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3-TRIANGULATIONS OF POLYHEDRA AND THEIR CONNECTION GRAPHS

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Abstract. Here we investigate the properties of 3-triangulation of polyhedra, when possible. Namely, it is known that 3-triangulation of convex polyhedra is always possible, but this is not the case with all non-convex ones. This is the reason to consider the decomposition of non-convex polyhedra into convex pieces if possible. After that, we introduce the connection graph for the 3-triangulable polyhedron in such a way that these pieces are represented by the nodes of the graph. First, our attention shall be focused on toroids, a special class of non-convex polyhedra, and the minimal number of tetrahedra necessary to 3-triangulate them. As another application of connection graphs, we shall also consider those corresponding to convex polyhedra, especially to conic triangulation of them.
Keywords: 3-triangulation, conic triangulation, non-convex polyhedra.

1. Introduction

Classical triangulation is defined as dividing a polygon with n vertices by $n-3$ diagonals into $n-2$ triangles without gaps and overlaps. By generalizing the term polygon, in higher dimensions we get a polyhedron and a d-dimensional polytope. The consequence is the generalization of the triangulation process to higher $d \geq 3$ dimensions, which we can also call triangulation or more specific 3-triangulation, d-triangulation. For this purpose, using only the original vertices, for 3-triangulation

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we divide a polyhedron into tetrahedra, and for d-triangulation a d-polytope into d-simplices.

New problems arise within the higher dimensions. Thus, contrary to the 2-triangulation that can be done for each polygon, it is known that we can 3-triangulate each convex polyhedra, but this is not the case with non-convex ones. The first example of a polyhedron whose interior cannot be triangulated without new vertices was given by Lennes [3] an the most famous one by Schönhardt [6, 8]. Another problem is that different 3-triangulations of polyhedra can have different numbers of tetrahedra [2, 9, 10]. In this sense, it is shown that the smallest and the largest number of tetrahedra in a 3-triangulation (the minimal and the maximal 3-triangulation) depend linearly, i.e. squarely on the number of vertices.

As in the case of 2-triangulation, 3-triangulation problems have significant applications in engineering and other fields of research [22]. Also, other types of polyhedron decomposition and space modeling have their role in applications [20, 21].

Here we first consider 3-triangulation of p-toroids, a special class of polyhedra. Then we consider other application of connection graph, connected also to some problems in graph theory [4].

Namely, the term 'polyhedron' usually means a simple polyhedron solid, topologically equivalent to a ball. On the other hand, there are classes of polyhedra topologically equivalent to torus or p-torus (ball with one or p handles). We call such polyhedra 1-toroids and p-toroids, following Szilassi's definition [18]. He called torus-like polyhedra, toroids. Thus, we use the term p-toroid ($p \in \mathbb{N}$ is a given natural number) for p-torus-like polyhedron, and the term toroid as a common name for any p-toroid (the Szilassi's toroid would be 1-toroid).

Toroids are not convex, but under certain conditions it is possible to 3-triangulate them. A well-known example of a 1-toroid is the Császár polyhedron [1, 19] which has 7 vertices and is triangulable with 7 tetrahedra. It is also discussed as a polyhedron without diagonals [1, 16, 17]. Introducing the concept of connection graph, the minimal 3-triangulations and other properties of 1-, 2- and p-toroids are considered in [5, 11–14].

Some characteristic cases of polyhedra and some necessary terms from graph theory are given in Section 2., while in Section 3. some necessary definitions and properties of 3-triangulation are presented. Section 4. gives a proof of Theorem 4.3 on the minimal necessary number of tetrahedra for 3-triangulation of a p-toroid, which differs from that in [11] and provides a different point of view, giving also additional properties of the connection graph. In Section 5. we consider applications of connection graph to 3-triangulation of convex polyhedra: besides examples of different triangulations of icosahedron and forming corresponding connection graphs, we shall especially consider the cone triangulation of convex polyhedra and prove the Theorem 5.1.

2. Preliminaries

2.1. Basic properties of 3-triangulations

The smallest number of tetrahedra in a 3-triangulation of a polyhedron with n vertices is $n-3$. For example, such a polyhedron is a pyramid V_{n-1} with $n-1$ vertices at the basis and the apex, which means a total of n vertices. We can 3-triangulate it as follows: do any 2-triangulation of the basis into $(n-1)-2=n-3$ triangles. The apex together with each of such triangles forms one of the tetrahedra in 3-triangulation.

But not all polyhedra have the same property to have a 3-triangulation with $n-3$ tetrahedra. For example, in Section 5. two triangulations of the icosahedron ($n=12$) are given. The first triangulation has 17 tetrahedra, while the second, which is minimal for icosahedron [9], has 15 tetrahedra. This example also shows that a polyhedron can have triangulations with different numbers of tetrahedra.

Although it is possible to 3-triangulate all convex polyhedra, this is not the case for all non-convex ones. E.g. the famous example of a non-convex polyhedron given by Schönhardt ([8]) shows that it is not possible to triangulate all polyhedra.

On the other hand, the Császár polyhedron [1] is non-convex one for which in Wolfram Demonstrations Project [19] Szilassi shows to be 3-triangulable with 7 tetrahedra. This polyhedron is an example of 1-toroid with the smallest number of vertices. It has 7 vertices and no diagonals, i.e. each vertex is connected to six others by edges.

According to Szilassi's definition given in [18], in [11, 14] we introduce the term p-toroid ($p \in \mathbb{N}$).

Definition 2.1. A polyhedron as a solid is called *p-toroid*, $p \in \mathbb{N}$, if it is topologically equivalent to a ball with p handles (solid p-torus).

We use term *toroid* as a common name for all p-toroids.

Let us remind here that in the surface theory *p-torus* is a cyclic polygon with paired sides. Any side s and its pair S are oppositely directed related to the fixed orientation of the polygon, and then glued together. By a standard combinatorial procedure, the polygon can be divided and glued to a cyclic normal form $a_1b_1A_1B_1a_2b_2A_2B_2...a_pb_pA_pB_p$, as a p-torus. This combinatorial procedure is independent of the future spatial placement of the surface. Thus, we can form a torus from any spatial knot (as a topological circle in the space). Of course, its surface can be 2-triangulated, to be the surface of a polyhedron.

2.2. Overview of concepts from graph theory

A *graph* is a couple $G=(V,E)$ where V is a set of *nodes* (vertices) and E is a set of *edges* connecting nodes from V. *Degree* of a node u is the number of edges with u as one of its endpoints.

Undirected graphs are used in this paper and will be called 'graphs' for short. These are graphs that have edges that have no direction. A *path*, a *cycle* and a *loop* are defined as usual in graph theory.

A graph is *connected* if for each two distinct nodes u and v there exists at least one path joining them. Otherwise, graph is *disconnected* with two or more connected components.

A *tree* is a graph in which any two nodes are connected by exactly one path. This means that a tree is a connected graph. A *spanning tree* is a subset of connected graph G, which has all the nodes covered with minimum possible number of edges. It is possible to conclude that every connected graph G has at least one spanning tree.

For a spanning tree, we have to look for all edges which are present in the graph but not in the tree. Adding one of the missing edges to the tree will form a cycle which is called *fundamental cycle* ([7]). All fundamental cycles form a *cycle basis*. Note that a graph can have more different spanning trees. Consequently, each spanning tree constructs its own cycle basis. We can also form a cycle basis for any disconnected graph, using one spanning tree for each connected component.

However, the number of fundamental cycles is always the same and can be calculated as follows. For any given undirected graph having V nodes, E edges and r connected components, the number of fundamental cycles N_{FC} is:

$$N_{FC} = E - V + r$$

This number is also called *cycle rank* or *circuit rank*.

Two graphs G_1 and G_2 are said to be *isomorphic* if they have the same number of vertices and edges, and their edge connectivity is preserved.

3. Decomposition of polyhedra to convex pieces

We consider 3-triangulations (when possible) of polyhedra by decomposing them into convex pieces and forming a graph representing that decomposition. For this reason, we need the following definitions.

Definition 3.1. A polyhedron is *piecewise convex* if it can be divided into finitely many convex polyhedra P_i, $i = 1, \dots, m$, with disjoint interiors. A pair of above polyhedra P_i, P_j is said to be *neighbouring* if they have a common face called *contact face.*

If the above polyhedra P_i and P_j are not neighbouring, they may have a common edge e or a common vertex v. That is possible iff there is a sequence of neighbouring polyhedra $P_i, P_{i+1}, \dots, P_{i+k} \equiv P_j$ such that the edge e, or the vertex v belongs to each contact face f_l common to P_l and P_{l+1}, $l \in \{i, \dots, i + k - 1\}$. Otherwise, polyhedra P_i and P_j do not have common points.

Remark 3.1. A piecewise convex polyhedron, especially a piecewise convex toroid can be 3-triangulated, because the same is true for its convex pieces. Namely, we can first make a common 2-triangulation of the contact faces, and then, taking into account the new triangular faces, 3-triangulate convex pieces.

On the other hand, each 3-triangulable polyhedron is a collection of connected tetrahedra and so it is piecewise convex.

Definition 3.2. If a polyhedron P is piecewise convex, its *connection graph* (or its *graph of connection*) is a graph whose *nodes* represent convex polyhedra P_i, $i = 1, \dots, m$, the pieces of P, and *edges* represent contact faces between them.

It is obvious that connection graphs are connected and undirected. Some special types of connection graphs will be useful for our further consideration.

Definition 3.3. An *m-division* of a polyhedron is a division in which the tetrahedra participating in the minimal 3-triangulations of the pieces are at the same time participating in the minimal 3-triangulation of the whole polyhedron. A connection graph of a given polyhedron is an *m-graph* if it represents an m-division of that polyhedron.

Remark 3.2. Note that the m-division and thus the m-graph of a polyhedron is not unique. The convex pieces of the m-division can be either separate tetrahedra or their different collections. Besides that, more possibilities for minimal 3-triangulation of the same polyhedron may appear.

On the other hand, it is obvious that there exists at least one m-division of a given 3-triangulable polyhedron. It is its partition into tetrahedra that participate in minimal 3-triangulation.

In order to have the same number of handles for the considered toroid P and number of basic cycles of the corresponding connection graph G we introduce term *optimized graph* of connection. If G has some of the cycles which do not correspond to some handle of P, such situation can interfere us in proving the theorem. We call such a cycle *false*. An example of a toroid P with connection graph that has false cycle is given in [11].

Let us consider a toroid P and its connection graph G that have one or more false cycles. For each of the false cycles, note all the nodes that belong to it and the corresponding convex pieces of P. The union of such convex pieces for each false cycle builds a new node of optimized graph $\hat{G}$. The other nodes of the graph G remain in $\hat{G}$ and we call them the old ones. The edges between the old nodes remain in $\hat{G}$. The edges of G between some old node and some node belonging to a false cycle are converted to the edge of $\hat{G}$ between that old node and the new one.

The optimized graph $\hat{G}$ has the same number of basic cycles as the number of handles of the starting toroid P. Note that it is not necessary that the new nodes of the optimized graph to correspond to convex polyhedra, they only correspond to simple piecewise convex polyhedra. Also, if the graph G is an m-graph, the same property holds for the graph $\hat{G}$.

4. The minimal number of tetrahedra in 3-triangulation of toroids

In [12] the next theorem for 1-toroids is proved:

Theorem 4.1. *If a 1-toroid with $n \geq 7$ vertices can be 3-triangulated, then the minimal number of tetrahedra in that triangulation is $T_{min} \geq n$.*

The corresponding theorem for 2-toroids is given in [13].

Theorem 4.2. *If it is possible to 3-triangulate a 2-toroid with $n \geq 10$ vertices, then the minimal number of tetrahedra for that triangulation is $T_{min} \geq n+3$.*

Using the previous two Theorems in [11] the Theorem 4.3 is proved.

Theorem 4.3. *If a p-toroid with n vertices can be 3-triangulated, then the minimal number of tetrahedra necessary for its 3-triangulation is $T_{min} \geq n+3(p-1)$.*

Here, we give the second proof of the same theorem. As it was mentioned before, this proof provides a different point of view, showing also additional properties of the connection graph.

Proof. As in the first proof, here we also use the mathematical induction for this purpose. Theorems 4.1 and 4.2 are again the starting steps, which guarantee that the statement is true for $k = 1, 2$. Note that statement is also true for a simple polyhedron, where we can assume that $k = 0$, because it holds $T_{min} \geq n-3$.

Let us suppose that statement is true for any $h \leq k$, ($k \in \mathbb{N} \cup \{0\}$) i.e.

> *If a h-toroid ($h \leq k \in \mathbb{N} \cup \{0\}$) with n vertices can be 3-triangulated, then the minimal number of tetrahedra necessary for its 3-triangulation is $T_{min} \geq n+3(h-1)$.*

In the optimized m-graph G of the $(k+1)$-toroid P with n vertices, observe a node d that belongs to some of the cycles (Figure 4.1). Let us denote by c degree of the node d.

We introduce a subgraph $\tilde{G}$ of G that contains only the node d, and subgraph $\bar{G}$ a of G obtained after excluding the node d and all c edges of G with d as an endpoint.

The graph $\bar{G}$ can have more components, say $r \geq 1$. We can assume that edges with d as an endpoint are arranged into r groups, one for each component. So we'll mark the graph components of $\bar{G}$ with G_j, $1 \leq j \leq r$, and edges connecting node d with this component with e_{j,i_j}, $1 \leq i_j \leq c_j$, where c_j is the number of edges in group j. In some groups there may be only one edge $e_{j,1}$ (i.e. $c_j = 1$), but not in all of them, since d belongs to some cycles. Moreover, as node d belongs to cycle(s), it follows that $r < c$. Each subgraph G_j have at least c_j nodes, and (if $c_j \geq 2$) the edges connecting them.

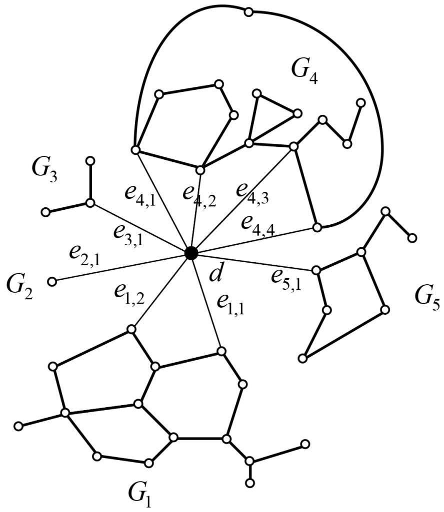

FIG. 4.1: Graph G and node d, where $k+1 = 11$, $c = 9$, $r = 5$, $q = 7$

Let us first consider groups with $c_j \geq 2$. After the eventual change of the cycle base, edges $e_{j,1}, e_{j,2}, \ldots, e_{j,c_j}$, belong to $c_j - 1$ basic cycles, placed between e_{j,i_j} and e_{j,i_j+1}, $1 \leq i_j \leq c_j - 1$ (see also Remark 4.1). After excluding the edges $e_{j,1}, e_{j,2}, \ldots, e_{j,c_j}$, the number of basic cycles in the rest of graph G would decrease by $c_j - 1$.

If $e_{j,1}$ is the only one edge in the group, i.e. $c_j = 1$, its exclusion would keep the number of basic cycles, so again $c_j - 1 = 0$. The corresponding disconnected component G_j of $\bar{G}$ consists of at least another node of $e_{j,1}$, opposite to the node d.

As excluding the edges $e_{j,1}, e_{j,2}, \ldots, e_{j,c_j}$ reduces the number of basic cycles, the sum of the basic cycles of all components of $\bar{G}$ is $q \leq k$. More precisely,

$$q = k + 1 - \sum_{j=1}^{r} (c_j - 1) = k + 1 - \sum_{j=1}^{r} c_j + r = k + 1 - c + r.$$

The observed subgraphs $\tilde{G}$ and $G_1, G_2, \ldots G_r$ are the connection graphs for some subpolyhedra of the polyhedron P. Moreover, the polyhedron $\tilde{S}$ obtained from $\tilde{G}$ is simple, while the components $G_1, G_2, \ldots G_r$ lead to toroids or simple polyhedra (i.e. 0-toroids) $P_1, P_2, \ldots P_r$, with resp. $q_1, q_2, \ldots q_r$ handles where $q_1 + q_2 + \ldots + q_r = q (\leq k)$.

Denote by n_S the number of vertices of $\tilde{S}$ and by $n_1, n_2, \ldots n_r$ those of $P_1, P_2, \ldots, P_r$. These pieces are connected by c contact faces, one for each of the edges e_{j,i_j}, $1 \leq i_j \leq c_j$, $1 \leq j \leq r$. Contact faces have resp. $t_1, t_2, \ldots, t_c$ ($t_1, t_2, \ldots, t_c \geq 3$) vertices. Taking into account that the separation of $\tilde{S}$ from the toroids P_j, $1 \leq j \leq r$, duplicates the contact faces, we conclude

$$n = n_S + n_1 + n_2 + \ldots + n_r - (t_1 + t_2 + \ldots + t_c).$$

Then using the induction hypothesis

$$\begin{aligned} T_{min}(P) &= T_{min}(P_1) + T_{min}(P_2) + \ldots + T_{min}(P_r) + T_{min}(\tilde{S}) \geq \\ &\geq (n_1 + 3(q_1 - 1)) + \ldots + (n_r + 3(q_r - 1)) + (n_S - 3) = \\ &= n_1 + n_2 + \ldots + n_r + n_S + 3(q - r - 1) = \\ &= n + t_1 + t_2 + \ldots + t_c + 3(q - r - 1) \geq \\ &\geq n + 3(c + q - r - 1) = \\ &= n + 3(c + (k + 1 - c + r) - r - 1) = \\ &= n + 3\left((k+1) - 1\right). \end{aligned}$$

Thus the statement is proved. □

Remark 4.1. Note that if, for example, the edges $e_{j,1}, e_{j,2}, \ldots, e_{j,c_j}$, $1 \leq j \leq r$, $c_j \geq 2$ belong to c_j cycles: $c_j - 1$ of them between e_{j,i_j} and e_{j,i_j+1}, $1 \leq i_j \leq c_j - 1$ and one of them between e_{j,c_j} and $e_{j,1}$, then after excluding (node d and) the edges $e_{j,1}, e_{j,2}, \ldots, e_{j,c_j}$, the previous c_j cycles would merge into a new one and the number of basic cycles would decrease by $c_j - 1$. So, in the initial cycle basis, we can replace the cycle between e_{j,c_j} and $e_{j,1}$ with the merged cycle and the situation would be the same as the one mentioned in the proof of the Theorem 4.3.

Using graph theory, we can conclude that if we exclude (one) node d with c edges and if after that process r connected components remain in the graph, then the number of fundamental cycles after these removals would be

$$q = (k + 1) - c + 1 + (r - 1)$$

which is the same result as the one obtained in the proof of the Theorem 4.3.

5. Connection graphs of convex polyhedra

For any convex polyhedron, the simplest connection graph consists of a single node. Of course, other more precise connection graphs can be formed, showing e.g. how the polyhedron is 3-triangulated. In this case, the nodes of the graph would represent the tetrahedra in the considered triangulation. Here, we shall consider two types of triangulations described in [10], and their connection graphs.

1. The *"Greedy Peeling"* (GP) *algorithm* for triangulating a given polyhedron P is iterative and is described below. Take a vertex of the smallest order (an arbitrary one in the case that there are more such vertices) and discard the triangulated "pyramid" which consists of the mentioned vertex and its neighbour vertices. 2-triangulate the new surface of the remaining polyhedron in such a way that the whole triangulation would be face to face. All this has to be done in such a way to get a new polyhedron, which is convex. Then repeat everything with the new polyhedron. At the end of such triangulation there remains only one tetrahedron.

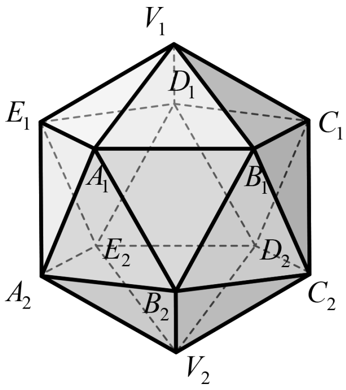

FIG. 5.1: Icosahedron

Let's apply the GP algorithm to the icosahedron (Figure 5.1). Denote by V_1 and V_2 its opposite vertices, and by A_1, B_1, C_1, D_1, E_1 and A_2, B_2, C_2, D_2, E_2 vertices of space pentagons connected with V_1, resp. V_2, and chosen in such a way that the icosahedron contains edges A_1A_2 and A_1B_2. Then a GP triangulation with 17 tetrahedra was obtained in [10] by discarding vertices in the following order and adding new edges marked in parentheses: A_1 (V_1A_2, V_1B_2), B_1 (V_1C_2), C_1 (V_1D_2), D_1 (D_2E_1), E_1 (V_1E_2), E_2 (A_2D_2), V_1 (A_2C_2), B_2. There remains the tetrahedron $A_2C_2D_2V_2$.

Here, for the described triangulation the nodes of the connection graph are introduced by the following list, while the graph is given in Figure 5.2.

Node number	1	2	3	4	5
Tetrahedron	$A_1V_1A_2E_1$	$A_1V_1A_2B_2$	$A_1V_1B_1B_2$	$B_1V_1C_2C_1$	$B_1V_1C_2B_2$

6	7	8	9	10	11
$C_1V_1D_2D_1$	$C_1V_1D_2C_2$	$D_1D_2E_1V_1$	$D_1D_2E_1E_2$	$E_1V_1E_2D_2$	$E_1V_1E_2A_2$

12	13	14	15	16	17
$E_2A_2D_2V_1$	$E_2A_2D_2V_2$	$V_1A_2C_2B_2$	$V_1A_2C_2D_2$	$B_2A_2C_2V_2$	$A_2C_2D_2V_2$

As we can notice, the graph in the Figure 5.2 is not plane graph (it has intersecting edges). If we modify this graph by introducing a new node that represents three collected tetrahedra, that form a new polyhedron P, then we can get a new graph that is plane graph (Figure 5.3). Collected tetrahedra originally correspond to nodes 14, 15, 17. Observe that the original node 16 is doubly connected to the new node. This means that corresponding tetrahedra $B_2A_2C_2V_2$ to the node 16 is connected to the new polyhedron P with two faces, moreover P is not convex. If we annex the tetrahedron $B_2A_2C_2V_2$ to P to form a larger polyhedron, we get a graph that is at the same time plane graph and all its nodes represent convex polyhedra.

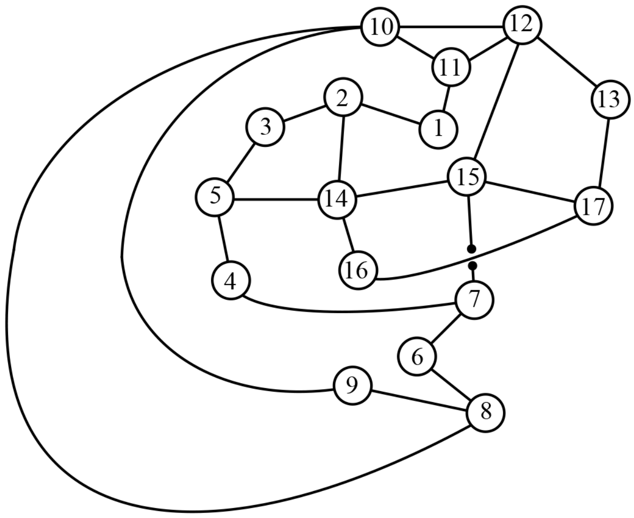

Fig. 5.2: Connection graph for GP triangulation of icosahedron

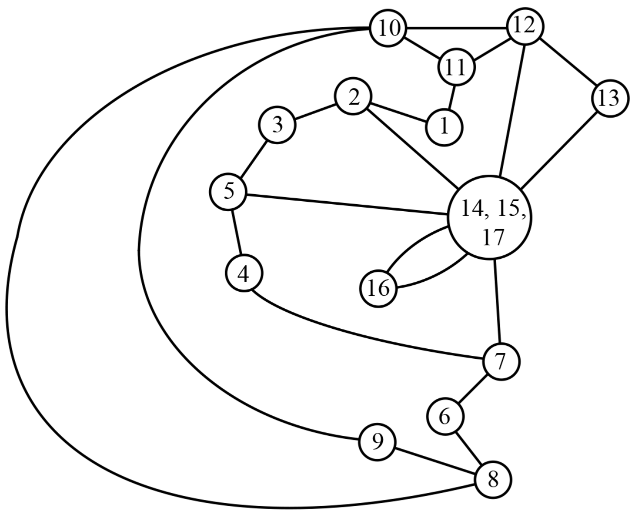

Fig. 5.3: Modified graph for GP triangulation of icosahedron

2. Let us consider the connection graph for the cone triangulation defined as follows.

Definition 5.1. 3-Triangulation in which one of the vertices V is the common apex, which builds a tetrahedron with each triangular face of the polyhedron, except these containing apex is *cone triangulation.*

The cone triangulation is known as that which gives a small number of tetrahedra [9, 10, 15]. By the Euler's theorem, if a polyhedron P with n vertices has only triangular faces, then the number of faces is $2n - 4$. Each polyhedron with $n \geq 13$ vertices has at least one vertex of order 6 or more. Therefore the number of tetrahedra in cone triangulation of P with $n \geq 13$ is at most $2n - 10$.

As before, for the cone triangulation of a convex polyhedron P we can form the connection graph G in such a way that nodes represent tetrahedra of the triangulation. E.g. the connection graph of the cone triangulation for the icosahedron is given in Figure 5.4.

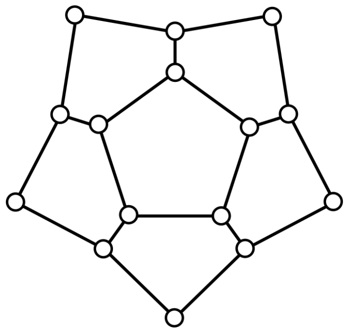

FIG. 5.4: Connection graph for cone triangulation of icosahedron

We claim that the graph G is related to the dual polyhedron of the P. Duality is introduced as follows.

Definition 5.2. For a convex polyhedron P, the Q is its *dual polyhedron* if the vertices of one of them correspond to the faces of the other, and the edges between pairs of vertices of one correspond to the edges between pairs of faces of the other.

We can consider vertices and edges of a convex polyhedron P as a graph G_P. Then vertices and edges of its dual polyhedron Q form a graph G_Q. If we exclude vertices of Q that correspond to the triangular faces of P with the apex V as a one of vertices, and edges of Q containing these vertices, then it remains the subgraph G'_Q of G_Q. We claim that G'_Q is isomorphic to connection graph G representing cone triangulation of P.

If we denote with G'_P subgraph of G_P, obtained from G_P by excluding the vertex V and the edges containing it, then we can also think about G_Q as dual graph of G_P and about G'_Q as weak dual graph of G'_P. Namely in graph theory

Definition 5.3. The *dual graph* of a planar graph G is a graph that has a node for each 'face' of G. The dual graph has an edge for each pair of faces in G that are separated from each other by an edge, and a self-loop when the same face appears on both sides of an edge.

The *weak dual graph* of a plane graph is the subgraph of the dual graph whose nodes correspond to the bounded faces of the original graph.

In introducing this definition we have problem with term 'face' of the graph. More precisely, the definition of the dual graph (and so of weak dual) depends on the choice of embedding of the graph G. That is the reason the dual graph is rather a property of plane graphs (graphs that are already embedded in the plane) than planar graphs (graphs that may be embedded but for which the embedding is not yet known). For planar graphs generally, there may be multiple dual graphs, depending on the choice of planar embedding of the graph. In our case, the structure of the polyhedron P uniquely determines G'_Q as weak dual graph of G'_P. The polyhedral graphs and their dual ones are also considered in [4].

So we can formulate Theorem 5.1 about connection graph of cone triangulation. The proof is illustrated in Figure 5.5.

Theorem 5.1. *If G_Q is the graph which consists of vertices and edges of polyhedron Q dual to a given convex polyhedron P, then the connection graph G representing cone triangulation of P is the subgraph of G_Q. Moreover, graph G is planar.*

Proof. For each tetrahedron of cone triangulation of P with apex V, chose one inner point A_i. Then project whole polyhedron P and chosen points A_i by the central projection π from the apex V, to a plane α ($V \notin \alpha$). If G_P is the graph which consists of vertices and edges of polyhedron P then the central projection is giving graph $\bar{G}'_P$ which is isomorphic to a subgraph of G_P. In fact, each edge of P containing V as a end-point is whole projected to one point, the same one as the projection of its other end-point. Projection of other vertices and edges is giving similar structure as original ones in corresponding 'part' of starting graph G_P, i.e. it is making isomorphic subgraph of G_P. More precisely, $\bar{G}'_P$ is isomorphic to previously introduced graph G'_P. Furthermore, $\bar{G}'_P$ is plane graph.

The projection of points A_i to $\bar{A}_i \in \alpha$ is done by the rays from V passing through inner points B_i of the triangular faces of P. Note that the points B_i can be considered as the vertices of the dual polyhedron Q of P. Also, the edges between B_i's in Q are corresponding to lines connecting A_i's from neighbour tetrahedra in triangulation of P. This means the projection π of points A_i and lines between them (and so of B_i and edges between them), form the graph $G \equiv \bar{G}$ representing cone triangulation of P. At the same time, this projection gives a graph that is obviously isomorphic to the subgraph of G_Q or, more precisely, according to our earlier notation, it is isomorphic to G'_Q. So, as we claimed, G is planar graph isomorphic to subgraph of G_Q. $\square$

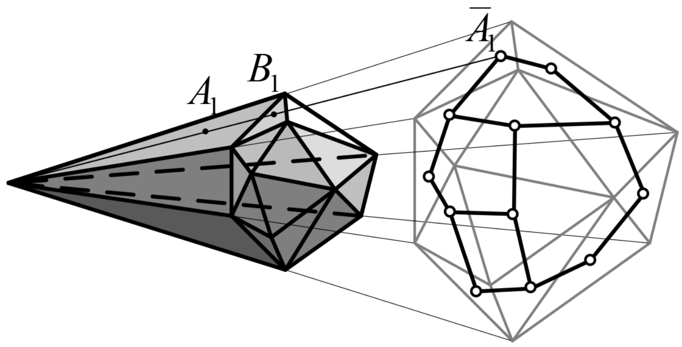

FIG. 5.5: Connection graph for cone triangulation of polyhedron P

6. Summary

Here, properties of the connection graph for 3-triangulation were investigated. Moreover, in addition to considering triangulations of toroids and other non-convex polyhedra, new applications of connection graphs for convex polyhedra are given.

As shown by the discussions in [5, 14] as well as in this paper, the space of realization of a connection graph can be an important property, so it would be good to take it into account in further investigation.

Also, in some future investigations, based on the algorithm for finding triangular faces of a given polyhedron P, given in [15], an algorithm for forming the connection graph G that represents the cone triangulation of P can be given.

Some future topic can also be, e.g. forming connection graphs for n-dimensional triangulations of polytopes, with the aim to consider them more easily.

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