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RICCI SOLITONS AND RICCI BI-CONFORMAL VECTOR FIELDS ON THE MODEL SPACE Sol_0^4

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Abstract. In this paper, we classify the Ricci solitons and the Ricci bi-conformal vector fields on model space Sol_0^4. We also show which of them are gradient vector fields and which one of those are Killing vector fields.
Keywords: Ricci solitons, vector fields, manifolds.

1. Introduction

The study of conformal vector fields in geometry and physics has been an important subject. In general conformal vector fields preserve angles and ratios of distances between points on the manifolds. Suppose (M, g) be a Riemannian manifold, X a smooth vector field, and f a smooth function on M. Thus (M, g) is called a conformal vector field when the following equation holds

$$\mathcal{L}_X g = fg,$$

$\mathcal{L}_X$ is the Lie derivative along X. So if the function $f = 0$, X is a Killing vector field. X is said to be a gradient conformal vector field when X is the gradient of a smooth function. Completely in [8, 9], the conformal vector field is explained. At first, bi-conformal vector fields were introduced by Garcia-Parrado and Senovilla

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[15], then Ricci bi-conformal vector fields were defined by De et al. in [7]. Presume that α and β are smooth functions and Y, Z are vector fields, the vector field X is called a Ricci bi-conformal vector field when the following equations are true

$$(\mathcal{L}_X g)(Y,Z) = \alpha g(Y,Z) + \beta S(Y,Z), \tag{1.1}$$

and

$$(\mathcal{L}_X S)(Y,Z) = \alpha S(Y,Z) + \beta g(Y,Z), \tag{1.2}$$

where S denotes the Ricci tensor of M. The relationship between the existence of certain holomorphic structures in complex manifolds is one of the geometric properties of bi-conformal vector fields. Also, in [15] the authors showed that the bi-conformal vector fields can be concluded with the flow of the bi-conformal vector fields, which is the geometrical interpretation of these vector fields of the generalized regular motions of two orthogonal projectors. In [2], [3], [4], [5] and [19] Ricci bi-conformal vector fields on Lorentzian five-dimensional two-step nilpotent Lie groups, Siklos spacetimes, homogeneous Gödel-type spacetimes, $\mathbb{H}^2 \times \mathbb{R}$, and Lorentzian Walker manifolds, have been studied respectively.

Studies about Ricci's solitons play important roles in geometry and physics that are natural generalizations of Einstein's metrics. At first, Ricci soliton was studied in Lorentzian manifolds and was offered by Hamilton [16]. Therefore, on a pseudo-Riemannian manifold (M, g) we define Ricci soliton as follows

$$\mathcal{L}_X g + S = \lambda g, \tag{1.3}$$

where X is a smooth vector field on M, and λ is a real number [6]. If we consider λ as a smooth function on M, then it is said to be an almost Ricci soliton. A Ricci soliton was developed as a self-similar solution to the Ricci flow. Solving Poincare century-old conjecture is the first importance of using the Ricci soliton. Then, its applications were investigated in various fields of economics and science. Ricci solitons are useful in various sciences such as physics [14], biology, chemistry [17], and economics [21]. Also, the importance of Ricci soliton and Ricci flow can be seen in medical imaging of brain surfaces [25]. Moreover, the algebraic Ricci solitons of three-dimensional Lie group $\mathbb{H}^2 \times \mathbb{R}$, have been studied in [1]. In addition, the presence of sol-solitons on the three-dimensional Lie group Sol_3, has been checked.

The connected pseudo-Riemannian manifold (M, g) is called homogeneous, if the group of isometries of (M, g) transitively acts on M. Study Riemannian homogeneous spaces are common in geometry, algebra, and group theory. A Thurston geometry (G, X) is a homogeneous space where we have is connected and simply connected X; let G be a group, and there is a compact point stabilizer that G acts on X transitively, and only one compact manifold exists that it is composed of (G, X), so G is not included in any larger group of diffeomorphisms of X. Thurston geometry is studied in dimension three for three-manifolds; that is a subset of Riemannian homogeneous spaces. Therefore, there is a similarity between the possible Riemannian structures of orientable compact three-manifolds and the uniformization theorem for surfaces that are compact and orientable. We can divide any three-manifold into slices. Therefore each of them accepts a Riemannian metric in a locally isometric manner in one of the eight three-dimensional model spaces, the Thurston geometries $\mathbb{R}^3, \mathbb{S}^3, \mathbb{H}^3, \mathbb{S}^2 \times \mathbb{R}, \mathbb{H}^2 \times \mathbb{R}, \tilde{SL}(2, \mathbb{R}), Nil^3$ and Sol^3. Eight three-dimensional Thurston spaces are explained completely in [22, 23].

One of the four-dimensional Thurston geometries is the model space (Sol_0^4, g). Filipkiewicz studied spaces in dimension four and he obtained 19 homogeneous model spaces in it [13]. In addition, Wall checked that between these model spaces, the space (Sol_0^4, g) depends on 14 spaces that accept a complex structure that is according to the geometric structure [24]. Also, Sol_0^4 has been found to have a locally conformal Kahler (LCK) structure [11]. Moreover, in this space, the generalization of geodesics, which are J-trajectories, that represent the analog of magnetic curves in LCK spaces, have been investigated. Also, in non-geodesic J-trajectories in the arbitrary LCK manifold where the anti-Lee field has an invariable length, the first and second curvatures have been investigated [12]. In a homogeneous space, there are hypothetical isometrics that map every point to every other point. In addition, in some homogeneous spaces, there is a different translation called the translation curve. This new translation moves the given unit vector at the origin to any point by mappings its tangent. Molnár and Szilagyi studied translation curves in [20]. Moreover, Erjavec [10] classified geodesics and translation curves in Sol_0^4 spaces. In [18], the hypersurfaces of the four-dimensional Thurston geometry Sol_0^4, that is a Riemannian homogeneous space and a solvable Lie group, are investigated. Especially, it presents a complete classification of hypersurfaces whose second fundamental form is the Codazzi tensor- including fully geodesic hypersurfaces and hypersurfaces with parallel second fundamental form- and of totally umbilical hypersurfaces of Sol_0^4.
The paper is arranged as follows: In Section 2, we recall the essential general ideas on (Sol_0^4, g) which will be used throughout the paper. In Section 3, we calculate the Ricci solitons, talk about a theorem of this equation on this space, and discuss the existence of Ricci solitons. Also, in Section 4, we check the Ricci bi-conformal vector fields on (Sol_0^4, g) spaces, and we investigated which of Ricci bi-conformal vector fields are Killing vector fields and gradient vector fields.

2. The model space Sol_0^4

The primary manifold of the model space Sol_0^4 is $\mathbb{R}^4(x, y, z, t)$ with the group operation

$$(x_1, y_1, z_1, t_1) * (x_2, y_2, z_2, t_2) = (x_1 + e^{t_1} x_2, y_1 + e^{t_1} y_2, z_1 + e^{-2t_1} z_2, t_1 + t_2).$$

This process is deduced from the matrix multiplications by the following definition

$$(x, y, z, t) := \begin{pmatrix} e^t & 0 & 0 & 0 & x \\ 0 & e^t & 0 & 0 & y \\ 0 & 0 & e^{-2t} & 0 & z \\ 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.1}$$

Notice $(0,0,0,0)$ is a neutral element. From the below equation, we have the inverse element of (x,y,z,t)

$$(x,y,z,t)^{-1} = (-e^{-t}x, -e^{-t}y, -e^{2t}z, -t). \tag{2.2}$$

Utilizing the inverse translation of (2.2), by the pullback of coordinate differentials,

$$\begin{pmatrix} e^{-t} & 0 & 0 & 0 & -e^{-t}x \\ 0 & e^{-t} & 0 & 0 & -e^{-t}y \\ 0 & 0 & e^{2t} & 0 & -e^{2t}z \\ 0 & 0 & 0 & 1 & -t \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \\ dt \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-t}dx \\ e^{-t}dy \\ e^{2t}dz \\ dt \\ 0 \end{pmatrix}. \tag{2.3}$$

The left invariant Riemannian metric g of Sol_0^4 is obtained as follows

$$g = e^{-2t}(dx^2 + dy^2) + e^{4t}dz^2 + dt^2. \tag{2.4}$$

Therefore, the left constant basis vector fields of the dual metric are considered as

$$e_1 = e^t \frac{\partial}{\partial x}, \quad e_2 = e^t \frac{\partial}{\partial y}, \quad e_3 = e^{-2t}\frac{\partial}{\partial z}, \quad e_4 = \frac{\partial}{\partial t}. \tag{2.5}$$

So basis vector fields are satisfied the following brackets:

$$[e_1,e_2] = [e_1,e_3] = [e_2,e_3] = 0, \ [e_4,e_1] = e_1, \ [e_4,e_2] = e_2,$$
$$[e_4,e_3] = -2e_3.$$

The Levi-Civita connection of manifold (M,g) is shown by ∇. We have the following equation, which is known as the curvature tensor R of (M,g)

$$R(X,Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$$

, so the Ricci tensor S by $S(X,Y) = tr(Z \to R(X,Z)Y)$ is defined. The details of Levi-Civita connection on Sol_0^4 are calculated by

$$\nabla_{e_i} e_j = \begin{pmatrix} e_4 & 0 & 0 & -e_1 \\ 0 & e_4 & 0 & -e_2 \\ 0 & 0 & -2e_4 & 2e_3 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.6}$$

and details of Ricci tensor are determined by

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -6 \end{pmatrix}. \tag{2.7}$$

For any vector field $X = X^k e_k$ by $(\mathcal{L}_X g)(e_i,e_j) = g(\nabla_{e_i}X, e_j) + g(e_i, \nabla_{e_j}X)$ the Lie derivative of the metric g along to the vector field X (see [26]), is given by

$$
\begin{aligned}
(\mathcal{L}_X g)_{11} &= -2X^4 + 2e_1X^1,\\
(\mathcal{L}_X g)_{12} &= e_1X^2 + e_2X^1,\\
(\mathcal{L}_X g)_{13} &= e_1X^3 + e_3X^1,\\
(\mathcal{L}_X g)_{14} &= X^1 + e_1X^4 + e_4X^1,\\
(\mathcal{L}_X g)_{22} &= 2e_2X^2 - 2X^4,\\
(\mathcal{L}_X g)_{23} &= e_2X^3 + e_3X^2,\\
(\mathcal{L}_X g)_{24} &= X^2 + e_2X^4 + e_4X^2,\\
(\mathcal{L}_X g)_{33} &= 2e_3X^3 + 4X^4,\\
(\mathcal{L}_X g)_{34} &= -2X^3 + e_3X^4 + e_4X^3,\\
(\mathcal{L}_X g)_{44} &= 2e_4X^4.
\end{aligned}
\tag{2.8}
$$

Further, by using $(\mathcal{L}_X S)(e_i, e_j) = X(S(e_i, e_j)) - S(\mathcal{L}_X e_i, e_j) - S(e_i, \mathcal{L}_X e_j)$ the Lie derivative of the Ricci tensor along X (see [26]), is determined by

$$
\begin{aligned}
(\mathcal{L}_X S)_{11} &= 0,\\
(\mathcal{L}_X S)_{12} &= 0,\\
(\mathcal{L}_X S)_{13} &= 0,\\
(\mathcal{L}_X S)_{14} &= -6e_1X^4,\\
(\mathcal{L}_X S)_{22} &= 0,\\
(\mathcal{L}_X S)_{23} &= 0,\\
(\mathcal{L}_X S)_{24} &= -6e_2X^4,\\
(\mathcal{L}_X S)_{33} &= 0,\\
(\mathcal{L}_X S)_{34} &= -6e_3X^4,\\
(\mathcal{L}_X S)_{44} &= -12e_4X^4.
\end{aligned}
\tag{2.9}
$$

3. Ricci solitons on the model space Sol_0^4

In this part, the equation (1.3) on the model space Sol_0^4 is solved. Substituting (2.7), (2.8), and (2.9) into (1.3), the following system is obtained

$$2e_1X^1 - 2X^4 = \lambda, \tag{3.1}$$

$$e_2X^1 + e_1X^2 = 0, \tag{3.2}$$

$$e_1X^3 + e_3X^1 = 0, \tag{3.3}$$

$$X^1 + e_1X^4 + e_4X^1 = 0, \tag{3.4}$$

$$2e_2X^2 - 2X^4 = \lambda, \tag{3.5}$$

$$e_2X^3 + e_3X^2 = 0, \tag{3.6}$$

$$X^2 + e_2X^4 + e_4X^2 = 0, \tag{3.7}$$

$$2e_3X^3 + 4X^4 = \lambda, \tag{3.8}$$
$$-2X^3 + e_3X^4 + e_4X^3 = 0, \tag{3.9}$$
$$2e_4X^4 - 6 = \lambda. \tag{3.10}$$

By taking the integral of the equation (3.10), X^4 is infered

$$X^4 = \frac{\lambda + 6}{2}t + F(x, y, z),$$

for some smooth function F. Integrating the equation (3.1), the following relation is deduced

$$X^1 = \frac{\lambda}{2}e^{-t}x + \frac{\lambda + 6}{2}e^{-t}xt + e^{-t}\int F(x, y, z)dx + G(y, z, t), \tag{3.11}$$

for some smooth function G. Now, by taking the integration of equation (3.5), X^2 is obtained as

$$X^2 = (\frac{\lambda}{2})e^{-t}y + (\frac{\lambda + 6}{2})e^{-t}ty + e^{-t}\int F(x, y, z)dy + K(x, z, t), \tag{3.12}$$

for some smooth function K. Also, integrating of the equation (3.8), X^3 is concluded

$$X^3 = (\frac{\lambda}{2})e^{2t}z - (\lambda + 6)e^{2t}tz - 2e^{2t}\int F(x, y, z)dz + L(x, y, t), \tag{3.13}$$

for some smooth function L. Substituting (2.5), (3.11), and (3.12) into (3.2), we obtain

$$\int \partial_y F(x, y, z)dx + e^t\partial_y G(y, z, t) + \int \partial_x F(x, y, z)dy + e^t\partial_x K(x, z, t) = 0, \tag{3.14}$$

also, by the same access for the equation (3.3), the following equation is gotten

$$e^{-3t}\int \partial_z F(x, y, z)dx + e^{-2t}\partial_z G(y, z, t) - 2e^{3t}\int \partial_x F(x, y, z)dz + e^t\partial_x L(x, y, t) = 0. \tag{3.15}$$

Next, from the equation (3.4), we have

$$\frac{\lambda + 6}{2}e^{-t}x + G(y, z, t) + \partial_t G(y, z, t) + e^t\partial_x F(x, y, z) = 0, \tag{3.16}$$

so $\lambda = -6$ and $F(x, y, z) = A(y, z)x + B(y, z)$ are obtained, for some smooth functions A and B, by derivation the last equation along to X. Thus, (3.16) can be rewritten as follows

$$G(y, z, t) + \partial_t G(y, z, t) + e^t A(y, z) = 0,$$

so we obtain $G(y, z, t)$ as

$$G(y, z, t) = -\frac{1}{2}e^t A(y, z) + e^{-t}C(y, z),$$

for some smooth function C. Now by taking the integration of (3.14), $K(x,z,t)$ is deduced

$$K(x,z,t) = -\partial_y A(y,z)e^{-t}\frac{x^3}{6} - e^{-t}\partial_y B(y,z)\frac{x^2}{2} + \frac{1}{2}e^t\partial_y A(y,z)x \\ -\partial_y C(y,z)xe^{-t} - e^{-t}x\int A(y,z)dy + K_1(z,t), \tag{3.17}$$

for some smooth function K_1, by derivation of the equation (3.17) along to y, we have

$$-\partial_{yy} A(y,z)e^{-t}\frac{x^3}{6} - e^{-t}\partial_{yy} B(y,z)\frac{x^2}{2} + \frac{1}{2}e^t\partial_{yy} A(y,z)x \\ -\partial_{yy} C(y,z)xe^{-t} - e^{-t}xA(y,z) = 0, \tag{3.18}$$

now we get a polynomial along to x. Since x is optional we infer $A = \partial_{yy}B = \partial_{yy}C = 0$. By the same process on the equation (3.15), $L(x,y,t)$ is infered

$$L(x,y,t) = -e^{-4t}(\partial_z A(y,z)\frac{x^3}{6} + \partial_z B(y,z)\frac{x^2}{2}) \\ -e^{-3t}(-\frac{1}{2}e^t\partial_z A(y,z) + \partial_z C(y,z)e^{-t})x + 2e^{2t}x\int A(y,z)dz + L_1(y,t), \tag{3.19}$$

for some smooth function L_1, by deriving of the equation (3.19) along to z, we have

$$-e^{-4t}(\partial_{zz} A(y,z)\frac{x^3}{6} + \partial_{zz} B(y,z)\frac{x^2}{2}) \\ -e^{-3t}(-\frac{1}{2}e^t\partial_{zz} A(y,z) + \partial_{zz} C(y,z)e^{-t})x + 2e^{2t}xA(y,z) = 0, \tag{3.20}$$

since (3.19) is a polynomial along to variable x, and x is optional, we find $\partial_{zz}B = \partial_{zz}C = 0$. By substituting (3.12), (3.13), and (2.4) in (3.6), we have

$$e^{-2t}(\partial_z K(x,z,t) - 2e^{3t}\int B(y,z)dz + e^t\partial_y L(x,y,t) + e^{3t}\int \partial_z B(y,z)dy = 0, \tag{3.21}$$

by derivation of the equation (3.21) along z, we obtain

$$\partial_{zz}K_1(z,t) - 2e^{3t}\partial_y B(y,z) = 0, \tag{3.22}$$

thus, we have a polynomial along to x. Since x is optional we infer $\partial_{zz}K_1 = \partial_y B = 0$. Therefore, we can be written $K(x,z,t)$ and $L(x,y,t)$ as follow

$$K(x,z,t) = -\partial_y C(y,z)xe^{-t} + K_1(z,t), \tag{3.23}$$

$$L(x,y,t) = -e^{-4t}\partial_z B(y,z)\frac{x^2}{2} - e^{-4t}\partial_z C(y,z)x + L_1(y,t). \tag{3.24}$$

By replacing $K(x,z,t)$ and $L(x,y,t)$ into the equation (3.21), we get a polynomial along to x. Since x is optional we get $\partial_z K_1 = \partial_y L_1 = \partial_{zy}B = \partial_{yz}C = 0$. Therefore, $K_1(z,t)$ and $L_1(y,t)$ are obtained as follows

$$K_1(z,t) = K_1(t)z + K_2(t), \tag{3.25}$$

$$L_1(y,t) = L_1(t)y + L_2(t). \tag{3.26}$$

So by substituting X^2 and X^4 into equation (3.7), the following equation is concluded

$$K_1(t) + K_1'(t) + e^t \partial_{yz} B(y,z) = 0, \tag{3.27}$$

thus, we deduce

$$K_1(t) = e^{-t} a_1, \tag{3.28}$$

$$K_2(t) = -\frac{1}{2} e^t \partial_y B(y,z) + e^{-t} a_2, \tag{3.29}$$

for some constant a_1 and a_2. Now (3.25) is written as follows

$$K_1(z,t) = e^{-t} z a_1 + e^{-t} a_2, \tag{3.30}$$

by derivation of the equation (3.30) along to z, we determine $a_1 = 0$. By replacing X^3 and X^4 into the equation (3.9) we have a polynomial along to x. Since x is optional, we deduce $\partial_z B = \partial_z C = 0$, and the following equations are obtained

$$L_1(y,t) = e^{2t} y a_3 + e^{2t} a_4, \tag{3.31}$$

for some constant a_3 and a_4, by derivation of the equation (3.31) along to y, we infer $a_3 = 0$. Because $\partial_{yz} = \partial_{yy} = 0$, then we have $\partial_y C(y,z) = a_5$, for some constant a_5, thus we deduce

$$C(y,z) = a_5 y + a_6, \tag{3.32}$$

for some constant a_6. Considering that $\partial_y B = \partial_z B = 0$, then $B = a_7$, for some constant a_7.

Briefly, we have

$$F = a_7, G = e^{-t}(a_5 y + a_6), K = a_5 x e^{-t} + e^{-t} a_2, L = e^{2t} a_4.$$

Consequently, X^1, X^2, X^3 and X^4 are listed as follows

$$\begin{aligned} X^1 &= e^{-t}(-x(-3+a_7) + y a_5 + a_6), \\ X^2 &= e^{-t}(y(-3+a_7) - a_5 x + a_2), \\ X^3 &= e^{2t}(-3z - 2a_7 z + a_4), \\ X^4 &= a_7. \end{aligned}$$

Therefore, the following theorem is stated:

Theorem 3.1. *The vector field X on (Sol_0^4, g) where g given by (2.4), is a Ricci soliton vector field if and only if*

$$X = (-x(-3+a_7)+y a_5+a_6)\frac{\partial}{\partial x} + (y(-3+a_7)-a_5 x+a_2)\frac{\partial}{\partial y} + (-3z-2a_7 z+a_4)\frac{\partial}{\partial z} + a_7 \frac{\partial}{\partial t}.$$

Now, we can investigate which of Ricci solitons on (Sol_0^4, g) is as gradient vector field. Also, consider $X = \nabla f$ on (Sol_0^4, g) with potential function f. Therefore, with respect to basis e_1, e_2, e_3, e_4 we have

$$\nabla f = e_1 f e_1 + e_2 f e_2 + e_3 f e_3 + e_4 f e_4.$$

From theorem (3.1), the Ricci soliton X on (Sol_0^4, g) is gradient vector field as ∇f if and only if

$$\begin{aligned}
\partial_x f &= e^{-2t}(-x(-3 + a_7) + y a_5 + a_6),\\
\partial_y f &= e^{-2t}(y(-3 + a_7) - a_5 x + a_2),\\
\partial_z f &= e^{4t}(-3z - 2a_7 z + a_4),\\
\partial_t f &= a_7.
\end{aligned} \tag{3.33}$$

Taking derivation of the first equation of the last system along to t we get $\partial_t \partial_x f = -2e^{-2t}(-x(-3+a_7)+ya_5+a_6)$. The derivation of the fourth equation of (3.33) along to x implies that $\partial_x \partial_t f = 0$. Therefore, from them, we deduce $a_5 = a_6 = 0, a_7 = 3$. By deriving the second equation and the fourth equation of (3.33) along to t and y, respectively, we get $a_2 = a_5 = 0, a_7 = 3$. Also, by deriving the third equation and the fourth equation along to t and z, respectively, we get $a_4 = 0, a_7 = -\frac{3}{2}$. Therefore, the gradient vector field on the Ricci soliton X on (Sol_0^4, g) has no solution.

Thus, we have the following corollary:

Corollary 3.1. *There is not any gradient Ricci soliton X on (Sol_0^4, g).*

4. Ricc bi-conformal vector fields on the model space Sol_0^4

In this section, we solve the equation (1.1) and (1.2) on the model space Sol_0^4. Replacing (2.4), (2.7), and (2.8) into (1.1), the following equations are obtained

$$-2X^4 + 2e_1 X^1 = \alpha, \tag{4.1}$$
$$e_1 X^2 + e_2 X^1 = 0, \tag{4.2}$$
$$e_1 X^3 + e_3 X^1 = 0, \tag{4.3}$$
$$X^1 + e_1 X^4 + e_4 X^1 = 0, \tag{4.4}$$
$$2e_2 X^2 - 2X^4 = \alpha, \tag{4.5}$$
$$e_2 X^3 + e_3 X^2 = 0, \tag{4.6}$$
$$X^2 + e_2 X^4 + e_4 X^2 = 0, \tag{4.7}$$
$$2e_3 X^3 + 4X^4 = \alpha, \tag{4.8}$$
$$-2X^3 + e_3 X^4 + e_4 X^3 = 0, \tag{4.9}$$
$$2e_4 X^4 = \alpha - 6\beta. \tag{4.10}$$

Also, substituting (2.4), (2.7), and (2.9) into (1.2), the following equations are obtained

$$-6e_1X^4 = 0, \tag{4.11}$$
$$-6e_2X^4 = 0, \tag{4.12}$$
$$-6e_3X^4 = 0, \tag{4.13}$$
$$-12e_4X^4 = -6\alpha + \beta. \tag{4.14}$$

In the following we solve the above equations. By integrating the equations (4.11), (4.12) and (4.13), X^4 is found

$$X^4 = F(t), \tag{4.15}$$

for some smooth function F. From the equations (4.10) and (4.14), we arrive at

$$\beta = 0. \tag{4.16}$$

So from (4.10), we get

$$F^{'}(t) = \frac{\alpha}{2}. \tag{4.17}$$

From (4.1), we obtain

$$X^1 = \frac{\alpha}{2}e^{-t}x + e^{-t}F(t)x + G(y, z, t), \tag{4.18}$$

for some smooth function G. Similarly, from (4.5), we calculate

$$X^2 = \frac{\alpha}{2}e^{-t}y + e^{-t}F(t)y + K(x, z, t), \tag{4.19}$$

for some smooth function K. Also from (4.8), we obtain

$$X^3 = \frac{\alpha}{2}e^{2t}z - 2e^{2t}F(t)z + L(x, y, t), \tag{4.20}$$

for some smooth function L. Now from (4.2), we deduce

$$\partial_x K(x, z, t) = -\partial_y G(y, z, t), \tag{4.21}$$

by derivation of the equation (4.21) along to x, we conclude

$$\partial_{xx} K(x, z, t) = 0, \tag{4.22}$$

also by derivation of the equation (4.21) along to y, we have

$$\partial_{yy} G(y, z, t) = 0, \tag{4.23}$$

then by taking integration of (4.22) and (4.23), the following relations are obtained

$$K(x, z, t) = A(z, t)x + B(z, t), \tag{4.24}$$
$$G(y, z, t) = C(z, t)y + D(z, t), \tag{4.25}$$

for some smooth functions A, B, C, and D. From equation (4.21), we get $C(z,t) = -A(z,t)$, so (4.25) can be rewritten as follow

$$G(y,z,t) = -A(z,t)y + D(z,t). \tag{4.26}$$

From (4.7), we obtain

$$\partial_t K(x,z,t) + K(x,z,t) = 0. \tag{4.27}$$

thus, we have

$$\partial_t A(z,t)x + \partial_t B(z,t) + A(z,t)x + B(z,t) = 0, \tag{4.28}$$

therefore, we calculate

$$A(z,t) = e^{-t} A_1(z), \tag{4.29}$$

$$B(z,t) = e^{-t} B_1(z), \tag{4.30}$$

for some smooth functions A_1, and B_1. Therefore, we can be written (4.24) as follow

$$K(x,z,t) = e^{-t}(A_1(z)x + B_1(z)). \tag{4.31}$$

From (4.4), we get

$$\partial_t G(y,z,t) + G(y,z,t) = 0. \tag{4.32}$$

thus, we have

$$\partial_t C(z,t)y + \partial_t D(z,t) + C(z,t)y + D(z,t) = 0, \tag{4.33}$$

therefore, we deduce

$$C(z,t) = e^{-t} C_1(z), \tag{4.34}$$

$$D(z,t) = e^{-t} D_1(z), \tag{4.35}$$

for some smooth functions C_1, and D_1. Therefore, we can be written (4.25) as follow

$$G(y,z,t) = e^{-t}(C_1(z)y + D_1(z)). \tag{4.36}$$

Now from the equation (4.9), the following relation is obtained

$$\partial_t L(x,y,t) - 2L(x,y,t) - e^{2t}\alpha z = 0, \tag{4.37}$$

thus, we have

$$L(x,y,t) = e^{-2t} L_1(x,y), \tag{4.38}$$

$$\alpha = 0, \tag{4.39}$$

for some smooth function L_1. Therefore, from (4.15) and (4.17), we infer

$$X^4 = a_1, \tag{4.40}$$

for some constant a_1. From (4.3), we arrive at

$$e^t \partial_x L(x, y, t) + e^{-2t} \partial_z G(y, z, t) = 0, \tag{4.41}$$

and derivation this with respect to x and z,respectively, we infer

$$\partial_{xx} L(x, y, t) = 0, \tag{4.42}$$

$$\partial_{zz} G(y, z, t) = 0, \tag{4.43}$$

therefore from (4.38), the following relations are deduced

$$L_1(x, y) = L_2(y)x + L_3(y). \tag{4.44}$$

for some smooth functions L_2, and L_3. Now by substituting (4.36) and (4.38) into (4.41), we get

$$e^{-t} \partial_x L_1 + e^{-3t}(C_1'(z)y + D_1'(z)) = 0, \tag{4.45}$$

thus, from (4.45), we get $C_1'(z) = D_1'(z) = \partial_x L_1(x, y) = L_2(y) = 0$. Therefore, we deduce

$$C_1(z) = a_2, \tag{4.46}$$

$$D_1(z) = a_3, \tag{4.47}$$

$$L_1(x, y) = L_3(y). \tag{4.48}$$

for some constant a_2 and a_3. Also, from (4.6), we have

$$e^{-t} \partial_y L_3(y) + e^t \partial_z A_1(z)x + e^t \partial_z B_1(z) = 0, \tag{4.49}$$

now we get a polynomial with respect to x. So we have

$$e^t \partial_z A_1(z) = 0, \tag{4.50}$$

$$e^{-t} \partial_y L_3(y) + e^t \partial_z B_1(z) = 0, \tag{4.51}$$

therefore, we get

$$A_1(z) = a_4, \tag{4.52}$$

$$B_1(z) = a_5, \tag{4.53}$$

$$L_3(y) = a_6, \tag{4.54}$$

for some constants a_4, a_5, and a_6. By substituting the obtained elements in (4.31), (4.36), and (4.38), we have

$$K(x, z, t) = e^{-t}(a_4 x + a_5), \tag{4.55}$$

$$G(y, z, t) = e^{-t}(a_2 y + a_3), \tag{4.56}$$

$$L(x, y, t) = e^{-2t} a_6, \tag{4.57}$$

and from (4.26), we have $a_4 = -a_2$.

Subsequently, $X^1, X^2, X^3, X^4, \alpha$ and β are listed as

$$\begin{aligned} X^1 &= (a_1 x + a_2 y + a_3)e^{-t}, \\ X^2 &= (a_1 y - a_2 x + a_5)e^{-t}, \\ X^3 &= -2(a_1 z + a_6)e^{2t}, \\ X^4 &= a_1, \\ \alpha &= \beta = 0. \end{aligned}$$

Therefore, the following theorem is stated:

Theorem 4.1. *The vector field X on (Sol_0^4, g) where g is given by (2.4), is Ricci bi-conformal vector field if and only if $\alpha = \beta = 0$ and*

$$X = (a_1 x + a_2 y + a_3)\frac{\partial}{\partial x} + (a_1 y - a_2 x + a_5)\frac{\partial}{\partial y} - 2(a_1 z + a_6)\frac{\partial}{\partial z} + a_1\frac{\partial}{\partial t}.$$

Now, consider $X = \nabla f$ on (Sol_0^4, g) with potential function f. Therefore,

$$\nabla f = e_1 f e_1 + e_2 f e_2 + e_3 f e_3 + e_4 f e_4.$$

From theorem (4.1), the Ricci bi-conformal vector field X on (Sol_0^4, g) is gradient vector field as ∇f if and only if

$$\begin{aligned} \partial_x f &= e^{-2t}(a_1 x + a_2 y + a_3), \\ \partial_y f &= e^{-2t}(a_1 y - a_2 x + a_5), \\ \partial_z f &= -2e^{4t}(a_1 z + a_6), \\ \partial_t f &= a_1. \end{aligned} \tag{4.58}$$

Taking the derivation of the first equation of the last system along to t, we get $\partial_t\partial_x f = -2e^{-2t}(a_1x+a_2y+a_3)$. The derivation of the fourth equation of (4.58) with respect to x implies that $\partial_x\partial_t f = 0$. Therefore, from them, we deduce $a_1 = a_2 = a_3 = 0$. By deriving the second equation and the fourth equation of (4.58) along to t and y, respectively, we get $a_1 = a_2 = a_5 = 0$. Thus, the third and the fourth equations of (4.58) along to t and z,respectively, becomes $\partial_t\partial_z f = -8e^{4t}(a_1 z + a_6)$ and $\partial_z\partial_t f = 0$, thus we have $a_1 = a_6 = 0$. Therefore (4.58) becomes

$$\partial_x f = \partial_y f = \partial_z f = \partial_t f = 0,$$

The direct integration yields to the following

$$f(x, y, z, t) = a_7.$$

for some constant a_7. As a result, we can state the following theorem:

Theorem 4.2. *Any Ricci bi-conformal vector field X on (Sol_0^4, g) is gradient vector field with potential function f if and only if $f = a_7$.*

In the end, we have:

Corollary 4.1. *Any Ricci bi-conformal vector field X on (Sol_0^4, g) is Killing vector field.*

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