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ON THE STABILITY OF GENERALIZED S-SPACE FORMS WITH TWO STRUCTURE VECTOR FIELDS

Crina-Daniela Neacşu[1] **and Gabriel-Eduard Vîlcu**[2]

[1] **Department of Mathematics and Informatics, Faculty of Applied Sciences University Politehnica of Bucharest, Bucharest, Romania**
[2] **Gheorghe Mihoc-Caius Iacob Institute of Mathematical Statistics and Applied Mathematics of the Romanian Academy, Bucharest, Romania**

ORCID IDs: Crina-Daniela Neacşu https://orcid.org/0000-0003-4622-7617
Gabriel-Eduard Vîlcu https://orcid.org/0000-0001-6922-756X

Abstract. The main purpose of this work is to derive the conditions that ensure the stability of the generalized S-space forms with two structure vector fields. In addition, some particular conditions under which a generalized S-space form with two structure vector fields is unstable are obtained. Several consequences are also discussed at the end of the article.
Keywords: generalized S-space forms, two structure vector fields.

1. Introduction

The concept of a generalized S-space form with two structure vector fields was introduced by Carriazo, Fernández and Fuentes in [10]. Roughly speaking, this is nothing but a metric f-manifold in the sense of Yano [45], equipped with two structure vector fields such that the Riemannian curvature tensor takes a certain form depending on eight functions. This important class of Riemannian spaces includes in particular the family of real, (generalized) complex and (generalized) Sasakian space forms, S-space forms and C-space forms. The geometry of these particular spaces is very interesting and many nice properties were derived in the last

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Corresponding Author: Gabriel-Eduard Vîlcu. E-mail addresses: crina.neacsu@upb.ro (C.-D. Neacşu), gabriel.vilcu@upb.ro (G.-E. Vîlcu)
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years (see, e.g., [3,4,11,15,21,22,26,31,37,39,42,43]. Note that generalized S-space forms with an arbitrary number of structure vector fields were also investigated in [9, 14, 28].

On the other hand, the notion of a harmonic map between Riemannian manifolds has been introduced by Eells and Sampson [12] as a generalisation of geodesics, these maps being defined as critical points of the Dirichlet energy. Harmonic maps are objects of high interests in physics, where they can be found under the appellation of generalized sigma models (see, e.g., [12, 13, 24], as well as the excellent review paper [35]). The harmonicity of some different kinds of maps between Riemannian spaces equipped with various remarkable geometric structures was investigated by many authors (see, e.g., [1,2,7,20,32–34,41,44]). In this context, a problem of prime importance is to investigate the stability property of harmonic maps (for example, see [5, 38, 40]). Recall at the moment that a harmonic map u is stable if the index of u, defined as the dimension of the largest subspace on which the Hessian of u is negative definite, is zero.

It is clear that the identity map 1_M of a Riemannian space (M, g_M) provides us one of the simplest examples of harmonic maps. If this map is stable, then the Riemannian space (M, g_M) is said to be stable. Otherwise, (M, g_M) is called unstable. Although 1_M has a simple form, the study of its stability is a non-trivial problem and a lot of interesting results can be found in the literature (see, e.g., [8, 16, 17, 19, 25, 27, 29–31, 36]) Very recently, Gherghe and the second author of the present paper proved in [18] a criterion for the stability of locally conformal almost cosymplectic manifolds of pointwise constant ϕ-holomorphic sectional curvature. Moreover, the first author of the present paper, established in [26] that a compact T-space form is unstable if the first eigenvalue of the Laplace-Beltrami operator has a certain upper bound.

Motivated by these works, we will investigate in the following the stability of the generalized S-space forms with two structure vector fields. We first obtain the conditions that ensure the stability of such spaces. Then we prove that in some particular conditions, a generalized S-space form with two structure vector fields is unstable. In the last part of the paper, we apply the previously found results to some particular classes of generalized S-space forms with two structure vector fields.

2. Preliminaries

2.1. Generalized S-space forms

A Riemannian manifold (M, g_M) of dimension $(2n+s)$ equipped with an f-structure, i.e. a $(1, 1)$ tensor field f of rank $2n$ on M satisfying $f^3 = -f$, is said to be a metric f-manifold if there exist s vector fields ξ_α, $\alpha = 1, ..., s$, on M, such that [45]

$$f\xi_\alpha = 0,\ \eta_\alpha \circ f = 0,\ f^2 = -I + \sum_{\alpha=1}^{s} \eta_\alpha \otimes \xi_\alpha \tag{2.1}$$

and the Riemannian metric g_M satisfies

$$g_M(X,Y) = g_M(fX,fY) + \sum_{\alpha=1}^{s} \eta_\alpha(X)\eta_\alpha(Y), \tag{2.2}$$

for any vector fields X, Y on M, where η_α is the 1-form dual to ξ_α $\alpha = 1, ..., s$. Hence

$$g_M(X, \xi_\alpha) = \eta_\alpha(X), \tag{2.3}$$

for $\alpha = 1, ..., s$.

Note that ξ_α, $\alpha = 1, ..., s$, are called the structure vector fields on M and due this a metric f-manifold is also named as a manifold with a metric f-structure with complemented frames ξ_α, $\alpha = 1, ..., s$, or as a metric f-manifold with s structure vector fields ξ_α, $\alpha = 1, ..., s$. Denoting by $\mathcal{M}$ the distribution spanned by the structure vector fields, then we have the decomposition

$$TM = \mathcal{L} \oplus \mathcal{M},$$

where $\mathcal{L}$ stands for the complementary orthogonal distribution of $\mathcal{M}$. It is clear from (2.3) that

$$\eta_\alpha(X) = 0 \tag{2.4}$$

for any $X \in \mathcal{L}$ and $\alpha = 1, ..., s$, while (2.1) implies

$$fX = 0 \tag{2.5}$$

for any $X \in \mathcal{M}$.

We recall now that a metric f-manifold (M, g_M) is called a K-manifold [6] if the 2-form Ω on M defined by

$$\Omega(X,Y) = g_M(X, fY),$$

is closed and the Nijenhuis tensor $N_f = [f, f]$ satisfies

$$N_f = -2\sum_{\alpha=1}^{s} \xi_\alpha \otimes d\eta^\alpha.$$

A K-manifold is said to be an S-manifold if $F = d\eta^\alpha$, for $\alpha = 1, ..., s$, and is said to be a C-manifold if η^α is closed, for $\alpha = 1, ..., s$ (for basic properties of K-manifolds, S-manifolds and C-manifolds see [6]). An S-manifold of constant f-sectional curvature is said to be an S-space form. Similarly, a C-manifold of constant f-sectional curvature is called a C-space form.

Suppose now we have a metric f-manifold (M, g_M) with two structure vector fields ξ_1, ξ_2. Such a manifold is said to be a generalized S-space form (with two structure vector fields) if there exists eight differentiable functions $F_1, ..., F_8$ on M

such that the Riemannian curvature tensor on M takes the form

$$
\begin{aligned}
R(X,Y)Z = {} & F_1\{g_M(Y,Z)X - g_M(X,Z)Y\} \\
& +F_2\{g_M(X,fZ)fY - g_M(Y,fZ)fX + 2g_M(X,fY)fZ\} \\
& +F_3\{\eta_1(X)\eta_1(Z)Y - \eta_1(Y)\eta_1(Z)X + g_M(X,Z)\eta_1(Y)\xi_1 - g_M(Y,Z)\eta_1(X)\xi_1\} \\
& +F_4\{\eta_2(X)\eta_2(Z)Y - \eta_2(Y)\eta_2(Z)X + g_M(X,Z)\eta_2(Y)\xi_2 - g_M(Y,Z)\eta_2(X)\xi_2\} \\
& +F_5\{\eta_1(X)\eta_2(Z)Y - \eta_1(Y)\eta_2(Z)X + g_M(X,Z)\eta_1(Y)\xi_2 - g_M(Y,Z)\eta_1(X)\xi_2\} \\
& +F_6\{\eta_2(X)\eta_1(Z)Y - \eta_2(Y)\eta_1(Z)X + g_M(X,Z)\eta_2(Y)\xi_1 - g_M(Y,Z)\eta_2(X)\xi_1\} \\
& +F_7\{\eta_1(X)\eta_2(Y)\eta_2(Z)\xi_1 - \eta_2(X)\eta_1(Y)\eta_2(Z)\xi_1\} \\
& +F_8\{\eta_2(X)\eta_1(Y)\eta_1(Z)\xi_2 - \eta_1(X)\eta_2(Y)\eta_1(Z)\xi_2\},
\end{aligned}
\tag{2.6}
$$

for any vector fields X, Y, Z on M.

As natural examples of generalized S-space forms with two structure vector fields we have real, (generalized) complex and (generalized) Sasakian space forms, S-space forms and C-space forms. We also have the following non-trivial examples constructed in [10]: pseudo-umbilical, totally contact-umbilical and totally umbilical hypersurfaces of a generalized Sasakian space form; principal toroidal bundles over a Kähler manifold and warped products of the real line and any generalized Sasakian space form. Note that other interesting examples of generalized S-space forms with two structure vector fields were constructed in [9].

2.2. Harmonic maps

Let (M, g_M) and (N, g_N) be two Riemannian manifolds and $u : (M, g_M) \to (N, g_N)$ a smooth map. The second fundamental form of the map u, denoted by α_u, is given by

$$\alpha_u(X,Y) = \widetilde{\nabla}_X u_* Y - u_* \nabla_X Y,$$

for any vector fields X, Y on M, where ∇ is the Riemannian connection on M and $\widetilde{\nabla}$ is the pullback of the Riemannian connection ∇' of N to the induced vector bundle $u^{-1}(TN)$. The trace of α_u, denoted by $\tau(u)$, is called the tension field of u. If $\tau(u) = 0$, then u is said to be a harmonic map. Equivalently, if M is compact, then the map u is harmonic if and only if for any smooth variation $\{u_t\}_{t\in(-\epsilon,\epsilon)}$ of u, with $u_0 = u$, one has

$$\frac{d}{dt}\mathcal{E}_t|_{t=0} = 0,$$

where $\mathcal{E}_t$ is the Dirichlet energy of u_t, that is

$$\mathcal{E}_t = \int_M e(u_t)\vartheta_{g_M},$$

where ϑ_{g_M} is the canonical measure associated with g_M and

$$e(u)_p = \frac{1}{2} trace(u^* g_N)_x, \ \forall x \in M.$$

Now, let $\{u_{s,t}\}_{s,t\in(-\epsilon,\epsilon)}$ be a smooth variation of u with two parameters s and t such that $u_{0,0}=u$. Then the Hessian of the map u is defined by:

$$Hess_u(V,W)=\frac{\partial^2}{\partial s\partial t}(\mathcal{E}(u_{s,t}))|_{(s,t)=(0,0)},$$

where $V,W\in\Gamma(u^{-1}(TN))$ are the associated variational vector fields.

Recall that the dimension of the largest subspace of $\Gamma(u^{-1}(TN))$ on which the Hessian of u is negative definite is called the index of u and denoted by $\mathfrak{i}(u)$. A harmonic map u having $\mathfrak{i}(u)=0$ is said to be stable. Otherwise, u is termed as unstable. For more details and basic properties of harmonic maps, see [5]. We only recall now the next formula obtained independently by Mazet and Smith [23, 36] which will be useful later:

$$Hess_u(V,W)=\int_M g_N(\mathcal{J}_u(V),W)\vartheta_{g_M}, \tag{2.7}$$

where J_u denotes the Jacobian operator of u given by [5]

$$\mathcal{J}_uV=-\sum_{i=1}^m\left(\widetilde{\nabla}_{E_i}\widetilde{\nabla}_{E_i}-\widetilde{\nabla}_{\nabla_{E_i}E_i}\right)V-\sum_{i=1}^m R^N(V,u_*E_i)u_*E_i, \tag{2.8}$$

for any $V\in\Gamma(u^{-1}(TN))$, where $\{E_1,...,E_m\}$ is a local orthonormal frame on M and R^N is the Riemannian curvature tensor of N.

Taking into account that the rough Laplacian $\bar{\Delta}_u$ of u is given by

$$\bar{\Delta}_uV=-\sum_{i=1}^m\left(\widetilde{\nabla}_{E_i}\widetilde{\nabla}_{E_i}-\widetilde{\nabla}_{\nabla_{E_i}E_i}\right)V \tag{2.9}$$

we obtain immediately from (2.8) that

$$\mathcal{J}_uV=\bar{\Delta}_uV-\sum_{i=1}^m R^N(V,u_*E_i)u_*E_i. \tag{2.10}$$

3. A stability criterion

The aim of this section is to obtain a stability criterion for generalized S-space forms with two structure vector fields.

Suppose $M=(M,g_M,f,\xi_1,\xi_2,F_1,...,F_8)$ is a compact generalized S-space form of dimension $(2n+2)$ and let 1_M be the identity map on M. Applying (2.7) for $u=1_M$ and using (2.8), we derive:

$$\begin{aligned} Hess_{1_M}(V,V) &= \int_M g_M(\mathcal{J}_{1_M}V,V)\vartheta_{g_M} \\ &= -\int_M\sum_{i=1}^n g_M((\widetilde{\nabla}_{E_i}\widetilde{\nabla}_{E_i}-\widetilde{\nabla}_{\nabla_{E_i}E_i})V,V)\vartheta_{g_M} \end{aligned}$$

$$
\begin{aligned}
& - \int_M \sum_{i=1}^{n} g_M(R^M(V,E_i)E_i,V)\vartheta_{g_M} \\
& - \int_M \sum_{i=1}^{n} g_M((\widetilde{\nabla}_{fE_i}\widetilde{\nabla}_{fE_i} - \widetilde{\nabla}_{\nabla_{fE_i}fE_i})V,V)\vartheta_{g_M} \\
& - \int_M \sum_{i=1}^{n} g_M(R^M(V,fE_i)fE_i,V)\vartheta_{g_M} \\
& - \int_M g_M((\widetilde{\nabla}_{\xi_1}\widetilde{\nabla}_{\xi_1} - \widetilde{\nabla}_{\nabla_{\xi_1}\xi_1})V,V)\vartheta_{g_M} \\
& - \int_M g_M(R^M(V,\xi_1)\xi_1,V)\vartheta_{g_M} \\
& - \int_M g_M((\widetilde{\nabla}_{\xi_2}\widetilde{\nabla}_{\xi_2} - \widetilde{\nabla}_{\nabla_{\xi_2}\xi_2})V,V)\vartheta_{g_M} \\
& - \int_M g_M(R^M(V,\xi_2)\xi_2,V)\vartheta_{g_M} \qquad (3.1)
\end{aligned}
$$

where $\{E_1,...,E_n,fE_1,..,fE_n,\xi_1,\xi_2\}$ is a local orthonormal frame on M. But, with respect to this frame, the rough Laplacian of 1_M is

$$
\begin{aligned}
\bar{\Delta}_{1_M}V = & -\sum_{i=1}^{n}(\widetilde{\nabla}_{E_i}\widetilde{\nabla}_{E_i} - \widetilde{\nabla}_{\nabla_{E_i}E_i})V \\
& -\sum_{i=1}^{n}(\widetilde{\nabla}_{fE_i}\widetilde{\nabla}_{fE_i} - \widetilde{\nabla}_{\nabla_{fE_i}fE_i})V \\
& -(\widetilde{\nabla}_{\xi_1}\widetilde{\nabla}_{\xi_1} - \widetilde{\nabla}_{\nabla_{\xi_1}\xi_1})V \\
& -(\widetilde{\nabla}_{\xi_2}\widetilde{\nabla}_{\xi_2} - \widetilde{\nabla}_{\nabla_{\xi_2}\xi_2})V. \qquad (3.2)
\end{aligned}
$$

Using now (3.2) in (3.1), it follows that we can rewtite $Hess_{1_M}(V,V)$ as follows:

$$
\begin{aligned}
Hess_{1_M}(V,V) = & \int_M g_M(\bar{\Delta}_{1_M}V,V)\vartheta_{g_M} \\
& - \int_M \sum_{i=1}^{n} g_M(R^M(V,E_i)E_i,V)\vartheta_{g_M} \\
& - \int_M \sum_{i=1}^{n} g_M(R^M(V,fE_i)fE_i,V)\vartheta_{g_M} \\
& - \int_M g_M(R^M(V,\xi_1)\xi_1,V)\vartheta_{g_M} \\
& - \int_M g_M(R^M(V,\xi_2)\xi_2,V)\vartheta_{g_M}. \qquad (3.3)
\end{aligned}
$$

Taking now $X = V$, $Y = E_i$, $Z = E_i$ in (2.6) and applying $g_M(\cdot,V)$ in the resulting formula, due to the fact that

$$g(E_i, E_i) = 1,\ g(E_i, fE_i) = 0$$

and taking account of (2.1)-(2.5), one obtains

$$\begin{aligned} g_M(R^M(V,E_i)E_i,V) &= F_1[g_M(V,V) - g_M^2(V,E_i)] + 3F_2 g_M^2(fV,E_i) \\ &\quad -F_3\eta_1^2(V) - F_4\eta_2^2(V) - (F_5+F_6)\eta_1(V)\eta_2(V). \end{aligned} \tag{3.4}$$

In a similar way, taking $X = V$, $Y = fE_i$, $Z = fE_i$ in (2.6) and applying $g_M(\cdot, V)$ in the resulting formula, one derives

$$\begin{aligned} g_M(R^M(V,fE_i)fE_i,V) &= F_1[g_M(V,V) - g_M^2(V,fE_i)] + 3F_2 g_M^2(fV,fE_i) \\ &\quad -F_3\eta_1^2(V) - F_4\eta_2^2(V) - (F_5+F_6)\eta_1(V)\eta_2(V). \end{aligned} \tag{3.5}$$

Now, we take $X = V$, $Y = \xi_1$, $Z = \xi_1$ in (2.6) and applying $g_M(\cdot, V)$ in the resulting formula, with the same arguments as in the previous computations, one finds

$$\begin{aligned} g_M(R^M(V,\xi_1)\xi_1,V) &= F_1[g_M(V,V) - g_M^2(V,\xi_1)] \\ &\quad +F_3[g_M^2(V,\xi_1) - g_M(V,V)] \\ &\quad -(F_4 - F_8)\eta_2^2(V). \end{aligned} \tag{3.6}$$

In a similar way, taking $X = V$, $Y = \xi_2$, $Z = \xi_2$ in (2.6) and applying $g_M(\cdot, V)$ in the resulting formula, one derives

$$\begin{aligned} g_M(R^M(V,\xi_2)\xi_2,V) &= F_1[g_M(V,V) - g_M^2(V,\xi_2)] \\ &\quad +F_4[g_M^2(V,\xi_2) - g_M(V,V)] \\ &\quad -(F_3 - F_7)\eta_1^2(V). \end{aligned} \tag{3.7}$$

Using now (3.4), (3.5), (3.6) and (3.7) in (3.3), one arrives at the next formula

$$\begin{aligned} Hess_{1_M}(V,V) &= \int_M g_M(\bar{\Delta}_{1_M}V,V)\vartheta_{g_M} \\ &\quad - \int_M \sum_{i=1}^{n} \{F_1[g_M(V,V) - g_M^2(V,E_i)] + 3F_2 g_M^2(fV,E_i) \\ &\quad -F_3\eta_1^2(V) - F_4\eta_2^2(V) - (F_5+F_6)\eta_1(V)\eta_2(V)\}\vartheta_{g_M} \\ &\quad - \int_M \sum_{i=1}^{n} \{F_1[g_M(V,V) - g_M^2(V,fE_i)] + 3F_2 g_M^2(fV,fE_i) \\ &\quad -F_3\eta_1^2(V) - F_4\eta_2^2(V) - (F_5+F_6)\eta_1(V)\eta_2(V)\}\vartheta_{g_M} \\ &\quad - \int_M \{F_1[g_M(V,V) - g_M^2(V,\xi_1)] \\ &\quad +F_3[g_M^2(V,\xi_1) - g_M(V,V)] - (F_4 - F_8)\eta_2^2(V)\}\vartheta_{g_M} \\ &\quad - \int_M \{F_1[g_M(V,V) - g_M^2(V,\xi_2)] \\ &\quad +F_4[g_M^2(V,\xi_2) - g_M(V,V)] - (F_3 - F_7)\eta_1^2(V)\}\vartheta_{g_M}. \end{aligned} \tag{3.8}$$

Combining and summing the terms having similar type in (3.8), in view of the fact that (see, e.g. [31, Theorem 1])

$$\int_M g_M(\bar{\Delta}_{1_M} V, V)\vartheta_{g_M} = \int_M g_M(\widetilde{\nabla}_V, \widetilde{\nabla}_V)\vartheta_{g_M}, \tag{3.9}$$

we obtain the following expression for $Hess_{1_M}(V,V)$:

$$\begin{aligned} Hess_{1_M}(V,V) &= \int_M g_M(\widetilde{\nabla}_V, \widetilde{\nabla}_V)\vartheta_{g_M} \\ &+ \int_M [-(2n+1)F_1 - 3F_2 + F_3 + F_4] g_M(V,V)\vartheta_{g_M} \\ &+ \int_M [2nF_3 + 3F_2 - F_7]\eta_1^2(V)\vartheta_{g_M} \\ &+ \int_M [2nF_4 + 3F_2 - F_8]\eta_2^2(V)\vartheta_{g_M} \\ &+ 2n\int_M (F_5 + F_6)\eta_1(V)\eta_2(V)\vartheta_{g_M}. \end{aligned} \tag{3.10}$$

and therefore we derive that the index $\mathfrak{i}(1_M)$ of the identity map 1_M of M is zero if the next conditions are satisfied:

(A) $-(2n+1)F_1 - 3F_2 + F_3 + F_4 \geq 0$;

(B) $2nF_3 + 3F_2 - F_7 \geq 0$;

(C) $2nF_4 + 3F_2 - F_8 \geq 0$;

(D) $F_5 + F_6 = 0$.

Therefore, we proved the following result.

Theorem 3.1. *Let $(M, g_M, f, \xi_1, \xi_2, F_1, ..., F_8)$ be a $(2n+2)$-dimensional compact generalized S-space form with two structure vector fields. If the eight defining functions $F_1, ..., F_8$ satisfy relations (A), (B), (C) and (D), then M is stable.*

4. Instability conditions

The aim of the current section is to derive conditions under which a generalized S-space form with two structure vector fields is unstable.

We start by recalling the well-known Weitzenböck formula. If $\mathcal{B}$ is a vector bundle over a Riemannian manifold (M, g_M) and η is a 1-form, then we have [17,31]:

$$\Delta_1\eta = \bar{\Delta}\eta - \rho(\eta),$$

where Δ_1 is the Laplacian of $\mathcal{B}$-valued 1-forms, $\bar{\Delta}$ denotes the rough Laplacian on 1-forms, while $\rho(\eta)$ is given by

$$\rho(\eta)(X) = \sum_{i=1}^{m} R^M(X, U_i)(\eta(U_i)) - \sum_{i=1}^{m} \eta(R^M(X, U_i)U_i),$$

for any vector field X on M, where $\{U_1, \ldots, U_m\}$ is an orthonormal frame on M.

Now, let us suppose that $M = (M, g_M, f, \xi_1, \xi_2, F_1, ..., F_8)$ is a compact generalized S-space form with two structure vector fields, having dimension $2n+2$. By applying Weitzenböck formula for $\mathcal{B} = M \times \mathbb{R}$, we obtain:

$$\begin{aligned} \Delta_1 V &= \bar{\Delta} V + \sum_{i=1}^{n} R^M(V, E_i)E_i \\ &\quad + \sum_{i=1}^{n} R^M(V, fE_i)fE_i + R^M(V, \xi_1)\xi_1 + R^M(V, \xi_2)\xi_2. \end{aligned} \tag{4.1}$$

where $\{E_1, ..., E_n, fE_1, ..., fE_n, \xi_1, \xi_2\}$ is a local orthonormal frame on M.

Using now (4.1) in (3.3), we obtain

$$\begin{aligned} Hess_{1_M}(V, V) &= \int_M g_M(\Delta_1 V, V)\vartheta_{g_M} \\ &\quad -2\int_M \sum_{i=1}^{n} g_M(R^M(V, E_i)E_i, V)\vartheta_{g_M} \\ &\quad -2\int_M \sum_{i=1}^{n} g_M(R^M(V, fE_i)fE_i, V)\vartheta_{g_M} \\ &\quad -2\int_M g_M(R^M(V, \xi_1)\xi_1, V)\vartheta_{g_M} \\ &\quad -2\int_M g_M(R^M(V, \xi_2)\xi_2, V)\vartheta_{g_M}. \end{aligned} \tag{4.2}$$

But it is known that (see, e.g. [18])

$$\int_M g_M(\Delta_1 V, V)\vartheta_{g_M} = \lambda_1 \int_M g_M(V, V)\vartheta_{g_M},$$

where λ_1 denotes the first eigenvalue of the Laplace-Beltrami operator. In view of the above equation, (4.2) reduces to

$$\begin{aligned} Hess_{1_M}(V, V) &= \lambda_1 \int_M g_M(V, V)\vartheta_{g_M} \\ &\quad -2\int_M \sum_{i=1}^{n} g_M(R^M(V, E_i)E_i, V)\vartheta_{g_M} \end{aligned}$$

$$-2\int_M \sum_{i=1}^{n} g_M(R^M(V,fE_i)fE_i,V)\vartheta_{g_M}$$
$$-2\int_M g_M(R^M(V,\xi_1)\xi_1,V)\vartheta_{g_M}$$
$$-2\int_M g_M(R^M(V,\xi_2)\xi_2,V)\vartheta_{g_M}. \tag{4.3}$$

Using (3.4), (3.5), (3.6) and (3.7) in (4.3), we get

$$\begin{aligned} Hess_{1_M}(V,V) &= \int_M \{\lambda_1 + 2[-(2n+1)F_1 - 3F_2 + F_3 + F_4]\} g_M(V,V)\vartheta_{g_M} \\ &+2\int_M [2nF_3 + 3F_2 - F_7]\eta_1^2(V)\vartheta_{g_M} \\ &+2\int_M [2nF_4 + 3F_2 - F_8]\eta_2^2(V)\vartheta_{g_M} \\ &+4n\int_M (F_5+F_6)\eta_1(V)\eta_2(V)\vartheta_{g_M}. \end{aligned} \tag{4.4}$$

Based on (4.4), we are able now to prove the next result providing instability conditions for a generalized S-space form with two structure vector fields.

Theorem 4.1. *Let $(M, g_M, f, \xi_1, \xi_2, F_1, ..., F_8)$ be a $(2n+2)$-dimensional compact generalized S-space form with two structure vector fields. If the first eigenvalue λ_1 of the Laplace-Beltrami operator and the eight defining functions $F_1, ..., F_8$ satisfy the next three relations*

(E) $\lambda_1 < 2(2n+1)F_1 + 6F_2 - 2F_3 - 2F_4$,

(F) $(2F_3 + |F_5 + F_6|)n + 3F_2 - F_7 \leq 0$,

(G) $(2F_4 + |F_5 + F_6|)n + 3F_2 - F_8 \leq 0$,

then M is unstable.

Proof. Using the elementary inequality

$$(F_5+F_6)\eta_1(V)\eta_2(V) \leq \frac{1}{2}|F_5+F_6|[\eta_1^2(V)+\eta_2^2(V)]$$

in (4.4), we derive

$$\begin{aligned} Hess_{1_M}(V,V) &\leq \int_M \{\lambda_1 + 2[-(2n+1)F_1 - 3F_2 + F_3 + F_4]\} g_M(V,V)\vartheta_{g_M} \\ &+2\int_M [2nF_3 + 3F_2 - F_7 + n|F_5+F_6|]\eta_1^2(V)\vartheta_{g_M} \\ &+2\int_M [2nF_4 + 3F_2 - F_8 + n|F_5+F_6|]\eta_2^2(V)\vartheta_{g_M}. \end{aligned} \tag{4.5}$$

Taking account of relations (E), (F) and (G), we conclude easily from (4.5) that

$$Hess_{1_M}(V,V) < 0.$$

Thus, we have $i_{1_M} \neq 0$ and it is clear that M is unstable. □

5. Consequences and further developments

In this section we will discuss the applicability of Theorems 3.1 and 4.1 for two particular classes of generalized S-space forms with two structure vector fields.

Remark 5.1. It is known that any S-space form M of constant f-sectional curvature c with two structure vector fields is a generalized S-space form. In this case, we have (see [10, page 210])

$$F_1 = \frac{1}{4}(c+6),\ F_2 = F_7 = F_8 = \frac{1}{4}(c-2),\ F_3 = F_4 = \frac{1}{4}(c+2),\ F_5 = F_6 = -1.$$

Since $F_5 + F_6 = -2 \neq 0$, it is clear that one cannot apply Theorem 3.1 as condition (D) is not valid. On the other hand, a simple computation shows us that condition (E) in Theorem 4.1 is equivalent to

$$\lambda_1 < c(n+1) + 6n - 2, \tag{5.1}$$

while each of the conditions (F) and (G) is equivalent to $c \leq -2n$. Hence, we have the next result.

Theorem 5.1. *If M is an $(2n+2)$-dimensional compact S-space form of constant f-sectional curvature $c \leq -2n$ with two structure vector fields such that first eigenvalue λ_1 of the Laplace-Beltrami operator satisfies (5.1), then M is unstable.*

Remark 5.2. It is known that any C-space form M with two structure vector fields is a generalized S-space form. In this case, if c is the constant f-sectional curvature of M, then we have (see [10, page 210])

$$F_1 = ... = F_4 = \frac{c}{4},\ F_5 = F_6 = 0,\ F_7 = F_8 = \frac{c}{4}.$$

Now, we can see immediately that condition (D) in Theorem 3.1 is automatically satisfied. On the other hand, one can directly checked that condition (A) in Theorem 3.1 is equivalent to $c \leq 0$, while each of the conditions (B) and (C) is equivalent to $c \geq 0$. Hence, we have the following result.

Theorem 5.2. *Any compact C-space form of vanishing f-sectional curvature with two structure vector fields is stable.*

On the other hand, if we focuss on the conditions appearing in Theorem 4.1, we can easily see that condition (E) *is equivalent to*

$$\lambda_1 < c(n+1), \tag{5.2}$$

while each of the conditions (F) *and* (G) *is equivalent to* $c \leq 0$. *Hence, we have the next result.*

Theorem 5.3. *If* M *is an* $(2n+2)$*-dimensional compact* C*-space form of constant* f*-sectional curvature* $c \leq 0$ *with two structure vector fields such that first eigenvalue* λ_1 *of the Laplace-Beltrami operator satisfies (5.2), then* M *is unstable.*

Remark 5.3. An open problem for further research is to investigate the stability of generalized S-space forms with an arbitrary number of structure vector fields. For definition, examples and basic properties of these spaces, see [28].

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