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CHARACTERISTICS OF RANDERS METRICS OF ISOTROPIC PROJECTIVE RICCI CURVATURE

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Abstract. In this paper, we delve into the exploration of projective Ricci curvature, with a specific focus on characterizing Finsler metrics possessing isotropic projective Ricci curvature and isotropic **S**-curvature. Notably, our investigation reveals a compelling result: every Randers metric featuring isotropic **S**-curvature and constant projective Ricci curvature emerges as a weak Einstein metric. Furthermore, we pinpoint the conditions under which such a metric exhibits isotropic projective Ricci curvature. Remarkably, on a closed Einstein Randers manifold, we establish that being **PRic**-flat is equivalent to being **Ric**-flat. This intriguing equivalence sheds light on the intricate interplay between projective and Riemannian geometry, offering valuable insights into the geometric structures underlying Finsler metrics.

Keywords: Finsler metric, Randers metric, Projective Ricci curvature, Weak Einstein metrics.

1. Introduction

Two connections on the smooth manifold M that share the same geodesics (as unparameterized curves) are termed projectively equivalent. This implies that for

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every geodesic $\gamma(t)$ of any of these connections, one can find a reparametrization $\gamma(\phi(t))$ that is a geodesic of the other connection. In Riemannian geometry, it is more common to compare connections in the family of Levi-Civita connections corresponding to Riemannian metrics on the same manifold. However, in a general setting, one can also consider the projective transformation of a Riemannian structure as being a Riemannian metric on a different underlying manifold.

Locally, any geodesic is identified as the unique solution of a system of second-order differential equations, involving connection components These reduce to equations involving spray coefficients when restricted to the family of Finsler metrics. The reason is that, components $\Gamma^k_{ij}(x, y)$ of Finsler connections (Berwald, Cartan, or Chen connection) satisfy the relation $\Gamma^k_{ij}(x, y)y^iy^j = G^k(x, y)$, where G^k are homogeneous functions of degree 2 with respect to y, defining the canonical spray induced by the Finsler metric F. Thus, the condition for being projectively related translates into a relation between spray coefficients G^k and $\widetilde{G}^k$ corresponding to Finsler metrics F and $\widetilde{F}$, respectively.
This reformulation allows us to generalize the concept of being projectively related to the family of sprays, characterizing two sprays as projectively related whenever their coefficients satisfy a specific relation. Instead of solely examining the projective transformation of the Finsler metric F, we explore the entire orbit of the spray G induced by F under projective deformations, which includes sprays that may not correspond to any Finsler metrics.

While our primary interest lies in comparing geodesics induced from a metric structure on the manifold, we extend our scope to include geodesics defined by sprays.

This extended viewpoint paves the way to discover more projectively invariant objects, such as Douglas curvature, Weyl curvature [2], generalized Douglas-Weyl curvature [3], and another projective invariant defined by Akbar-Zadeh in [1]. We refer to [13], [17] for yet another set of special projective invariants.

These projectively invariant objects mainly correspond to the spray G, either directly or through different connections defined with respect to G. However, Z. Shen introduced a spray $\widetilde{G}$, corresponding to G and the **S**-curvature, in [16] which is uniquely determined in each projective class. This implies that the so-called *projective spray* $\widetilde{G}$ is invariant under the projective deformation of G. The Ricci curvature of the projective spray $\widetilde{G}$, termed *projective Ricci curvature* and denoted by **PRic**, then poses as a new projective invariant characterizing Finsler manifolds with respect to some geometric properties.

For instance, in [6], Cheng et al. characterized **PRic**-flat Randers metrics and **PRic**-flat Randers metrics with isotropic **S**-curvature, later corrected in [5]. A Finsler metric F is said to be projective Ricci flat (i.e. **PRic**-flat) if the projective Ricci curvature of F vanishes, and is said to have isotropic **S**-curvature if $\mathbf{S} = (n + 1)c(x)F$ for some scalar function c on M.

PRic-flat spherically symmetric Finsler metrics and **PRic**-flat square metrics were characterized in [19, 20].

Recently, Rezaei et al. studied Randers metrics of isotropic **PRic**-curvature, c.f. [11]. According to them, a Finsler metric F is of isotropic (resp. constant) **PRic**-curvature if $\mathbf{PRic} = (n-1)kF^2$ for some function (resp. constant function) k on M.

In this paper, concentrating on Randers metrics of isotropic **S**-curvature, we obtain equations that characterize these metrics and then identify weak Einstein metrics in this class by their **PRic**-curvature. More precisely, we prove the following theorem:

Theorem 1.1. *Let F be a Finsler metric of isotropic **S**-curvature on a manifold M. Then F is of isotropic **PRic**-curvature if and only if F is weak Einstein.*

A Finsler metric F on an n-dimensional manifold M is called a weak Einstein metric if its Ricci curvature satisfies the following equation

$$\mathbf{Ric}_F = (n-1)\left(\frac{3\theta}{F} + \sigma\right)F^2, \tag{1.1}$$

where σ is a scalar function and $\theta = \theta_i y^i$ is a $1-$form on M.

Corollary 1.1. *Let F be a weak Einstein Randers metric with $1-$form $\theta = \theta_i y^i$ and scalar function $\sigma = \sigma(x)$. Then F has isotropic **PRic**-curvature with scalar function $l = l(x)$ if and only if θ and σ satisfy the following equations*

$$\begin{aligned} \sigma(x) &= l(x) - c^2(x), \\ \theta &= -\tfrac{1}{3}c_0. \end{aligned}$$

In [14], C. Robles shows that every non-Riemannian Einstein Randers metric on a closed manifold is $\mathbf{Ric}_F$-flat if and only if it is Berwaldian.

In this paper, we generalize this result to the case of **PRic**-flat Einstein Randers metrics and prove the following statement

Theorem 1.2. *Let (M,F) be a connected, closed Einstein Randers manifold with Ricci scalar $\mathbf{Ric}_F$, and **PRic**-flat. Then F is either Riemannian metric or $\mathbf{Ric}_F$-flat.*

Using this result, we conclude

Corollary 1.2. *Let $F = \alpha + \beta$ be a non-Riemannian Einstein Randers metric on a closed manifold M. Then F is **PRic**-flat if and only if it is $\mathbf{Ric}_F$-flat.*

2. Preliminaries

A Finsler structure on M is a function $F : TM \to [0,\infty)$ with the following properties:

(a) F is C^∞ on TM_0, where $\pi : TM_0 \to M$ is the slit tangent bundle;

(b) F is positively homogeneous of degree one in y, that is

$$F(x,\lambda y)=\lambda F(x,y),\forall\lambda>0$$

(c) The Hessian matrix of F^2, i.e. $(g_{ij})=\Big(1/2[\frac{\partial^2}{\partial y^i\partial y^j}F^2]\Big)$, is positive definite on TM_0.

A Finsler manifold is then a pair consisting of a (smooth) manifold M and a Finsler structure F on M.
For each $y\in T_xM_0$, a non-degenerate bilinear form g_y on T_xM is defined by

$$g_y(u,v):=\frac{1}{2}\frac{\partial^2}{\partial s\partial t}|_{s=t=0}F^2(y+su+tv),\ \forall u,v\in T_xM.$$

In a local coordinates, $g_y(u,v)=g_{ij}(x,y)u^iv^j$, where $u=u^i\frac{\partial}{\partial x^i}$ and $v=v^i\frac{\partial}{\partial x^i}$. This defines a Riemannian metric on the vector bundle π^*TM.
The energy functional corresponding to the Finsler metric F is given by

$$\mathcal{E}_F(\alpha)=\int_0^1 F^2(\dot\alpha(t))dt,$$

with extremal points satisfying the Euler-Lagrange equations

$$E_i(F^2):=\frac{\partial F^2}{\partial x^i}-\frac{d}{dt}\frac{\partial F^2}{\partial y^i}=0. \tag{2.1}$$

These extremal points coincide with the trajectories of the geodesic spray:

$$G=y^i\frac{\partial}{\partial x^i}-2G^i\frac{\partial}{\partial y^i},$$

where spray coefficients G^i are given by

$$G^i:=\frac{1}{4}g^{il}\Big\{[F^2]_{x^ky^l}y^k-[F^2]_{x^l}\Big\},$$

c.f. [12].
Various affine connections in Finsler geometry are defined in a way that their Christoffel symbols $\bar\Gamma^k{}_{ij}$ satisfy

$$\bar\Gamma^k{}_{ij}(x,y)y^iy^j=G^k(x,y).$$

The Christoffel symbols split into two components $\Gamma^k{}_{ij}(x,y)$ and $\bar B^k{}_{ij}(x,y)$, with the formal component defined as:

$$\Gamma^k{}_{ij}(x,y)=\frac{1}{2}g^{kl}\left(\frac{\partial g^{il}}{\partial x^j}+\frac{\partial g^{jl}}{\partial x^i}-\frac{\partial g^{ij}}{\partial x^l}\right)(x,y),$$

and the second component $\bar{B}^k{}_{ij}(x,y)$ vanishes by contracting with y^iy^j.

In the spacial case where F corresponds to the norm of a Riemannian metric tensor g, the spray coefficients are determined by the Christoffel symbols $\Gamma^i{}_{jk}$ via $G^i(x,y) = \frac{1}{2}\Gamma^i{}_{jk}(x)y^jy^k$.

The Riemannian curvature is represented by a family of linear maps,

$$\{\mathbf{R}_y : T_xM^n \longrightarrow T_xM^n,\ y \in T_0M\}$$

defined on a local chart as

$$\mathbf{R}_y = R^i{}_j dx^j \otimes \frac{\partial}{\partial x^i}|_x,$$

where the components $R^i{}_j$ are given by

$$R^i{}_j \quad = \quad 2\frac{\partial G^i}{\partial x^j} - y^k\frac{\partial^2 G^i}{\partial x^j \partial x^k} + 2G^k\frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^k}\frac{\partial G^k}{\partial y^j}.$$

The Ricci curvature is the trace of the Riemann curvature and is defined by

$$\mathbf{Ric} \quad = \quad R^m_{\ m}.$$

A well-known non-Riemannian object associated with the Finsler metric F is the $\mathbf{S}$-curvature, expressed as

$$\mathbf{S} = \frac{\partial G^m}{\partial y^m} - y^m\frac{\partial}{\partial x^m}\Big[\ln \sigma_{BH}\Big], \tag{2.2}$$

where σ_{BH} is the component of the Busemann-Hausdorff volume form $dV_F = \sigma_{BH}(x)\ dx^1\ \wedge\cdots\wedge\ dx^n$, c.f. [15].
F is termed to have isotropic $\mathbf{S}$-curvature if there exists a scalar function $c = c(x)$ on M such that $\mathbf{S} = (n+1)cF$. If c is constant, F is said to have constant $\mathbf{S}$-curvature.
The projective spray $\widetilde{G}$ associated with a spray G on an n-dimensional Finsler manifold (M, F) is defined as

$$\widetilde{G} \quad = \quad G + \frac{2\mathbf{S}}{n+1}Y,$$

where $\mathbf{S}$ denotes the $\mathbf{S}$-curvature of F and $Y := y^i\frac{\partial}{\partial y^i}$ is a vertical vector field on TM. The vector files $\widetilde{G}$ is a projective invariant on TM_0. The Ricci curvature of $\widetilde{G}$, referred to as the projective Ricci curvature and denoted by $\mathbf{PRic}$, is given by

$$\mathbf{PRic} \quad := \quad \mathbf{Ric}_F + \frac{n-1}{n+1}\mathbf{S}_{|i}y^i + \frac{n-1}{(n+1)^2}\mathbf{S}^2, \tag{2.3}$$

where the vertical line "|" indicates the horizontal derivative with respect to the Berwald connection of F.

The concept of projective Ricci curvature, which is projectively invariant according to the fixed volume form, was introduced by Z. Shen in [16].
In the case where $\mathbf{PRic} = 0$, the Finsler metric F is termed projective Ricci flat. By relaxing the assumption, we define

Definition 2.1. Let F be a Finsler metric on an n-dimensional smooth manifold M. Then

(a) F is said to have isotropic **PRic**-curvature if

$$\mathbf{PRic} = (n-1)l(x)F^2, \tag{2.4}$$

where $l = l(x)$ is a scalar function on M;

(b) F is of constant **PRic**-curvature if

$$\mathbf{PRic} = (n-1)kF^2, \tag{2.5}$$

where $k \in \mathbb{R}$ is a constant function on M.

The class of (α, β)-metrics forms a special and important subset of Finsler metrics, comprising those Finsler functions F expressed in the form

$$F := \alpha\phi(s), \quad s = \frac{\beta}{\alpha}$$

where

i) $\alpha := \alpha(x, y) = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric,

ii) $\beta := \beta(y) = b_i(x)y^i$ is a $1-$form with $||\beta||_\alpha < b_0$ on M,

iii) ϕ is a smooth positive function on some open symmetric interval $(-b_0, b_0)$ satisfying the following condition:

$$\phi(s) > 0, \quad \phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, |s|^2 \leq b^2 < b_0^2, \tag{2.6}$$

where $b^2 := ||\beta||_\alpha^2$.

In the following, we use some common notations for (α, β)-metrics and define

$$r_{ij} := \frac{1}{2}(b_{i;j} + b_{j;i}), \quad s_{ij} := \frac{1}{2}(b_{i;j} - b_{j;i}),$$

where ";" denotes the covariant derivative with respect to the Levi-Civita connection of α. Further, we have

$$r^i{}_j := a^{im}r_{mj}, \quad s^i{}_j := a^{im}s_{mj}, \quad r_j := b^m r_{mj}, \; s_j := b^m s_{mj},$$

$$q_{ij} := r_{im}s^m_{\ j}, \quad t_{ij} := s_{im}s^m_{\ j}, \quad q_j := b^i q_{ij} = r_m s^m_{\ j}, \quad t_j := b^i t_{ij} = s_m s^m_{\ j}, \quad t^m_{\ m} := s^i_{\ j} s^j_{\ i}.$$

Throughout this paper, the matrix (a^{ij}) is the inverse of the matrix (a_{ij}), $b := ||\beta||_\alpha$ and $\rho := ln\sqrt{1-b^2}$.
As is customary in tensor computations in Finsler geometry, zero in the lower index indicates the contraction with the tangential coordinate. For instance,

$$r_{i0} := r_{ij}y^j, \ s_{i0} := s_{ij}y^j, \ r_{00} := r_{ij}y^i y^j, \ r_0 := r_i y^i, \ s_0 := s_i y^i.$$

The spray coefficients of (α,β)-metrics, as derived in [8], can be expressed as

$$G^i = {}^\alpha G^i + \alpha Q s^i_{\ 0} + \Theta(r_{00} - 2\alpha Q s_0)\frac{y^i}{\alpha} + \Psi(r_{00} - 2\alpha Q s_0)b^i,$$

where

$$Q = \frac{\phi'}{\phi - s\phi'}, \ \Theta = \frac{(\phi - s\phi')\phi' - s\phi\phi''}{2\phi[\phi - s\phi' + (b^2 - s^2)\phi'']}, \ \Psi = \frac{\phi''}{2[\phi - s\phi' + (b^2 - s^2)\phi'']}.$$

For details about the projective Ricci curvature of (α,β)-metrics, we refer to [19, 10]. In particular, for a Randers metric, a special case of (α,β)-metrics with $\phi(s) = 1+s$, the spray coefficients of F and α are related as per [7]

$$G^i = G^i_\alpha + \alpha s^i_{\ 0} + \frac{-2\alpha s_0 + r_{00}}{2(\alpha+\beta)}y^i. \tag{2.7}$$

In [5, 6], the projective Ricci curvature of a Randers metric with isotropic **S**-curvature is given by

$$\mathbf{PRic} = \mathbf{Ric}_\alpha + 2\alpha s^m_{\ 0;m} - 2t_{00} - \alpha^2 t^m_{\ m} + (n-1)\big[2\alpha(\rho_m s^m_{\ 0}) - \rho_{0;0} + \rho_0^2\big]. \tag{2.8}$$

We recall that a Finsler metric F on an n-dimensional manifold M is classified as a weak Einstein metric if it conforms to the equation (1.1) governing the Ricci curvature. Specifically, F is termed an Einstein metric if $\theta = 0$ in (1.1), expressed as

$$\mathbf{Ric}_F = (n-1)\sigma F^2. \tag{2.9}$$

In particular, a Finsler metric F is labeled as having constant Ricci if it satisfies(2.9) with a constant σ. In the context of [7], a Randers metric $F = \alpha + \beta$ on an n-dimensional manifold M qualifies as a weak Einstein metric if and only if α, β meet the conditions defined by the following equations

$$\begin{aligned} \mathbf{Ric}_\alpha &= (n-1)\big[(\sigma - 3c^2)\alpha^2 + (\sigma + c^2)\beta^2 + (3\theta - c_0)\beta - s_{0;0} - s_0^2\big] \\ &\quad + 2t_{00} + \alpha^2 t^m_m, \end{aligned} \tag{2.10}$$

$$s^m_{\ 0;m} = (n-1)\big[(\sigma + c^2)\beta + 2cs_0 + t_0 + \frac{3\theta + c_0}{2}\big], \tag{2.11}$$

$$r_{00} = -2s_0\beta + 2c(\alpha^2 - \beta^2), \tag{2.12}$$

where c is a scalar function and $c_0 = c_{x^i} y^i$.

Consequently, according to (2.12), F possesses isotropic **S**-curvature. As a special case, we can derive the necessary and sufficient conditions for a Randers metric to be an Einstein metric.

3. Constant PRic-curvature of Randers metrics of isotropic S-curvature

In this section, we initially derive a formula for the projective Ricci curvature of Randers metrics with isotropic **S**-curvature. Subsequently, we demonstrate that **PRic**-flat Randers metrics with isotropic **S**-curvature represent a special case of Randers metrics exhibiting constant **PRic**-curvature.

Now, consider a Randers metric $F = \alpha + \beta$ with isotropic **S**-curvature, where $\mathbf{S} = (n+1)c(x)F$. According to [6], α and β then satisfy the following equation

$$r_{00} + 2\beta s_0 = 2c(\alpha^2 + \beta^2), \tag{3.1}$$

which can be expressed as

$$r_{ij} = -b_i s_j - b_j s_i + 2c(a_{ij} - b_i b_j). \tag{3.2}$$

From (3.2), we deduce

$$r_i = -b^2 s_i + 2c(1 - b^2) b_i, \tag{3.3}$$

$$r_0 = -b^2 s_0 + 2c(1 - b^2)\beta. \tag{3.4}$$

Furthermore, we have

$$r_{0;0} = -b^2 s_{0;0} + 2(1 - b^2)\big(- s_0^2 - 6c\beta s_0 + c_0\beta + 2c^2\alpha^2 - 6c^2\beta^2\big), \tag{3.5}$$

$$q_i = -b^2 t_i + 2c(1 - b^2) s_i, \tag{3.6}$$

$$q_0 + t_0 = (1 - b^2)(t_0 + 2cs_0). \tag{3.7}$$

By substituting (3.4), (3.5) and (3.7) into (2.8), we arrive at the following formula for the projective Ricci curvature of Randers metrics with isotropic **S**-curvature:

$$\begin{aligned} \mathbf{PRic} &= \mathbf{Ric}_\alpha + 2\alpha s^m{}_{0;m} - 2t_{00} - \alpha^2 t^m{}_m \\ &+ (n-1)\big[- 2\alpha(t_0 + 2cs_0) + s_{0;0} + s_0^2 + 2c_0\beta + 4c^2\alpha^2\big]. \end{aligned} \tag{3.8}$$

Consider Randers metrics with isotropic **S**-curvature and constant **PRic**-curvature. Substitute (2.5) into (3.8) and organize the resulting equation with respect to α as follows

$$A_2\alpha^2 + A_1\alpha + A_0 = 0, \tag{3.9}$$

where

$$A_2 = -t^m{}_m + (n-1)\big[4c^2 - k\big], \tag{3.10}$$

$$A_1 = 2s^m{}_{0;m} - 2(n-1)\big[t_0 + 2cs_0 + k\beta\big], \tag{3.11}$$

$$A_0 = \mathbf{Ric}_\alpha - 2t_{00} + (n-1)\big[s_{0;0} + s_0^2 + 2c_0\beta - k\beta^2\big], \tag{3.12}$$

where $\mathbf{Ric}_\alpha$ is the Ricci curvature of α, $c = c(x)$ is a scalar function and ";" denotes the covariant derivative with respect to the Levi-Civita connection of α.

Equation (3.9) yield two fundamental equations:

$$A_1 = 0, \tag{3.13}$$

and

$$A_2\alpha^2 + A_0 = 0. \tag{3.14}$$

By combining (3.13) and (3.14), we derive equations characterizing Randers metrics of constant **PRic**-curvature as follows:

$$\mathbf{Ric}_\alpha = t^m_{\ m}\alpha^2 + 2t_{00} - (n-1)\big[4c^2\alpha^2 - k\alpha^2 + s_{0;0} + 2c_0\beta + s_0^2 - k\beta^2\big], \tag{3.15}$$

$$s^m_{\ 0;m} = (n-1)\big[t_0 + 2cs_0 + k\beta\big]. \tag{3.16}$$

This leads to the following result:
If $F = \alpha + \beta$ is a Randers metric of isotropic **S**-curvature, then F has constant **PRic**-curvature if and only if α and β satisfy the equations (3.15), (3.16).

It is noteworthy that equations (3.15), (3.16) with ($k = 0$) are equivalent to equations (2.5) and (2.6) in Cheng-Rezaei theorem, c.f. [5, Theorem 2.1.]. Consequently, we establish the following corollary:

Corollary 3.1. *Let $F = \alpha+\beta$ be a Randers metric on an n-dimensional manifold M. Then F is a* **PRic***-flat metric if and only if α and β satisfy (3.15) and (3.16), with $k = 0$.*

Importantly, this result aligns with the equations characterizing weak Einstein Randers metrics. Specifically, equations (3.15), (3.16) can also be deduced from (2.10), (2.11) with $3\theta + c_0 = 0$ and $\sigma + c^2 = k$.

4. Proof of main theorems

Proof of Theorem 1.1: By assumption, the **S**-curvature of F satisfies $\mathbf{S} = (n+1)c(x)F$. In this case, the projective curvature **PRic** is given by (2.3)

$$\mathbf{PRic} \quad = \quad \mathbf{Ric}_F + (n-1)c_0F + (n-1)c^2F^2. \tag{4.1}$$

Recall that a Finsler metric F is of isotropic **PRic**-curvature if and only if

$$\mathbf{PRic} = (n-1)k(x)F^2. \tag{4.2}$$

Combining (4.1) and (4.2), we find

$$\mathbf{Ric}_F = (n-1)\{-c_0F + (k-c^2)F^2\}. \tag{4.3}$$

Since a Finsler metric is called a weak Einstein metric if

$$\mathbf{Ric}_F = (n-1)(\frac{3\theta}{F} + \sigma)F^2 = (n-1)(3\theta F + \sigma F^2), \tag{4.4}$$

then from (4.3), F is a weak Einstein metric with

$$\theta = -\frac{1}{3}c_0, \ \sigma = k - c^2. \tag{4.5}$$

Conversely, if F is has isotropic **S**-curvature and is weak Einstein with (4.5), the Ricci curvature of F is given by (4.3). Substituting the Ricci curvature (4.3) into (4.1) yields

$$\mathbf{PRic} = (n-1)k(x)F^2.$$

Thus, F is of isotropic **PRic**-curvature. □

Proof of Theorem 1.2: Let F be an Einstein Randers metric (so F is isotropic **S**-curvature) with $\mathbf{PRic} = 0$. We can easily derive $c_0 = 0$, meaning c is a constant. Then, by (4.1), F satisfies

$$\mathbf{Ric}_F = -(n-1)c^2F^2. \tag{4.6}$$

We split the proof into two cases.

Case (1): If $c \neq 0$, then $\mathbf{Ric}_F < 0$. According to [4], on a closed manifold M, Einstein Randers metrics with negative Ricci scalar are Riemannian.
Case (2): if $c = 0$, then F is $\mathbf{Ric}_F$-flat. □

By Theorem 1.2, it is established that on closed manifolds, non-Riemannian Einstein Randers metrics with **PRic**-flat property are indeed $\mathbf{Ric}_F$-flat. Conversely, a corollary in [14] suggests that a non-Riemannian Einstein Randers metric on a closed manifold is $\mathbf{Ric}_F$-flat if and only if it is a Berwald metric. It is worth noting that Berwald metrics are associated with **S**-flatness. Therefore, Theorem 1.2 leads to Corollary 1.2.

Corollary 4.1. *Assume F is a non-Riemannian Einstein Randers metric with* **PRic***-flatness on a closed manifold M. Then* $\mathbf{S} = 0$.

Although the converse of Corollary 4.1 does not hold universally, we establish a weak result:

If F is a non-Riemannian Einstein Randers metric with **S**-flatness on a closed manifold M of dimension $n \geqslant 3$, then F exhibits constant **PRic**-curvature. The following example illustrates and confirms this result.

Example 4.1. Consider the generic tangent vectors on S^3 given by

$$u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}.$$

The Finsler function for Bao-Shen's Randers space is defined as

$$F(x, y, z; u, v, w) = \alpha(x, y, z; u, v, w) + \beta(x, y, z; u, v, w),$$

where

$$\alpha = \frac{\sqrt{\eta(cu - zv + yw)^2 + (zu + cv - xw)^2 + (-yu + xv + cw)^2}}{1 + x^2 + y^2 + z^2},$$

$$\beta = \frac{\pm\sqrt{\eta - 1}(cu - zv + yw)}{1 + x^2 + y^2 + z^2},$$

and $\eta > 1$ is a real constant. The family of Randers metrics on S^3 constructed by Bao-Shen satisfies **S**=0. Given that these metrics possess constant flag curvature K, we have $\mathbf{Ric}_F = 2KF^2$. Hence, Bao-Shen's metrics exhibit constant projective Ricci curvature with $k = K = constant$, i.e. $\mathbf{PRic} = 2KF^2$.

In Theorem 2, the assumption is that the manifold is closed, which is a crucial condition for establishing the equivalence of $\mathbf{Ric}_F$-flatness, **PRic**-flatness, and Berwald metrics. However, in the following example, the manifold is not closed. As a result, the metric is not Berwaldian, yet it transitions from $\mathbf{Ric}_F$-flat to **PRic**-flat.

This example serves as evidence that the Berwaldian condition is not a prerequisite for transitioning from $\mathbf{Ric}_F$-flat to **PRic**-flat.

Example 4.2. Consider the Randers metric defined in the vicinity of the origin in $\mathbf{R}^n$ as follows

$$F := \frac{\sqrt{|y|^2 - (|xQ|^2|y|^2 - < y, xQ >^2)}}{1 - |xQ|^2} - \frac{< y, xQ >}{1 - |xQ|^2}, \tag{4.7}$$

where $Q = (q^i_j)$ is an antisymmetric matrix. Despite Q not being equal to zero, the metric F satisfies $\mathbf{S} = 0$. In fact, $R^i_k = 0$ and the Ricci curvature $\mathbf{Ric}_F = 0$. However, it is important to note that F does not qualify as a Berwald metric when $Q \neq 0$. Nevertheless, F is a **PRic**-flat metric.

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