{2n+1}, v_{2n+1}], [u_{2n}, v_{2n}])\big] + \gamma\big[d([\bar{u},\bar{v}] \oplus \varpi_1, [u_{2n+1}, v_{2n+1}]) \\ & \qquad + d([\bar{u},\bar{v}] \oplus \varpi_1, [u_{2n}, v_{2n}])\big] \\ &= \alpha d([\bar{u},\bar{v}], [u_{2n}, v_{2n}]) + \beta\big[d([\bar{u},\bar{v}],[\bar{u},\bar{v}]) + d([u_{2n+1}, v_{2n+1}], [u_{2n}, v_{2n}])\big] \\ & \qquad + \gamma\big[d([\bar{u},\bar{v}], [u_{2n+1}, v_{2n+1}]) + d([\bar{u},\bar{v}], [u_{2n}, v_{2n}])\big]. \end{aligned} \tag{4.67}$$

Notice that according to Lemma 4.1 and (4.58), we have $\lim_{n\to\infty} d([\bar{u},\bar{v}],[u_{2n},v_{2n}]) = d([\bar{u},\bar{v}],[\tilde{u},\tilde{v}])$. Therefore, taking $n\to\infty$ in (4.66) and making use of (4.45), (4.58) and (4.67) and using the continuity of ϕ and Lemma 4.1, we have

$$\begin{aligned} d([\bar{u},\bar{v}],[\tilde{u},\tilde{v}]) &\leq \lim_{n\to\infty} \phi\big(M([a,b],[k_{2n+1},l_{2n+1}])\big) + \lim_{n\to\infty} d([u_{2n+1},v_{2n+1}],[\tilde{u},\tilde{v}]) \\ &= \phi\big((\alpha+2\gamma)d([\bar{u},\bar{v}],[\tilde{u},\tilde{v}])\big) \\ &\leq \phi\big(d([\bar{u},\bar{v}],[\tilde{u},\tilde{v}])\big) \ \ (\texttt{since}\ \alpha+2\gamma\leq 1). \end{aligned}$$

Since $\phi(t) < t$ for each $t > 0$, we obtain

$$d([\bar{u},\bar{v}],[\tilde{u},\tilde{v}]) \leq \phi\big(d([\bar{u},\bar{v}],[\tilde{u},\tilde{v}])\big) < d([\bar{u},\bar{v}],[\tilde{u},\tilde{v}]),$$

which is a contradiction. So $d([\bar{u},\bar{v}],[\tilde{u},\tilde{v}]) = 0$ for all $[\tilde{u},\tilde{v}] \in \big\langle [u,v] \big\rangle$. Hence, $\big\langle [\bar{u},\bar{v}] \big\rangle = \big\langle [u,v] \big\rangle$. Thus F and S have a unique class of near-points of coincidence. Therefore, we proved (i).

According to $F\mathcal{I} \subseteq T\mathcal{I}$ and (4.63), there is $[c,d] \in \mathcal{I}$ such that $[\tilde{u},\tilde{v}] \overset{\Omega}{=} T[c,d]$, i.e.,

$$T[c,d] \oplus \omega_5 = [\tilde{u},\tilde{v}] \oplus \omega_6, \quad \texttt{for some}\ \omega_5,\omega_6 \in \Omega. \tag{4.68}$$

Now we show that $d(G[c,d],[\tilde{u},\tilde{v}]) = 0$ for all $[\tilde{u},\tilde{v}] \in \big\langle [u,v] \big\rangle$. Assume that $d(G[c,d],[\tilde{u},\tilde{v}]) > 0$. Then from (4.37), (4.63) and the null equality, we have

$$\begin{aligned} d([\tilde{u},\tilde{v}],G[c,d]) &= d([\tilde{u},\tilde{v}]\oplus\omega_4,G[c,d]) = d(F[a,b]\oplus\omega_3,G[c,d]) \\ &= d(F[a,b],G[c,d]) \leq \phi\big(M([a,b],[c,d])\big). \end{aligned} \tag{4.69}$$

Due to (4.60), (4.63), (4.68) and the null equality, we have

$$\begin{aligned} & M([a,b],[c,d]) = \alpha d(S[a,b],T[c,d]) + \beta\big[d(F[a,b],S[a,b]) \\ &\quad + d(G[c,d],T[c,d])\big] + \gamma\big[d(S[a,b],G[c,d]) + d(F[a,b],T[c,d])\big] \\ = \ & \alpha d(S[a,b]\oplus\omega_2,T[c,d]\oplus\omega_5) \\ &\quad + \beta\big[d(F[a,b]\oplus\omega_3,S[a,b]\oplus\omega_2) + d(G[c,d],T[c,d]\oplus\omega_5)\big] \\ &\quad + \gamma\big[d(S[a,b]\oplus\omega_2,G[c,d]) + d(F[a,b]\oplus\omega_3,T[c,d]\oplus\omega_5)\big] \\ = \ & \alpha d([\tilde{u},\tilde{v}]\oplus\omega_1,[\tilde{u},\tilde{v}]\oplus\omega_6) + \beta\big[d([\tilde{u},\tilde{v}]\oplus\omega_4,[\tilde{u},\tilde{v}]\oplus\omega_1) \\ & + d(G[c,d],[\tilde{u},\tilde{v}]\oplus\omega_6)\big] + \gamma\big[d([\tilde{u},\tilde{v}]\oplus\omega_1,G[c,d]) + d([\tilde{u},\tilde{v}]\oplus\omega_4,[\tilde{u},\tilde{v}]\oplus\omega_6)\big] \\ = \ & \alpha d([\tilde{u},\tilde{v}],[\tilde{u},\tilde{v}]) + \beta\big[d([\tilde{u},\tilde{v}],[\tilde{u},\tilde{v}]) + d(G[c,d],[\tilde{u},\tilde{v}])\big] \\ & + \gamma\big[d([\tilde{u},\tilde{v}],G[c,d]) + d([\tilde{u},\tilde{v}],[\tilde{u},\tilde{v}])\big] = (\beta+\gamma)d([\tilde{u},\tilde{v}],G[c,d]). \end{aligned} \tag{4.70}$$

Therefore, from (4.69), (4.70) and the nondecreasing property of ϕ, we obtian

$$\begin{aligned} d([\tilde{u},\tilde{v}],G[c,d]) &\leq \phi\big((\beta+\gamma)d([\tilde{u},\tilde{v}],G[c,d])\big) \\ &\leq \phi\big(d([\tilde{u},\tilde{v}],G[c,d])\big) \ \ (\texttt{since}\ \beta+\gamma\leq 1) \\ &< d([\tilde{u},\tilde{v}],G[c,d]) \ \ (\texttt{since}\ \phi(t)<t\ \texttt{for}\ t>0), \end{aligned}$$

which is a contradiction. Thus, we have $d(G[c,d],[\tilde{u},\tilde{v}])=0$ and $G[c,d]\overset{\Omega}{=}[\tilde{u},\tilde{v}]$, i.e.,

$$G[c,d]\oplus\omega_7=[\tilde{u},\tilde{v}]\oplus\omega_8,\quad \texttt{for some } \omega_7,\omega_8\in\Omega. \tag{4.71}$$

Due to (4.68), Proposition 2.1 and (4.71), we have

$$[\tilde{u},\tilde{v}]\overset{\Omega}{=}G[c,d]\overset{\Omega}{=}T[c,d],\quad \texttt{for all } [\tilde{u},\tilde{v}]\in\big\langle [u,v]\big\rangle. \tag{4.72}$$

Hence, any $[\tilde{u},\tilde{v}]\in\big\langle [u,v]\big\rangle$ is a near-point of coincidence for G and T. Therefore, from Proposition 2.1, (4.64) and (4.72), we have

$$[\tilde{u},\tilde{v}]\overset{\Omega}{=}G[c,d]\overset{\Omega}{=}T[c,d]\overset{\Omega}{=}F[a,b]\overset{\Omega}{=}S[a,b],\quad \texttt{for all } [\tilde{u},\tilde{v}]\in\big\langle [u,v]\big\rangle. \tag{4.73}$$

Now we show that $\big\langle [u,v]\big\rangle$ is a unique class of near-points of coincidence for G and T. Suppose that $\big\langle [\bar{u},\bar{v}]\big\rangle$ is another class of near-points of coincidence for G and T such that $\big\langle [\bar{u},\bar{v}]\big\rangle\neq\big\langle [u,v]\big\rangle$. Then, we have $[\bar{u},\bar{v}]\notin\big\langle [u,v]\big\rangle$, $[\bar{u},\bar{v}]\overset{\Omega}{=}G\big[\bar{c},\bar{d}\big]\overset{\Omega}{=}T\big[\bar{c},\bar{d}\big]$ and $d([\tilde{u},\tilde{v}],[\bar{u},\bar{v}])>0$ for all $[\tilde{u},\tilde{v}]\in\big\langle [u,v]\big\rangle$. In other words, there exist $\varpi_4,\varpi_5,\varpi_6\in\Omega$ such that

$$[\bar{u},\bar{v}]\oplus\varpi_4=G\big[\bar{c},\bar{d}\big]\oplus\varpi_5=T\big[\bar{c},\bar{d}\big]\oplus\varpi_6. \tag{4.74}$$

Using (4.37), (4.39) and (4.74) and making use of the triangle inequality and the null equality, we obtain

$$\begin{aligned} d([u_{2n},v_{2n}],[\bar{u},\bar{v}]) &= d([u_{2n},v_{2n}],[\bar{u},\bar{v}]\oplus\varpi_4)=d(F[k_{2n},l_{2n}],G\big[\bar{c},\bar{d}\big]\oplus\varpi_5)\\ &= d(F[k_{2n},l_{2n}],G\big[\bar{c},\bar{d}\big])\le\phi\big(M([k_{2n},l_{2n}],\big[\bar{c},\bar{d}\big])\big). \end{aligned} \tag{4.75}$$

Due to (4.38), (4.39), (4.74) and the null equality, we have

$$\begin{aligned} M([k_{2n},l_{2n}],\big[\bar{c},\bar{d}\big]) &= \alpha d(S[k_{2n},l_{2n}],T\big[\bar{c},\bar{d}\big])\\ &\quad+\beta\big[d(F[k_{2n},l_{2n}],S[k_{2n},l_{2n}])+d(G\big[\bar{c},\bar{d}\big],T\big[\bar{c},\bar{d}\big])\big]\\ &\quad+\gamma\big[d(S[k_{2n},l_{2n}],G\big[\bar{c},\bar{d}\big])+d(F[k_{2n},l_{2n}],T\big[\bar{c},\bar{d}\big])\big]\\ &= \alpha d(S[k_{2n},l_{2n}],T\big[\bar{c},\bar{d}\big]\oplus\varpi_6)\\ &\quad+\beta\big[d(F[k_{2n},l_{2n}],S[k_{2n},l_{2n}])+d(G\big[\bar{c},\bar{d}\big]\oplus\varpi_5,T\big[\bar{c},\bar{d}\big]\oplus\varpi_6)\big]\\ &\quad+\gamma\big[d(S[k_{2n},l_{2n}],G\big[\bar{c},\bar{d}\big]\oplus\varpi_5)+d(F[k_{2n},l_{2n}],T\big[\bar{c},\bar{d}\big]\oplus\varpi_6)\big]\\ &= \alpha d([u_{2n-1},v_{2n-1}],[\bar{u},\bar{v}]\oplus\varpi_4)\\ &\quad+\beta\big[d([u_{2n},v_{2n}],[u_{2n-1},v_{2n-1}])+d([\bar{u},\bar{v}]\oplus\varpi_4,[\bar{u},\bar{v}]\oplus\varpi_4)\big]\\ &\quad+\gamma\big[d([u_{2n-1},v_{2n-1}],[\bar{u},\bar{v}]\oplus\varpi_4)+d([u_{2n},v_{2n}],[\bar{u},\bar{v}]\oplus\varpi_4)\big]\\ &= \alpha d([u_{2n-1},v_{2n-1}],[\bar{u},\bar{v}])+\beta\big[d([u_{2n},v_{2n}],[u_{2n-1},v_{2n-1}])+d([\bar{u},\bar{v}],[\bar{u},\bar{v}])\big]\\ &\quad+\gamma\big[d([u_{2n-1},v_{2n-1}],[\bar{u},\bar{v}])+d([u_{2n},v_{2n}],[\bar{u},\bar{v}])\big], \end{aligned} \tag{4.76}$$

Note that according to Lemma 4.1 and (4.58), we have $\lim_{n\to\infty}d([\bar{u},\bar{v}],[u_{2n},v_{2n}])=d([\bar{u},\bar{v}],[\tilde{u},\tilde{v}])$. Taking $n\to\infty$ in (4.75) and making use of (4.45), (4.58) and (4.76) and using the continuity of ϕ and Lemma 4.1, we have

$$d([\bar{u},\bar{v}],[\tilde{u},\tilde{v}])\le\phi\big((\alpha+2\gamma)d([\bar{u},\bar{v}],[\tilde{u},\tilde{v}])\big)\le\phi\big(d([\bar{u},\bar{v}],[\tilde{u},\tilde{v}])\big)\ \ (\texttt{since } \beta+2\gamma\le 1).$$

Since $\phi(t) < t$ for each $t > 0$, we obtain

$$d([\bar{u},\bar{v}],[\tilde{u},\tilde{v}]) \leq \phi\big(d([\bar{u},\bar{v}],[\tilde{u},\tilde{v}])\big) < d([\bar{u},\bar{v}],[\tilde{u},\tilde{v}]),$$

which is a contradiction. So $d([\bar{u},\bar{v}],[\tilde{u},\tilde{v}]) = 0$ for all $[\tilde{u},\tilde{v}] \in \big\langle [u,v] \big\rangle$. Hence, $\big\langle [\bar{u},\bar{v}] \big\rangle = \big\langle [u,v] \big\rangle$. Thus G and T have a unique class of near-points of coincidence. Therefore, we proved (ii).

Since F and S are weakly compatible self-mappings and $\big\langle [u,v] \big\rangle$ is a unique class of near-points of coincidence for F and S, then according to Lemma 4.2, there is a unique class of common near-fixed points $\big\langle [u,v] \big\rangle$ for F and S, i.e.,

$$[\tilde{u},\tilde{v}] \overset{\Omega}{=} F[\tilde{u},\tilde{v}] \overset{\Omega}{=} S[\tilde{u},\tilde{v}], \quad \texttt{for all } [\tilde{u},\tilde{v}] \in \big\langle [u,v] \big\rangle. \tag{4.77}$$

On the other hand, since G and T are weakly compatible self-mappings and $\big\langle [u,v] \big\rangle$ is a unique class of near-points of coincidence for G and T, then according to Lemma 4.2, there is a unique class of common near-fixed points $\big\langle [u,v] \big\rangle$ for G and T, i.e.,

$$[\tilde{u},\tilde{v}] \overset{\Omega}{=} G[\tilde{u},\tilde{v}] \overset{\Omega}{=} T[\tilde{u},\tilde{v}], \quad \texttt{for all } [\tilde{u},\tilde{v}] \in \big\langle [u,v] \big\rangle. \tag{4.78}$$

Therefore due to Proposition 2.1, (4.77) and (4.78), there is a unique class of common near-fixed points $\big\langle [u,v] \big\rangle$ for F,T,G and S. □

Example 4.3. Let $T,S : \mathcal{I} \to \mathcal{I}$ and $F,G : \mathcal{I} \to \mathcal{I}$ in the MIS be defined by $T[k,l] = S[k,l] = [k,l]$ and $F[k,l] = G[k,l] = \left[\frac{k}{4},\frac{l}{4}\right]$, respectively. Then we have $F\mathcal{I} \subseteq T\mathcal{I} = \mathcal{I}$ and $G\mathcal{I} \subseteq S\mathcal{I} = \mathcal{I}$. Define $d : \mathcal{I} \times \mathcal{I} \to \mathbb{R}^+$ by $d([k,l],[u,v]) = |(k+l)-(u+v)|$ for all $[k,l],[u,v] \in \mathcal{I}$. Suppose that $\alpha = \frac{1}{2}$, $\beta = \gamma = \frac{1}{8}$ and $\phi(t) = \frac{3}{4}t$. Therefore by (4.37), we have

$$\begin{aligned} d\bigg(\Big[\frac{k}{4},\frac{l}{4}\Big],\Big[\frac{u}{4},\frac{v}{4}\Big]\bigg) \leq \phi\bigg(&\alpha d([k,l],[u,v]) \\ &+\beta\bigg[d\bigg(\Big[\frac{k}{4},\frac{l}{4}\Big],[k,l]\bigg) + d\bigg(\Big[\frac{u}{4},\frac{v}{4}\Big],[u,v]\bigg)\bigg] \\ &+\gamma\bigg[d\bigg([k,l],\Big[\frac{u}{4},\frac{v}{4}\Big]\bigg) + d\bigg(\Big[\frac{k}{4},\frac{l}{4}\Big],[u,v]\bigg)\bigg]\bigg). \end{aligned}$$

$$\begin{aligned} \Rightarrow \frac{1}{4}\Big|(k+l)-(u+v)\Big| \leq \frac{3}{4}\bigg(&\frac{1}{2}\Big|(k+l)-(u+v)\Big| + \frac{3}{32}\Big|k+l\Big| + \frac{3}{32}\Big|u+v\Big| \\ &+\frac{1}{8}\bigg(\Big|(k+l)-\frac{1}{4}(u+v)\Big| + \Big|\frac{1}{4}(k+l)-(u+v)\Big|\bigg)\bigg). \end{aligned}$$

Thus, this example satisfies all conditions Theorem 4.2. Note that any $[-k,k] \in \Omega$ is a coincidence near-point for S and F. Therefore, S and F have a unique equivalence class of near-points of coincidence $\Big\langle [-k,k] \Big\rangle$, where $k \in \mathbb{R}$. Notice that according to definition of the equivalence classes, we have $\Big\langle [-k,k] \Big\rangle = \Big\langle [-1,1] \Big\rangle = \Big\langle [-2,2] \Big\rangle$. Moreover, the pairs $\{F,S\}$ and $\{G,T\}$ are weakly compatible. Thus, any $[-k,k] \in \Omega$ is a common near-fixed point for T, G, F and S. Hence, T, G, F and S have a unique equivalence class of common near-fixed points $\Big\langle [-k,k] \Big\rangle$ in $\mathcal{I}$.

Corollary 4.3. *Let $(\mathcal{I}, \|\cdot\|)$ be a BIS and the null equality holds for $\|\cdot\|$. Assume that $F, G, T, S : (\mathcal{I}, \|\cdot\|) \to (\mathcal{I}, \|\cdot\|)$ are self-mappings of $\mathcal{I}$ such that $F\mathcal{I} \subseteq T\mathcal{I}$, $G\mathcal{I} \subseteq S\mathcal{I}$ and*

$$\|F[k,l] \ominus G[u,v]\| \leq \phi\big(M([k,l],[u,v])\big),$$

for all $[k,l],[u,v] \in \mathcal{I}$, where a function $\phi : [0,\infty) \to [0,\infty)$ is continuous and nondecreasing with $\phi(t) < t$ for each $t > 0$ and

$$\begin{aligned} M([k,l],[u,v]) \;&=\; \alpha\|S[k,l] \ominus T[u,v]\| \\ &\quad +\beta\big[\|F[k,l] \ominus S[k,l]\| + \|G[u,v] \ominus T[u,v]\|\big] \\ &\quad +\gamma\big[\|F[k,l] \ominus T[u,v]\| + \|G[u,v] \ominus S[k,l]\|\big], \end{aligned}$$

with $\alpha, \beta, \gamma \geq 0$, $\alpha + 2\beta + 2\gamma \leq 1$. If one of the sets $F\mathcal{I}$, $G\mathcal{I}$, $S\mathcal{I}$ and $T\mathcal{I}$ is a closed subset of $(\mathcal{I}, \|\cdot\|)$, then
(i) F and S have a unique equivalence class of near-points of coincidence.
(ii) G and T have a unique equivalence class of near-points of coincidence.
In addition, if the pairs $\{F, S\}$ and $\{G, T\}$ are weakly compatible self-mappings, then a unique class of common near-fixed points exists for F, G, T and S.

5. Conclusion

Today, fixed points are used in various fields, and so far, excellent results have been obtained. Recently, Wu proposed the notion of a fixed point called a "near-fixed point." He defined the null set and presented the MIS and the NIS [14]. Continuing this research, we studied some near-fixed point theorems for contractive mappings in the MIS and the NIS. We also provided examples to demonstrate the correctness of the results.

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