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ON A GENERALIZED VERSION OF m-TOPOLOGY AND U-TOPOLOGY IN THE OVER-RING $C(X)_{\triangle}$ OF $C(X)$ *

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Abstract. In this paper, we generalize the m-topology and the U-topology of $C(X)$ to its over-ring $C(X)_{\triangle}$. The generalized versions will be referred to as the $m_I^{\triangle}$-topology and the $u_I^{\triangle}$-topology respectively. We define $A_I^{\triangle} = \{f \in C(X)_{\triangle} : |f(x)| \leq M$, for all $x \in Z \setminus H$, for some $Z \in Z_{\triangle}(I)$ and $H \in \Delta\}$, which turns out to be the component of $\underline{0}$ in $C(X)_{\triangle}$ with the $u_I^{\triangle}$-topology. Next we define $I_{\psi_{\triangle}}(X) = \{f \in C(X)_{\triangle} : |fg(x)| \leq M$, for all $g \in C(X)_{\triangle}$ and for all $x \in Z \setminus H$, for some $Z \in Z_{\triangle}(I)$ and $H \in \Delta\}$. This set will be seen to play a key role in determining the connected ideals in $C(X)_{\triangle}$ with the $m_I^{\triangle}$-topology. It is observed that $C(X)_{\triangle}$ with the $m_I^{\triangle}$-topology is a topological ring, whereas $C(X)_{\triangle}$ with the $u_I^{\triangle}$-topology is not so. Finally, we give several necessary and sufficient conditions for the coincidence of the $u_I^{\triangle}$-topology and the $m_I^{\triangle}$-topology on $C(X)_{\triangle}$.
Keywords: $C(X)_{\triangle}$, $u_I^{\triangle}$-topology, $m_I^{\triangle}$-topology.

1. Introduction

All topological spaces are assumed to be T_1. Let $\mathbb{R}^X$ be the ring of all real-valued functions defined on a nonempty topological space X with pointwise addition and

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multiplication. Also, the collection of all continuous members of $\mathbb{R}^X$ is denoted by $C(X)$, and the collection of all bounded members of $C(X)$ is denoted by $C^*(X)$. In this connection, we refer to the reader [5], where these two rings have been studied extensively. Our interest of study in this paper is an over-ring $C(X)_\Delta$ of $C(X)$. In [10], we have introduced the ring $C(X)_\Delta$, which consists of those real-valued functions, whose discontinuity set is a member of Δ, where Δ is a subcollection of $\mathcal{P}(X)$ with the properties: (i) For each $x \in X$, $\{x\} \in \Delta$, (ii) for $A, B \in \Delta$, $A \cup B \in \Delta$, (iii) for $A, B \in \mathcal{P}(X)$ with $A \subseteq B$, if $B \in \Delta$, then $A \in \Delta$. Then $C(X)_\Delta$ becomes a commutative ring with unity. Now for any $f, g \in C(X)_\Delta$, $f \vee g = \frac{1}{2}(f + g + |f - g|) \in C(X)_\Delta$ and $f \wedge g = \frac{1}{2}(f + g - |f - g|) \in C(X)_\Delta$. Therefore, $C(X)_\Delta$ is actually a lattice-ordered ring and a sublattice of $\mathbb{R}^X$. More properties of the ring $C(X)_\Delta$ have been studied in [9].

The m-topology on $C(X)$ is first introduced in the late 40's in [7] and later the research in this area became active over the last 20 years, for example, the works in [3, 6] and [8]. $C(X)$ endowed with the m-topology is denoted by $C_m(X)$ which is a Hausdorff topological ring. In [5], further studies have been done regarding the m-topology on $C(X)$. Also, the uniform norm topology on $C^*(X)$ has been studied and investigated in [5]. A generalization of m-topology on $C(X)$ has been defined and thoroughly studied in [1]. Also in [2], the authors have given a generalization of the U-topology of $C(X)$. In this paper, we introduce the same kind of topologies, i.e. the m_I^Δ-topology and the u_I^Δ-topology in the setting of $C(X)_\Delta$. We here aim to investigate the similarities and dissimilarities between the ring $C(X)$ and its over-ring $C(X)_\Delta$ via these two topologies.

Let us first recall some necessary definitions and results for a smooth continuation of the paper.

Definition 1.1. [10] For a topological space X and a subcollection Δ of $\mathcal{P}(X)$ ($\equiv$ the power set of X), where Δ is closed under forming subsets, finite unions and containing all singletons, we define,

$$C(X)_\Delta = \{f \in \mathbb{R}^X : \text{the set of points of discontinuities of } f \text{ is a member of } \Delta\}.$$

It can be easily observed that $C(X)_\Delta$ is a commutative ring with unity (with respect to pointwise addition and multiplication) containing $C(X)$, i.e. $C(X)_\Delta$ is a superring or an over-ring of $C(X)$. We note that $C(X)_\Delta$ is a sublattice of $\mathbb{R}^X$, in fact, $(C(X)_\Delta, +, ., \vee, \wedge)$ is a lattice-ordered ring if for any $f, g \in C(X)_\Delta$, one defines $(f \vee g)(x) = f(x) \vee g(x)$ and $(f \wedge g)(x) = f(x) \wedge g(x)$, $x \in X$. Also $f \vee g = \frac{f+g+|f-g|}{2} \in C(X)_\Delta$, for all $f, g \in C(X)_\Delta$. For $f \in C(X)_\Delta$ and $f > 0$, we note that there exists $h \in C(X)_\Delta$ such that $f = h^2$. Also, whenever $f \in C(X)_\Delta$ and f^r is defined where $r \in \mathbb{R}$, then $f^r \in C(X)_\Delta$. Also an element f in $C(X)_\Delta$ is called a unit if and only if $Z_\Delta(f) = \emptyset$ (see Theorem 2.17 of [10]).

Definition 1.2. [10] For $f \in C(X)_\Delta$, the set $f^{-1}(0) = \{x \in X : f(x) = 0\}$ will be called the zero set of f, to be denoted by $Z_\Delta(f)$.

We will use the notation $Z_\triangle(C(X)_\triangle)$ (or, $Z_\triangle(X)$) for the collection $\{Z_\triangle(f) : f \in C(X)_\triangle\}$ of all zero sets in X.

Remark 1.1. [10] (i) Unlike $C(X)$, $Z_\triangle(f)$ is not necessarily closed.
(ii) $Z_\triangle(f)$ need not be a G_δ-set as in the case of $C(X)$.

Throughout this paper whenever we speak of an ideal unmodified, we will always mean a proper ideal in the ring under consideration. For an ideal I of $C(X)_\triangle$, we shall denote $\{Z_\triangle(f) : f \in I\}$ by $Z_\triangle(I)$.

Definition 1.3. [10] An ideal I of $C(X)_\triangle$ is called a $Z_\triangle$-ideal if $Z_\triangle^{-1}Z_\triangle(I) = I$. Equivalently, $Z_\triangle(f) = Z_\triangle(g)$, for $f \in I$ and $g \in C(X)_\triangle$ implies that $g \in I$.

For a $Z_\triangle$-ideal I of $C(X)_\triangle$, we consider $U_I^{\triangle^+} = \{u \in C(X)_\triangle : u(x) > 0$, for all $x \in Z$, for some $Z \in Z_\triangle(I)\}$. For each f in $C(X)_\triangle$ and each $u \in U_I^{\triangle^+}$, we set $m_I^\triangle(f,u) = \{g \in C(X)_\triangle : |f(x)-g(x)| < u(x)$, for all $x \in Z\backslash H$, for some $Z \in Z_\triangle(I)$ and $H \in \Delta\}$. It is routine to check that there exists a unique topology which we call the $m_I^\triangle$-topology on $C(X)_\triangle$ in which for each $f \in C(X)_\triangle$ and each $u \in U_I^{\triangle^+}$, $\{m_I^\triangle(f,u) : f \in C(X)_\triangle$ and $u \in U_I^{\triangle^+}\}$ is a base for its open neighborhoods. It can be easily proved that $C(X)_\triangle$ with this $m_I^\triangle$-topology is a topological ring. Similarly, for $f \in C(X)_\triangle$ and $\epsilon > 0$ in $\mathbb{R}$, we define $u_I^\triangle(f,\epsilon) = \{g \in C(X)_\triangle : |f(x)-g(x)| < \epsilon$, for all $x \in Z \setminus H$, for some $Z \in Z_\triangle(I)$ and $H \in \Delta\}$. It needs some routine computations to show that $\{u_I^\triangle(f,\epsilon) : f \in C(X)_\triangle, \epsilon > 0\}$ is an open base for some topology on $C(X)_\triangle$, which we wish to call the $u_I^\triangle$-topology on $C(X)_\triangle$. For any $f \in C(X)_\triangle$, the family $\{u_I^\triangle(f,\epsilon) : \epsilon > 0\}$ turns out to be an open neighborhood base about the point f in this topology. Let $I_{\psi_\triangle}(X) = \{f \in C(X)_\triangle :$ given $g \in C(X)_\triangle$, there exists $M > 0$, $Z \in Z_\triangle(I)$ and $H \in \Delta$ such that $|f(x).g(x)| \leq M$ for each $x \in Z \setminus H\}$. This set will be seen to play a key role in determining the connected ideals in $C(X)_\triangle$ with the $m_I^\triangle$-topology. Next we define $A_I^\Delta = \{f \in C(X)_\triangle : |f(x)| \leq M$, for all $x \in Z \setminus H$, for some $Z \in Z_\triangle(I)$ and $H \in \Delta\}$, which turns out to be the component of $\underline{0}$ in $C(X)_\triangle$ with the $u_I^\triangle$-topology. We utilize this fact to establish that several statements individually are necessary and sufficient for $C(X)_\triangle$ with the $u_I^\triangle$-topology to be a topological ring.

In section 2, we at first investigate the clopen sets of $C(X)_\triangle$ with the $m_I^\triangle$-topology. As in [1], the map $\phi_f : \mathbb{R} \to C(X)_\triangle$ defined by $\phi_f(r) = r.f$ is introduced for any $f \in I_{\psi_\triangle}$. Then it has been shown that such ϕ_f is continuous, for any $f \in I_{\psi_\triangle}$ when $C(X)_\triangle$ is equipped with the $m_I^\triangle$-topology. With the help of this, we then prove that $I_{\psi_\triangle}(X)$ is the component of $\underline{0}$ in $C(X)_\triangle$ with the $m_I^\triangle$-topology. We next define for any $f \in A_I^\Delta$, $\|f\|_\triangle = inf\{M : |f(x)| \leq M$ for all $x \in Z \setminus H$, for some $Z \in Z_\triangle(I)$ and $H \in \Delta\}$. It can be easily checked that $(A_I^\Delta, \|.\|_\triangle)$ is a pseudo-normed linear space over $\mathbb{R}$. The relative topology on A_I^Δ induced by the

$m_I^\triangle$-topology is shown to be stronger than the pseudo-norm topology on A_I^Δ. For the coincidence of these two topologies, we then find a set of necessary and sufficient conditions involving A_I^Δ and $I_{\psi_\triangle}(X)$. Then we have proved that $C(X)_\triangle$ with this $m_I^\triangle$-topology is regular but not Hausdorff. Finally, we give a complete description of the connected ideals in $C(X)_\triangle$ with the $m_I^\triangle$-topology.

In section 3, we prove results concerning the $u_I^\triangle$-topology on $C(X)_\triangle$. Since the map $\phi_f : \mathbb{R} \to C(X)_\triangle$ defined by $\phi_f(r) = r.f$ is proved to be continuous for each $f \in I_{\psi_\triangle}$ in the $m_I^\triangle$-topology and the $u_I^\triangle$-topology is weaker than the $m_I^\triangle$-topology on $C(X)_\triangle$, it follows that the same map is also continuous for each $f \in I_{\psi_\triangle}$, if $C(X)_\triangle$ is equipped with the $u_I^\triangle$-topology. It is then proved that A_I^Δ is the component of $\underline{0}$ in $C(X)_\triangle$ equipped with the $u_I^\triangle$-topology. Next, an example has been furnished to show that for a free $Z_\triangle$-ideal, $C(X)_\triangle$ with the $u_I^\triangle$-topology is not a topological ring. Finally it has been proved that $C(X)_\triangle$ with $u_I^\triangle$-topology is a topological ring if and only if the $u_I^\triangle$-topology and the $m_I^\triangle$-topology coincide on $C(X)_\triangle$. Next endeavour has been made for the coincidence of the $u_I^\triangle$-topology and the $m_I^\triangle$-topology on $C(X)_\triangle$. For this, we define $U_I^{\triangle^{++}} = \{u \in C(X)_\triangle : u(x) > \lambda$, for all $x \in Z \setminus H$, for some $Z \in Z_\triangle(I)$ and $H \in \Delta, \lambda > 0\}$. It has been proved that for any $Z_\triangle$-ideal I in $C(X)_\triangle$, the $u_I^\triangle$-topology coincides with the $m_I^\triangle$-topology if and only if $U_I^{\triangle^{+}} \subseteq U_I^{\triangle^{++}}$. Finally, we give several other necessary and sufficient conditions for the coincidence of the $u_I^\triangle$-topology and the $m_I^\triangle$-topology on $C(X)_\triangle$.

Throughout the paper, for any literature on topology one may go through [4].

2. $m_I^\triangle$-topology on $C(X)_\triangle$

We start this section by investigating the nature of clopen sets of $C(X)_\triangle$ with the $m_I^\triangle$-topology.

Theorem 2.1. *The set A_I^Δ is a clopen set in $C(X)_\triangle$ with the $m_I^\triangle$-topology.*

Proof. For $f \in C(X)_\triangle \setminus A_I^\Delta$, $m_I^\Delta(f, 1) \subseteq C(X)_\triangle \setminus A_I^\Delta$. So A_I^Δ is a closed set. Again for $f \in A_I^\Delta$, $m_I^\Delta(f, 1) \subseteq A_I^\Delta$ which shows that A_I^Δ is an open set in $C(X)_\triangle$ with the $m_I^\triangle$-topology. □

Theorem 2.2. *For any $Z_\triangle$-ideal I, $I \subseteq I_{\psi_\triangle}(X) \subseteq A_I^\Delta \subseteq C(X)_\triangle$.*

Proof. It is straightforward. □

Theorem 2.3. *For $f \in I_{\psi_\triangle}(X)$, the map $\phi_f : \mathbb{R} \to C(X)_\triangle$ defined by $\phi_f(r) = r.f$ is continuous, when $C(X)_\triangle$ is equipped with the $m_I^\triangle$-topology.*

Proof. Let $r.f \in m_I^\Delta(f,u)$, where $u \in U_I^{\Delta^+}$. Then $u(x) > 0$, for all $x \in Z_\Delta(h)$, for some $Z_\Delta(h) \in Z_\Delta(I)$. Consider $g = |u| + |h|$. Then $g(x) > 0$ for all $x \in X$ and $g \in C(X)_\Delta$. So g is a unit of $C(X)_\Delta$. Therefore $\frac{1}{g} \in C(X)_\Delta$. Since $f \in I_{\psi_\Delta}(X)$, there exists $M > 0$ such that $|\frac{f(x)}{g(x)}| \leq M$ for all $x \in Z \setminus H$, for some $Z \in Z_\Delta(I)$ and $H \in \Delta$. Let $s \in (r - \frac{1}{M}, r + \frac{1}{M})$. Then $|\phi_f(r) - \phi_f(s)| = |r-s||f| = |r-s|\frac{|f|}{|g|}|g| < |g| = |u| = u$, for all $x \in (Z_\Delta(h) \cap Z) \setminus H$. So ϕ_f is continuous. □

Using the above theorem we can now have an idea about the component of $\underline{0}$ in $C(X)_\Delta$ with the m_I^Δ-topology as follows.

Theorem 2.4. *$I_{\psi_\Delta}(X)$ is the component of $\underline{0}$ in $C(X)_\Delta$ with the m_I^Δ-topology.*

Proof. It can be easily seen that $I_{\psi_\Delta}(X) = \bigcup_{f \in I_{\psi_\Delta}(X)} \phi_f(\mathbb{R})$ and $\underline{0} \in \bigcap_{f \in I_{\psi_\Delta}(X)} \phi_f(\mathbb{R})$. Hence $I_{\psi_\Delta}(X)$ is a connected ideal of $C(X)_\Delta$ with respect to the m_I^Δ-topology. Let J be a connected ideal of $C(X)_\Delta$ containing $I_{\psi_\Delta}(X)$. If $f \in J \setminus I_{\psi_\Delta}(X)$, then there exists $g \in C(X)_\Delta$ such that $fg \notin A_I^\Delta$. But A_I^Δ is a clopen set containing $\underline{0}$ and as J is an ideal, $fg \in J \setminus A_I^\Delta$. So $J = (J \cap A_I^\Delta) \cup (J \setminus A_I^\Delta) =$ the union of two non-empty disjoint open sets in the space J, a contradiction to the fact that J is connected. Hence $I_{\psi_\Delta}(X)$ is the component of $\underline{0}$ in $C(X)_\Delta$ with the m_I^Δ-topology. □

For $f \in A_I^\Delta$, we define $\|f\|_\Delta = inf\{M : |f(x)| \leq M$ for all $x \in Z \setminus H$, for some $Z \in Z_\Delta(I)$ and $H \in \Delta\}$. It can be easily checked that $(A_I^\Delta, \|.\|_\Delta)$ is a pseudo-normed linear space over $\mathbb{R}$. Next for $f \in A_I^\Delta$, we define $B(f,\epsilon) = \{g \in A_I^\Delta : \|f - g\|_\Delta < \epsilon\}$. Then the family $\{B(f,\epsilon) : f \in A_I^\Delta, \epsilon > 0\}$ forms a base of open sets for the pseudo-norm topology on A_I^Δ.

Lemma 2.1. *The relative topology on A_I^Δ induced by the m_I^Δ-topology is stronger than the pseudo-norm topology on A_I^Δ.*

Proof. It follows from the fact that for $f \in A_I^\Delta$, $A_I^\Delta \cap m_I^\Delta(f, \frac{\epsilon}{2}) \subseteq B(f,\epsilon)$. □

Theorem 2.5. *The following statements are equivalent:*

(i) $A_I^\Delta = C(X)_\Delta$.

(ii) $I_{\psi_\Delta}(X) = C(X)_\Delta$.

(iii) The pseudo-norm topology on A_I^Δ is identical with the relative m_I^Δ-topology.

(iv) The m_I^Δ-topology on $C(X)_\Delta$ is connected.

(v) The m_I^Δ-topology on $C(X)_\Delta$ is locally connected.

Proof. $(i) \Rightarrow (ii)$: Let $f, g \in C(X)_\Delta$. Then $f, g \in A_I^\Delta$ also. So there exist $M_1, M_2 > 0$ such that $|f(x)| \leq M_1$, for all $x \in Z_1 \setminus H_1$ and $|g(x)| \leq M_2$, for all $x \in Z_2 \setminus H_2$ for some $Z_1, Z_2 \in Z_\Delta(I)$ and $H_1, H_2 \in \Delta$. This implies that $|fg(x)| \leq M_1 M_2$, for all $x \in (Z_1 \cap Z_2) \setminus (H_1 \cup H_2)$, where $Z_1 \cap Z_2 \in Z_\Delta(I)$ and $H_1 \cup H_2 \in \Delta$. Hence $f \in I_{\psi_\Delta}(X)$.

$(ii) \Rightarrow (i)$: Follows from Theorem 2.2.

$(i) \Rightarrow (iii)$: From Lemma 2.1 it follows that the pseudo-norm topology on A_I^Δ is weaker than the m_I^Δ-topology on A_I^Δ. So it is sufficient to show that the m_I^Δ-topology is weaker than the pseudo-norm topology on A_I^Δ. Let $m_I^\Delta(f, u)$ be a basic element for m_I^Δ-topology on A_I^Δ, where $f \in A_I^\Delta$ and $u \in {U_I^\Delta}^+$. Then $u(x) > 0$ for all $x \in Z_\Delta(g)$, for some $Z_\Delta(g) \in Z_\Delta(I)$, where $g \in I$. Let $v = |u| + |g|$. Then $v \in C(X)_\Delta$ and $v(x) > 0$ for all $x \in X$. So v is a unit of $C(X)_\Delta$. This implies that $\frac{1}{v} \in C(X)_\Delta = A_I^\Delta$. Hence there exists $M > 0$ such that $|\frac{1}{v(x)}| \leq M$, for all $x \in Z_1 \setminus H_1$, for some $Z_1 \in Z_\Delta(I)$ and $H_1 \in \Delta$ which implies that $|v(x)| \geq \frac{1}{M}$, for all $x \in Z_1 \setminus H_1$ for some $Z_1 \in Z_\Delta(I)$ and $H_1 \in \Delta$. Now consider $B(f, \frac{1}{M}) = \{h \in A_I^\Delta : \|f - h\|_\Delta < \frac{1}{M}\}$. Choose $h \in B(f, \frac{1}{M})$. Then $\|f - h\|_\Delta < \frac{1}{M}$. Therefore there exist $Z_2 \in Z_\Delta(I)$ and $H \in \Delta$ such that $|f(x) - h(x)| < \frac{1}{M}$, for all $x \in Z_2 \setminus H_2$. This implies that $|f(x) - h(x)| < \frac{1}{M} \leq v(x) = |u(x)| = u(x)$, for all $x \in (Z_\Delta(g) \cap Z_1 \cap Z_2) \setminus (H_1 \cup H_2)$, where $Z_\Delta(g) \cap Z_1 \cap Z_2 \in Z_\Delta(I)$ and $H_1 \cup H_2 \in \Delta$. Hence $B(f, \frac{1}{M}) \subseteq m_I^\Delta(f, u)$. So the pseudo-norm topology on A_I^Δ is weaker than the m_I^Δ-topology on A_I^Δ.

$(iii) \Rightarrow (i)$: Let $A_I^\Delta \subsetneq C(X)_\Delta$. Then there exists $f \in C(X)_\Delta \setminus A_I^\Delta$ with $f \geq 1$ such that for each $Z \in Z_\Delta(I)$ and $H \in \Delta$, there exists $x \in Z \setminus H$ with $f(x) > n$, for all $n \in \mathbb{N}$. We consider $g = \frac{1}{f}$. Then $g \in C(X)_\Delta$ with $g \leq 1$. Therefore for each $Z \in Z_\Delta(I)$ and $H \in \Delta$, there exists $x \in Z \setminus H$ such that $g(x) < \frac{1}{n}$, for all $n \in \mathbb{N}$. Now $m_I^\Delta(\underline{0}, g)$ is a neighbourhood of $\underline{0}$ in the relative m_I^Δ-topology. We will show that there does not exist any $\epsilon > 0$ such that $B(\underline{0}, \epsilon) \subseteq m_I^\Delta(\underline{0}, g) \cap A_I^\Delta$. Since $\frac{\epsilon}{2} \in B(\underline{0}, \epsilon)$ but $\frac{\epsilon}{2} \notin m_I^\Delta(\underline{0}, g)$, $B(\underline{0}, \epsilon) \nsubseteq m_I^\Delta(\underline{0}, g) \cap A_I^\Delta$ for any $\epsilon > 0$, a contradiction. Hence $A_I^\Delta = C(X)_\Delta$.

$(i) \Rightarrow (iv)$: A pseudo-norm topology is path connected, hence connected.

$(iv) \Rightarrow (i)$: If $A_I^\Delta \neq C(X)_\Delta$, then by Theorem 2.1, A_I^Δ is a proper clopen set in $C(X)_\Delta$, a contradiction. Hence $A_I^\Delta = C(X)_\Delta$.

$(ii) \Rightarrow (v)$: To show that the m_I^Δ-topology is locally connected, it is sufficient to show that $m_I^\Delta(\underline{0}, \epsilon)$ is connected. Since $I_{\psi_\Delta}(X) = C(X)_\Delta$, for any $f \in C(X)_\Delta$ the map $\phi_f : \mathbb{R} \to C(X)_\Delta$ defined by $\phi_f(r) = r.f$ is continuous, by Theorem 2.3. Now $m_I^\Delta(\underline{0}, \epsilon) = \bigcup_{f \in m_I^\Delta(\underline{0}, \epsilon)} \phi_f([-1, 1])$ and $\underline{0} \in \bigcap_{f \in m_I^\Delta(\underline{0}, \epsilon)} \phi_f([-1, 1])$. This implies that $m_I^\Delta(\underline{0}, \epsilon)$ is connected. Hence the m_I^Δ-topology is locally connected.

$(v) \Rightarrow (ii)$: If possible, let $I_{\psi_\Delta}(X) \neq C(X)_\Delta$. Let U be any connected open neighborhood of $\underline{0}$. Then $U \subseteq I_{\psi_\Delta}(X)$ (since $I_{\psi_\Delta}(X)$ is a component of $\underline{0}$ by Theorem 2.4). Again there exists $u \in {U_I^\Delta}^+$ such that $m_I^\Delta(\underline{0}, u) \subseteq U \subseteq I_{\psi_\Delta}(X) \subsetneq$

$C(X)_\Delta$. Since $u \in U_I^{\Delta^+}$, $u(x) > 0$ for all $x \in Z$, for some $Z \in Z_\Delta(I)$, where $Z = Z_\Delta(g)$ for some $g \in I$. Let $v = |u| + |g|$. Then $v \in C(X)_\Delta$ and $v(x) > 0$, for all $x \in X$. Let $f \in C(X)_\Delta \setminus I_{\psi_\Delta}(X)$. Now $\frac{f}{1+|f|}v \in m_I^\Delta(\underline{0}, u) \subseteq U \subseteq I_{\psi_\Delta}(X)$. Therefore $f = \frac{1+|f|}{v}.\frac{f}{1+|f|}v \in I_{\psi_\Delta}(X)$ (as $I_{\psi_\Delta}(X)$ is an ideal), a contradiction. Hence $I_{\psi_\Delta}(X) = C(X)_\Delta$. □

Regarding the separation axioms of $C(X)_\Delta$ with the m_I^Δ-topology, we have the following results.

Proposition 2.1. *$C(X)_\Delta$ with the m_I^Δ-topology is a completely regular space.*

Proof. Let for each $u \in U_I^{\Delta^+}$, $S_u = \{(f,g) \in C(X)_\Delta \times C(X)_\Delta$: there exist $Z \in Z_\Delta(I)$ and $H \in \Delta$ such that $|f(x) - g(x)| < u(x)$ for each $x \in Z \setminus H\}$. Then the family $\{S_u : u \in U_I^{\Delta^+}\}$ is a base for some uniformity on $C(X)_\Delta$; furthermore the topology induced by this uniformity is the m_{I_Δ}-topology on $C(X)_\Delta$. Consequently, $C(X)_\Delta$ with the m_{I_Δ}-topology becomes a completely regular space. □

Remark 2.1. $C(X)_\Delta$ with the m_I^Δ-topology is not a Hausdorff space.

Example 2.1. Let X be a topological space and I be any Z_Δ-ideal of $C(X)_\Delta$. Consider $a, b \in X$ with $a \neq b$. Then $\chi_{\{a\}}, \chi_{\{b\}} \in C(X)_\Delta$ and $\chi_{\{a\}} \neq \chi_{\{b\}}$. Now $\chi_{\{a\}} \in m_I^\Delta(\chi_{\{a\}}, u)$, for any $u \in U_I^{\Delta^+}$. Since $u \in U_I^{\Delta^+}$, there exists $Z \in Z_\Delta(I)$ such that $u(x) > 0$, for all $x \in Z$. Hence $|\chi_{\{a\}}(x) - \chi_{\{b\}}(x)| = 0 < u(x)$, for all $x \in Z \setminus \{a, b\}$, where $Z \in Z_\Delta(I)$ and $\{a, b\} \in \Delta$. Therefore, $\chi_{\{b\}} \in m_I^\Delta(\chi_{\{a\}}, u)$. So $C(X)_\Delta$ with the m_I^Δ-topology is not a Hausdorff space.

Next we investigate the connected ideals of $C(X)_\Delta$ with the m_I^Δ-topology.

Proposition 2.2. *Let $J \subseteq C(X)_\Delta$ be an ideal containing a Z_Δ-ideal I. Then int $J = \emptyset$ in $C(X)_\Delta$ with the m_I^Δ-topology.*

Proof. Let $f \in$ *int* J. Then there exists $u \in U_I^{\Delta^+}$ such that $f \in m_I^\Delta(f, u) \subseteq J$. Since $u \in U_I^{\Delta^+}$, there exists $Z = Z_\Delta(|g|) \in Z_\Delta(I)$ such that $u(x) > 0$, for all $x \in Z_\Delta(|g|)$ with $|g| \in I$. Now $f + \frac{|u|}{2} \in m_I^\Delta(f, u) \subseteq J$ and $f \in J$ implies that $|u| \in J$. Now consider $v = |u| + |g|$. Then $v \in C(X)_\Delta$ as well as $v \in J$ and $v(x) > 0$ for all $x \in X$. Therefore v is a unit of $C(X)_\Delta$ contained in J, a contradiction. Hence *int* $J = \emptyset$. □

Proposition 2.3. *The following statements are equivalent for an ideal J of $C(X)_\Delta$ equipped with the m_I^Δ-topology.*

(i) J *is a connected ideal.*

(ii) $J \subseteq A_I^{\Delta}$.

(iii) $J \subseteq I_{\psi_\Delta}(X)$.

Proof. $(i) \Rightarrow (ii)$: Let $f \in J \setminus A_I^{\Delta}$. Since $\underline{0} \in J \cap A_I^{\Delta}$, it follows that $J \cap A_I^{\Delta}$ and $J \setminus A_I^{\Delta}$ are two nonempty disjoint open sets in the space J, as by Theorem 2.1 A_I^{Δ} is a clopen set in $C(X)_\Delta$. This contradicts the fact that J is a connected ideal. Hence $J \subseteq A_I^{\Delta}$.

$(ii) \Rightarrow (iii)$: Let $f \in J$. Then for all $g \in C(X)_\Delta$, $fg \in J$ (as J is an ideal). Therefore $fg \in A_I^{\Delta}$. Thus for all $g \in C(X)_\Delta$, $fg \in A_I^{\Delta}$ implies that $f \in I_{\psi_\Delta}(X)$. So $J \subseteq I_{\psi_\Delta}(X)$.

$(iii) \Rightarrow (i)$: For all $f \in J$, the map $\phi_f : \mathbb{R} \to C(X)_\Delta$ defined by $\phi_f(r) = r.f$ is continuous. Now $J = \bigcup_{f \in J} \phi_f(\mathbb{R})$ and $\underline{0} \in \bigcap_{f \in J} \phi_f(\mathbb{R})$. So J is connected. □

3. u_I^{Δ}-topology on $C(X)_\Delta$

Theorem 3.1. *For* $f \in A_I^{\Delta}$*, the map* $\phi_f : \mathbb{R} \to C(X)_\Delta$ *defined by* $\phi_f(r) = r.f$ *is continuous with respect to the* u_I^{Δ}*-topology.*

Proof. Let $r.f \in u_I^{\Delta}(f, \epsilon)$, for $\epsilon > 0$. Since $f \in A_I^{\Delta}$, there exists $M > 0$ such that $|f(x)| \leq M$, for all $x \in Z \setminus H$, for some $Z \in Z_\Delta(I)$ and $H \in \Delta$. Now consider the open set $(r - \frac{\epsilon}{M}, r + \frac{\epsilon}{M})$ containing r. If $s \in (r - \frac{\epsilon}{M}, r + \frac{\epsilon}{M})$, then $|\phi_f(r)(x) - \phi_f(s)(x)| = |rf(x) - sf(x)| = |r - s||f(x)| < \frac{\epsilon}{M} M = \epsilon$, for all $x \in Z \setminus H$, $Z \in Z_\Delta(I)$ and $H \in \Delta$. Hence ϕ_f is continuous whenever $f \in A_I^{\Delta}$. □

Theorem 3.2. A_I^{Δ} *is a clopen set in* $C(X)_\Delta$ *with respect to the* u_I^{Δ}*-topology.*

Proof. Can be done similarly as in Theorem 2.1. □

Lemma 3.1. A_I^{Δ} *is the component of* $\underline{0}$ *in* $C(X)_\Delta$ *with respect to the* u_I^{Δ}*-topology.*

Proof. It can be easily seen that $A_I^{\Delta} = \bigcup_{f \in A_I^{\Delta}} \phi_f(\mathbb{R})$ and $\underline{0} \in \bigcap_{f \in A_I^{\Delta}} \phi_f(\mathbb{R})$. Hence A_I^{Δ} is connected (as ϕ_f is continuous for $f \in A_I^{\Delta}$) and also A_I^{Δ} is a clopen set in $C(X)_\Delta$ with respect to the u_I^{Δ}-topology. So A_I^{Δ} is a maximal connected set containing $\underline{0}$. Hence A_I^{Δ} is the component of $\underline{0}$ in the u_I^{Δ}-topology. □

Since in a topological ring, the component of 0 is an ideal, in view of the following example we can make a remark that $C(X)_\Delta$ with the u_I^{Δ}-topology is not a topological ring.

Example 3.1. Consider $X = \mathbb{R}$ with the usual topology of reals and $\Delta = \{A \subseteq X : A$ is countable$\}$. Let $S = \{\chi_{\{x\}} : x \in X\}$ and $I =< S >$. Then I is a free $Z_\triangle$-ideal generated by S and for any $g \in I$, $Z_\triangle(g) = X \setminus A$, where A is a finite subset of X (where an ideal I is said to be free if $\bigcap_{f\in I} Z_\triangle(f) = \emptyset$). Now we show that $A_I^\triangle$ is not an ideal of $C(X)_\triangle$. Take the function $f : \mathbb{R} \to \mathbb{R}$ defined by,

$$f(x) = \begin{cases} x, & x \neq n \\ n+1, & x = n. \end{cases}$$

Clearly $f \in C(X)_\triangle$. But there does not exist any $Z \in Z_\triangle(I)$ and $H \in \Delta$ such that f is bounded on $Z \setminus H$. This implies that $f \notin A_I^\triangle$. Therefore $\underline{1} \in A_I^\triangle$ whereas $1.f \notin A_I^\triangle$, which implies that $A_I^\triangle$ is not an ideal. Hence $C(X)_\triangle$ with the $\overset{\triangle}{u}_I$-topology is not a topological ring.

In fact, we can say that

Remark 3.1. For any free $Z_\triangle$-ideal I, $C(X)_\triangle$ with the $\overset{\triangle}{u}_I$-topology may not be a topological ring.

We next have the nature of quasicomponent of $\underline{0}$ in $C(X)_\triangle$ with the $\overset{\triangle}{u}_I$-topology as follows.

Theorem 3.3. *The component and the quasicomponent of $\underline{0}$ in $C(X)_\triangle$ with the $\overset{\triangle}{u}_I$-topology are identical.*

Proof. By Lemma 3.1, we have that $A_I^\triangle$ is the component of $\underline{0}$ in $C(X)_\triangle$ with the $\overset{\triangle}{u}_I$-topology. So it is sufficient to prove that the quasicomponent of $\underline{0}$ is contained in $A_I^\triangle$. As $A_I^\triangle$ is a clopen set in $C(X)_\triangle$ with the $\overset{\triangle}{u}_I$-topology and the quasi-component of a point is the intersection of all clopen sets containing it, it thus follows that the quasi-component of $\underline{0}$ is contained in $A_I^\triangle$. Hence the component and the quasicomponent of $\underline{0}$ in $C(X)_\triangle$ with the $\overset{\triangle}{u}_I$-topology are identical. □

Proposition 3.1. *The pseudo-norm topology on $A_I^\triangle$ is identical with the relative $\overset{\triangle}{u}_I$-topology on it.*

Proof. For $f \in A_I^\triangle$, we have that $A_I^\triangle \cap u_I^\triangle(f, \frac{\epsilon}{2}) \subseteq B(f, \epsilon)$, for all $\epsilon > 0$. So the relative topology on $A_I^\triangle$ induced by the $\overset{\triangle}{u}_I$-topology is stronger than the pseudo-norm topology on $A_I^\triangle$. Again if $f \notin A_I^\triangle$, then $u_I^\triangle(f, \epsilon) \cap A_I^\triangle = \emptyset$, for any $\epsilon > 0$. Also for $f \in A_I^\triangle$, $B(f, \epsilon) \subseteq u_I^\triangle(f, \epsilon) \cap A_I^\triangle$, for all $\epsilon > 0$. So the pseudo-norm topology on $A_I^\triangle$ is stronger than the relative $\overset{\triangle}{u}_I$-topology on it. Hence the pseudo-norm topology on $A_I^\triangle$ is identical with the relative $\overset{\triangle}{u}_I$-topology on $A_I^\triangle$. □

Next we investigate about the connectedness of $C(X)_\triangle$ with the $\overset{\triangle}{u}_I$-topology. Also, we give a complete description of the connected ideals of $C(X)_\triangle$ with the $\overset{\triangle}{u}_I$-topology.

Proposition 3.2. *The $u_I^{\triangle}$-topology on $C(X)_{\triangle}$ is connected if and only if $A_I^{\triangle} = C(X)_{\triangle}$.*

Proof. Let the $u_I^{\triangle}$-topology on $C(X)_{\triangle}$ be connected. As $A_I^{\triangle}$ is a clopen set in $C(X)_{\triangle}$ with the $u_I^{\triangle}$-topology, we must have $A_I^{\triangle} = C(X)_{\triangle}$.

Conversely, let $A_I^{\triangle} = C(X)_{\triangle}$. As the $u_I^{\triangle}$-topology on $A_I^{\triangle}$ is identical with the pseudo-norm topology on the same and since the pseudo-norm topology is connected, so the $u_I^{\triangle}$-topology on $C(X)_{\triangle}$ must be connected. □

Proposition 3.3. *If $A_I^{\triangle} = C(X)_{\triangle}$, then $C(X)_{\triangle}$ with the $u_I^{\triangle}$-topology is locally connected.*

Proof. Since the pseudo-norm topology on $A_I^{\triangle}$ is ideantical with the $u_I^{\triangle}$-topology, it is sufficient to show that $u_I^{\triangle}(\underline{0}, \epsilon)$ is a connected open set, for any $\epsilon > 0$. As $A_I^{\triangle} = C(X)_{\triangle}$, by Theorem 3.1, the map $\phi_f : \mathbb{R} \to C(X)_{\triangle}$ defined by $\phi_f(r) = r.f$ is continuous for all $f \in C(X)_{\triangle}$. Now $u_I^{\triangle}(\underline{0}, \epsilon) = \bigcup_{f \in u_I^{\triangle}(\underline{0}, \epsilon)} \phi_f([-1, 1])$ and $\underline{0} \in \bigcap_{f \in u_I^{\triangle}(\underline{0}, \epsilon)} \phi_f([-1, 1])$. This implies that $u_I^{\triangle}(\underline{0}, \epsilon)$ is connected. Hence the $u_I^{\triangle}$-topology on $C(X)_{\triangle}$ is locally connected. □

Theorem 3.4. *$I_{\psi_{\triangle}}(X)$ is the maximal connected ideal containing $\underline{0}$ in the space $C(X)_{\triangle}$ endowed with the $u_I^{\triangle}$-topology.*

Proof. Obviously $\underline{0} \in I_{\psi_{\triangle}}(X)$. Let J be a connected ideal containing $I_{\psi_{\triangle}}(X)$. Then there exists $f \in J \setminus I_{\psi_{\triangle}}(X)$ such that $fg \notin A_I^{\triangle}$, for some $g \in C(X)_{\triangle}$. Again $A_I^{\triangle}$ is a clopen set in the $u_I^{\triangle}$-topology. But $fg \in J \setminus A_I^{\triangle}$ and $\underline{0} \in A_I^{\triangle} \cap J$. This contradicts the fact that J is connected. Hence $I_{\psi_{\triangle}}(X)$ is the maximal connected ideal containing $\underline{0}$. □

Proposition 3.4. *For an ideal J in $C(X)_{\triangle}$ with the $u_I^{\triangle}$-topology, the following statements are equivalent:*

(i) *J is a connected ideal.*

(ii) *$J \subseteq A_I^{\triangle}$.*

(iii) *$J \subseteq I_{\psi_{\triangle}}(X)$.*

Proof. $(i) \Rightarrow (ii)$: Let J be a connected ideal of $C(X)_{\triangle}$. Since $A_I^{\triangle}$ is the component of $\underline{0}$ in $C(X)_{\triangle}$ with the $u_I^{\triangle}$-topology, this implies that $J \subseteq A_I^{\triangle}$.

$(ii) \Rightarrow (i)$: Let $J \subseteq A_I^{\triangle}$. As the map $\phi_f : \mathbb{R} \to C(X)_{\triangle}$ defined by $\phi_f(r) = r.f$ is continuous in the $u_I^{\triangle}$-topology (by Theorem 3.1), for all $f \in J$ and $J = \bigcup_{f \in J} \phi_f(\mathbb{R})$, it follows that J is connected.

$(ii) \Rightarrow (iii)$: Let $J \subseteq A_I^{\Delta}$. If possible, let there exist $f \in J \setminus I_{\psi_{\Delta}}(X)$. Then there exists $g \in C(X)_{\vartriangle}$ such that $fg \notin A_I^{\Delta}$. But $fg \in J \subseteq A_I^{\Delta}$, a contradiction. Hence $J \subseteq A_I^{\Delta}$.

$(iii) \Rightarrow (ii)$: Let $J \subseteq I_{\psi_{\Delta}}(X)$. As $I_{\psi_{\Delta}}(X) \subseteq A_I^{\Delta} \subseteq C(X)_{\vartriangle}$, it follows that $J \subseteq A_I^{\Delta}$. □

Proposition 3.5. *Let $J \subseteq C(X)_{\vartriangle}$ be an ideal containing the $Z_{\vartriangle}$-ideal I. Then int $J = \emptyset$ with respect to the $u_I^{\vartriangle}$-topology on $C(X)_{\vartriangle}$.*

Proof. Since the $u_I^{\vartriangle}$-topology on $C(X)_{\vartriangle}$ is weaker than the $m_I^{\vartriangle}$-topology on the same and by Proposition 2.2, *int* $J = \emptyset$ with respect to the $m_I^{\vartriangle}$-topology on $C(X)_{\vartriangle}$, $intJ = \emptyset$ with respect to the $u_I^{\vartriangle}$-topology on $C(X)_{\vartriangle}$ also. □

Now for the coincidence of the $u_I^{\vartriangle}$-topology and the $m_I^{\vartriangle}$-topology on $C(X)_{\vartriangle}$, we define $U_I^{\vartriangle^{++}} = \{u \in C(X)_{\vartriangle} : u(x) > \lambda$, for all $x \in Z \setminus H$, for some $Z \in Z_{\vartriangle}(I)$ and $H \in \Delta, \lambda > 0\}$.

Proposition 3.6. *For any $Z_{\vartriangle}$-ideal I in $C(X)_{\vartriangle}$, the $u_I^{\vartriangle}$-topology coincides with the $m_I^{\vartriangle}$-topology if and only if $U_I^{\vartriangle^{+}} \subseteq U_I^{\vartriangle^{++}}$.*

Proof. First let $U_I^{\vartriangle^{+}} \subseteq U_I^{\vartriangle^{++}}$. Then it is sufficient to show that the $m_I^{\vartriangle}$-topology is weaker than the $u_I^{\vartriangle}$-topology. Let $f \in m_I^{\Delta}(f,u)$, where $f \in C(X)_{\vartriangle}$ and $u \in U_I^{\vartriangle^{+}}$. Since $U_I^{\vartriangle^{+}} \subseteq U_I^{\vartriangle^{++}}$, there exists $\lambda > 0$ such that $u(x) > \lambda$, for all $x \in Z_1 \setminus H_1$, for some $Z_1 \in Z_{\vartriangle}(I)$ and $H_1 \in \Delta$. We now show that $u_I^{\Delta}(f,\lambda) \subseteq m_I^{\Delta}(f,u)$. Let $g \in u_I^{\Delta}(f,\lambda)$. Then $|g(x) - f(x)| < \lambda$, for all $x \in Z_2 \setminus H_2$, for some $Z_2 \in Z_{\vartriangle}(I)$ and $H_2 \in \Delta$. This implies that $|g(x)-f(x)| < \lambda < u(x)$, for all $x \in (Z_1 \cap Z_2) \setminus (H_1 \cup H_2)$, where $Z_1 \cap Z_2 \in Z_{\vartriangle}(I)$ and $H_1 \cup H_2 \in \Delta$. So $g \in m_I^{\Delta}(f,u)$. Hence $u_I^{\Delta}(f,\lambda) \subseteq m_I^{\Delta}(f,u)$.

Conversely, let the $u_I^{\vartriangle}$-topology coincide with the $m_I^{\vartriangle}$-topology on $C(X)_{\vartriangle}$ and $u \in U_I^{\vartriangle^{+}}$. Then $f \in m_I^{\Delta}(f,u)$, where $m_I^{\Delta}(f,u)$ is an open set in the $m_I^{\vartriangle}$-topology, for any $f \in C(X)_{\vartriangle}$. Hence there exists $\lambda > 0$ such that $u_I^{\Delta}(f,\lambda) \subseteq m_I^{\Delta}(f,u)$. Now $f + \frac{\lambda}{2} \in u_I^{\Delta}(f,\lambda)$ implies that $f + \frac{\lambda}{2} \in m_I^{\Delta}(f,u)$. Thus $u(x) > \frac{\lambda}{2}$, for all $x \in Z \setminus H$, for some $Z \in Z_{\vartriangle}(I)$ and $H \in \Delta$, which implies that $u \in U_I^{\vartriangle^{++}}$. Hence $U_I^{\vartriangle^{+}} \subseteq U_I^{\vartriangle^{++}}$. □

Next we give several other necessary and sufficient conditions for the coincidence of the $m_I^{\vartriangle}$-topology and the $u_I^{\vartriangle}$-topology on $C(X)_{\vartriangle}$.

Theorem 3.5. *The following statements are equivalent:*

(i) *The* $u_I^{\triangle}$*-topology and the* $m_I^{\triangle}$*-topology coincides on* $C(X)_{\triangle}$.

(ii) $C(X)_{\triangle}$ *with the* $u_I^{\triangle}$*-topology is a topological ring.*

(iii) $A_I^{\triangle} = C(X)_{\triangle}$.

(iv) $I_{\psi_{\triangle}}(X) = C(X)_{\triangle}$.

(v) $C(X)_{\triangle}$ *with the* $u_I^{\triangle}$*-topology is connected.*

Proof. $(i) \Rightarrow (ii)$: Obvious, since $C(X)_{\triangle}$ with the $m_{I_{\triangle}}$-topology is a topological ring.

$(ii) \Rightarrow (iii)$: Let $C(X)_{\triangle}$ with the $u_I^{\triangle}$-topology be a topological ring. Then $A_I^{\triangle}$ is an ideal of $C(X)_{\triangle}$ (as $A_I^{\triangle}$ is the component of $\underline{0}$ in $C(X)_{\triangle}$ with the $u_I^{\triangle}$-topology). Also $\underline{1} \in A_I^{\triangle} \Rightarrow C(X)_{\triangle} = A_I^{\triangle}$.

$(iii) \Leftrightarrow (iv)$: Follows from Theorem 2.5.

$(iii) \Leftrightarrow (v)$ Follows from Proposition 3.2.

$(iii) \Rightarrow (i)$: To show that the $u_I^{\triangle}$-topology and the $m_I^{\triangle}$-topology coincides on $C(X)_{\triangle}$, it is sufficient to show that ${U_I^{\triangle}}^{+} \subseteq {U_I^{\triangle}}^{++}$. Now, for all $f \in C(X)_{\triangle}$, f is bounded on some $Z \setminus H$, where $Z \in Z_{\triangle}(I)$ and $H \in \Delta$. Therefore $C(X)_{\triangle} = \bigcup_{\epsilon>0} u_I^{\triangle}(\underline{0}, \epsilon)$. Let $u \in {U_I^{\triangle}}^{+}$. Then $u(x) > 0$, for all $x \in Z_{\triangle}(g)$, for some $Z_{\triangle}(g) \in Z_{\triangle}(I)$, where $g \in I$. We consider $v = u^2 + g^2$. Then $v \in C(X)_{\triangle}$ and $v(x) > 0$, for all $x \in X$. Hence v is a unit of $C(X)_{\triangle}$ and thus $\frac{1}{v} \in C(X)_{\triangle}$. Therefore there exists $\epsilon > 0$ such that $\frac{1}{v} \in u_I^{\triangle}(\underline{0}, \epsilon)$. This implies that $\frac{1}{v(x)} < \epsilon$, for all $x \in Z \setminus H$, for some $Z \in Z_{\triangle}(I)$ and $H \in \Delta$. Therefore $(u^2 + g^2)(x) > \frac{1}{\epsilon}$, for all $x \in Z \setminus H$, for some $Z \in Z_{\triangle}(I)$ and $H \in \Delta$. Thus $u(x) > \frac{1}{\sqrt{\epsilon}}$, for all $x \in (Z \cap Z_{\triangle}(g)) \setminus H$, where $Z \cap Z_{\triangle}(g) \in Z_{\triangle}(I)$ and $H \in \Delta$. Hence $u \in {U_I^{\triangle}}^{++}$. □

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