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ON A GROUP APPROACH TO THE STUDY OF THE GENERAL RELATIVISTIC VACUUM CONSTRAINT EQUATIONS

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Abstract. In this article we will classify general relativistic vacuum constraint equations on a Riemannian manifold using a method based on the pointwise decomposition of tensor products (reducible with respect to the action of the orthogonal group) into irreducible components. Each selected class of equations will be described.
Keywords: compact Riemannian manifold, Ricci tensor, orthogonal decomposition, general relativistic constraint equations.

1. Introduction

The method of classifying tensor structures on an n-dimensional ($n \geq 2$) pseudo-Riemannian (or Riemannian) manifold (M, g) based on the decomposition of tensor products (reducible with respect to the action of the pseudo-orthogonal or orthogonal group $\mathcal{O}(g)$) into pointwize irreducible components has become traditional in differential geometry (see, e.g., [12, 18, 24, 25, 28]). This method is also used in theoretical physics (see, e.g., [6, 15, 23]). For example, in [6], Einstein-Cartan manifolds were classified based on the irreducible decomposition of the torsion tensor of an affine-metric connection (irreducible with respect to the action of the Lorentz

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group). Current results in this modern research direction in General Relativity Theory have been systematized in [6].

In the present paper, we return to the work [23] by one of the authors and consider the proposed classification of the Einstein equations based on the aforementioned method. We will apply this classification to the general relativistic vacuum coupling equations (see, for example, [7, 8, 13]; [2, pp. 47-48]).

2. The general relativistic vacuum constraint equations

Let $(\bar{M}, \bar{g})$ be a spacetime of dimension $n \geq 3$ solving the Einstein field equations (see [2, 4, 7])

$$\overline{Ric} - \frac{1}{2}\ \bar{s}\ \bar{g} + 2\Lambda g = \kappa T \tag{2.1}$$

where we denote by $\bar{g}$ the metric tensor with Lorentzian signature $(- + \cdots +)$, we also denote by $\overline{Ric}$ and $\bar{s}$ the Ricci tensor and the scalar curvature of $\bar{g}$, respectively.

We use the letter κ to denote a positive constant whose value (and physical dimensions) depends on the specific conventions one adopts. In addition, as is customary in the physical literature, T stands for the stress-energy tensor of the sources, while Λ represents the cosmological constant.

In the present paper, we focus on the vacuum case; that is, we consider the field equations with no sources ($T = 0$) and set $\Lambda = 0$ for the cosmological constant. Consequently, $(\bar{M}, \bar{g})$ is a Ricci-flat spacetime (see [7]). In this case, $(\bar{M}, \bar{g})$ is a special case of an Einstein manifold (see [3, p. 44]). An example of a Ricci-flat spacetime is the Schwarzschild spacetime, which describes a static black hole. In this geometry, the Ricci tensor is zero everywhere, but the spacetime is most certainly not flat.

On the other hand, let (M, g) be an n-dimensional ($n \geq 2$) Riemannian manifold with the Levi-Civita connection ∇ consistent with g. In the present paper, we consider the vacuum constraint equations on (M, g) (see [2, pp. 47-48] and [11]):

$$\begin{cases} s - g(K, K) + (trace_g K)^2 = 0; \\ div_g K - d(trace_g K) = 0, \end{cases} \tag{2.2}$$

where s is the scalar curvature of (M, g) and $K \in C^\infty(S^2 M)$ is a symmetric bilinear differential form defined on (M, g). The well-known problem is to construct solutions of the general relativistic vacuum constraint equations (see, for example, [7, 8, 13], [2, pp. 47- 48]). In turn, recall here that Bonnet's classical result on a local solution has the following form (see [7]): Given an initial data triple (M, g, K) there exists a vacuum spacetime $(\bar{M}, \bar{g})$, the local spacetime development, such that (M, g) is a spacelike hypersurface of $\bar{M}$ and g, K are the intrinsic metric and extrinsic curvature (i.e. first and second fundamental forms) induced by $\bar{g}$ on M. In this case, the Levi-Civita connection ∇ on the spacelike hypersurface (M, g) is compatible with the induced metric g.

Let $\bar{K} := K - (trace_g K)g$, then for this notation the second equation from (2.2) can be rewritten in the form $\delta\bar{K} = 0$ for the divergence operator δ defined by (see [13, p. 434])

$$(\delta\bar{K})(Z) := -trace_g[(X, Y) \to (\nabla_X \bar{K})(Y, Z)]$$

for arbitrary vector fields X, Y, Z on M.

3. Invariantly defined seven classes of the general relativistic vacuum constraint equations

Consider the subbundle $\mathfrak{K}(M) \subset T^*M \otimes S^2M$ on a Riemannian manifold (M, g), such that $T(X, Y, Z) = T(X, Z, Y)$ and $\sum_{k=1}^{n} T(X_k, X_k, Z) = 0$ for any $T \in \mathfrak{K}(M)$, arbitrary vector fields X, Y, Z and a local orthonormal basis $\{X_1, \ldots, X_n\}$ of vector fields on M.

In [23, 25], we proved that $\mathfrak{K}(M)$ has pointwise irreducible decomposition (irreducible with respect to the action of orthogonal group $\mathcal{O}(n)$)

$$\mathfrak{K}(M) = \mathfrak{K}_1(M) \oplus \mathfrak{K}_2(M) \oplus \mathfrak{K}_3(M),$$

where

$$\mathfrak{K}_1(M) = \{T \in \mathfrak{K}(M) |\, T(X, Y, Z) + T(Y, Z, X) + T(Z, X, Y) = 0\},$$

$$\mathfrak{K}_2(M) = \{T \in \mathfrak{K}(M) |\, T(X, Y, Z) - T(Y, X, Z) = 0\},$$

$$\mathfrak{K}_3(M) = \{T \in \mathfrak{K}(M) |\, T(X, Y, Z) =$$
$$\tfrac{1}{(n-1)(n+2)}\, [(n+1)T_{23}(X)g(Y, Z) + T_{23}(Y)g(X, Z) - T_{23}(Z)g(X, Y)]\},$$

$$T_{23}(Z) = \sum_{k=1}^{n} T(Z, X_k, X_k) = 0 \tag{3.1}$$

for arbitrary vector fields X, Y, Z and $\{X_1, \ldots, X_n\}$ is a local orthonormal basis of vector fields on M.

If we take a data triple (M, g, K), where (M, g) is a Riemannian manifold and $\bar{K} = K - (trace_g K)g$, then by (2.2) we conclude $\nabla\bar{K} \in \mathfrak{K}(M)$. In this case, the covariant derivative $\nabla\bar{K}$ of a symmetric bilinear differential form $\bar{K}$ is a cross-section of relevant invariant subbundles $\mathfrak{K}_1(M), \mathfrak{K}_2(M)$ and $\mathfrak{K}_3(M)$, their direct sums $\mathfrak{K}_1(M) \oplus \mathfrak{K}_2(M)$, $\mathfrak{K}_1(M) \oplus \mathfrak{K}_3(M)$, $\mathfrak{K}_2(M) \oplus \mathfrak{K}_3(M)$ and the subbundle $\mathfrak{K}_1(M) \cap \mathfrak{K}_2(M) \cap \mathfrak{K}_3(M)$.

To this pointwise irreducible decomposition of $\nabla\bar{K}$ there corresponds a rough classification of the general relativistic vacuum constraint equations in which each class includes the equation (2.2) for which $\nabla\bar{K}$ is a cross-section of one of the invariant subbundles $\mathfrak{K}_1(M), \mathfrak{K}_2(M)$ and $\mathfrak{K}_3(M)$ or of their direct sums. We supplement this list with an additional class for which $\nabla\bar{K}$ is a cross-section of the subbundle $\mathfrak{K}_1(M) \cap \mathfrak{K}_2(M) \cap \mathfrak{K}_3(M)$.

Theorem 3.1. *Let (M, g, K) be a data triple, where (M, g) is a Riemannian manifold and $K \in C^\infty(S^2M)$ be a symmetric bilinear differential form on (M, g). Then the seven classes of the vacuum constraint equations (2.2) defined on (M, g) can be singled out invariantly, where covariant derivatives of $\bar{K} = K - (trace_g K)g$ are cross-sections of the corresponding invariant subbundles $\mathfrak{K}_1(M), \mathfrak{K}_2(M)$ and $\mathfrak{K}_3(M)$, their direct sums $\mathfrak{K}_1(M) \oplus \mathfrak{K}_2(M)$, $\mathfrak{K}_1(M) \oplus \mathfrak{K}_3(M)$, $\mathfrak{K}_2(M) \oplus \mathfrak{K}_3(M)$ and the subbundle $\mathfrak{K}_1(M) \cap \mathfrak{K}_2(M) \cap \mathfrak{K}_3(M)$.*

4. The class $\mathfrak{K}_1 \oplus \mathfrak{K}_2$ and TT-tensors

Let (M, g, K) be a data triple, where (M, g) is a Riemannian manifold and $K \in C^\infty(S^2M)$ be a symmetric bilinear differential form on (M, g). The class $\mathfrak{K}_1 \oplus \mathfrak{K}_2$ of the vacuum constraint equations (2.2) is selected via condition (see [25])

$$(4.1) \qquad \sum_{k=1}^{n} (\nabla \bar{K})(X_k, X_k, Z) = 0, \qquad \sum_{k=1}^{n} (\nabla \bar{K})(Z, X_k, X_k) = 0$$

where $\{X_1, \dots, X_n\}$ is a local orthonormal basis of vector fields. Therefore, if the covariant derivative of $\bar{K} = K - (trace_g K)g$ is a cross-section of the subbundles $\mathfrak{K}_1(M) \oplus \mathfrak{K}_2(M)$, then

$$(4.2) \qquad \sum_{k=1}^{n} (\nabla K)(X_k, X_k, Z) = 0, \qquad \sum_{k=1}^{n} (\nabla K)(Z, X_k, X_k) = 0.$$

Obviously, the opposite statement is also true. In turn, if we consider (M, g) as a spacelike hypersurface of a vacuum spacetime $(\bar{M}, \bar{g})$, then K is the second fundamental forms of (M, g). In this case, $H := trace_g K$ is called the *mean curvature* of (M, g) (see [3, p. 38]). Therefore, in our case (M, g) is a spacelike hypersurface of constant mean curvature. As a result, the following theorem holds.

Theorem 4.1. *Let (M, g, K) be a data triple, where (M, g) is a Riemannian manifold and $K \in C^\infty(S^2M)$ be a symmetric bilinear differential form on (M, g). The corresponding vacuum constraint equations (2.2) belongs to the class $\mathfrak{K}_1 \oplus \mathfrak{K}_2$ if and only if equations (4.2) hold. Furthermore, if (M, g) is a spacelike hypersurface of a vacuum spacetime $(\bar{M}, \bar{g})$ with the second fundamental forms K, then it a spacelike hypersurface of constant mean curvature.*

We recall that a symmetric divergence free and traceless covariant two-tensor is called TT-tensor (see, for instance, [11]). As a consequence of a result of Bourguignon-Ebin-Marsden (see [3, p. 132]) the space of TT-tensors is an infinite-dimensional vector space on any compact Riemannian manifold (M, g). Such tensors are of fundamental importance in stability analysis in General Relativity (see, for instance, [10] and [16]) and in Riemannian geometry (see, for instance, [3, p. 346-347] and [5]). In turn, if K is a TT-tensor, then the covariant derivative of $\bar{K} = K - (trace_g K)g$

is a cross-section of the subbundles $\mathfrak{K}_1(M) \oplus \mathfrak{K}_2(M)$. At the same time, from (2.2) we deduce $s = g(K, K)$. Moreover, if we consider (M, g) as a spacelike hypersurface of a vacuum spacetime $(\bar{M}, \bar{g})$. In this case, $H := trace_g K = 0$ and hence (M, g) is a maximal spacelike hypersurface (that is, with zero mean curvature). The importance of these spacelike hypersurfaces in General Relativity is well-known and a summary of several reasons justifying this opinion can be found, for instance, in [16]. As a result, we have the following corollary.

Corollary 4.1. *Let (M, g, K) be a data triple, where (M, g) is a Riemannian manifold and $K \in C^\infty(S^2M)$ be a TT-tensor on (M, g). Then corresponding vacuum constraint equations (2.2) belongs to the class $\mathfrak{K}_1 \oplus \mathfrak{K}_2$ and the scalar curvature s of (M, g) has the form $s = g(K, K)$. Furthermore, if (M, g) is a spacelike hypersurface of a vacuum spacetime $(\bar{M}, \bar{g})$ with the second fundamental forms K, then it a maximal spacelike hypersurface.*

5. The class $\mathfrak{K}_2 \oplus \mathfrak{K}_3$ and Codazzi tensors

The class $\mathfrak{K}_2 \oplus \mathfrak{K}_3$ of the vacuum constraint equations (2.2) is selected via condition (see [25])

$$(\nabla_X \bar{K})(Y, Z) - (\nabla_Y \bar{K})(X, Z) =$$

$$= \frac{1}{n-1}\left[\sum_{k=1}^{n}(\nabla \bar{K})(X, X_k, X_k)g(Y, Z) - \sum_{k=1}^{n}(\nabla \bar{K})(Y, X_k, X_k)g(X, Z)\right], \tag{5.1}$$

$$\sum_{k=1}^{n}(\nabla \bar{K})(X_k, X_k, Z) = 0$$

for arbitrary vector fields X, Y, Z on M and a local orthonormal basis $\{X_1, \ldots, X_n\}$ of vector fields on M. Using the identity $\bar{K} = K - (trace_g K)g$, we can rewrite (5.1) in the following form:

$$(\nabla_X K)(Y, Z) - (\nabla_Y K)(X, Z) =$$

$$= \frac{1}{n-1}\left[\sum_{k=1}^{n}(\nabla K)(X, X_k, X_k)g(Y, Z) - \sum_{k=1}^{n}(\nabla K)(Y, X_k, X_k)g(X, Z)\right]. \tag{5.2}$$

It is well-known from [3, pp. 436-440]) that a symmetric 2-tensor field B on (M, g) is called a *Codazzi tensor* if $(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) = 0$ for arbitrary tangent vectors X, Y, Z. In this case, $B = K - \frac{1}{n-1}(trace_g K)g$ is a Codazzi tensor. Following theorem was proved.

Theorem 5.1. *Let (M, g, K) be a data triple, where (M, g) is a Riemannian manifold and $K \in C^\infty(S^2M)$ be a symmetric bilinear differential form on (M, g). The corresponding vacuum constraint equations (2.2) belongs to the class $\mathfrak{K}_2 \oplus \mathfrak{K}_3$ if and only if $K - \frac{1}{n-1}(trace_g K)g$ is a Codazzi tensor.*

The following local result is well-known (see [3, p. 436]): an arbitrary Codazzi tensor $B \in C^{\infty}(S^2M)$ defined on a Riemannian manifold (M, g) of constant curvature C has the form $B = \nabla df + Cfg$ for an arbitrary function $f \in C^{\infty}(M)$. In our case, $B = K - \frac{1}{n-1}(trace_g K)g$. Therefore, on a Riemannian manifold (M, g) of constant curvature C, the symmetric bilinear differential form K from the data triple (M, g, K) of the class $\mathfrak{K}_2 \oplus \mathfrak{K}_3$ has the form $K = \nabla df + (\Delta f - (n-1)Cf)g$ for the function $f \in C^{\infty}(M)$ and its Laplacian $\Delta f = trace_g(\nabla df)$. Therefore, the following statement holds.

Corollary 5.1. *Let (M, g, K) be a data triple, where (M, g) is a Riemannian manifold of constant curvature C and $K \in C^{\infty}(S^2M)$ be a symmetric bilinear differential form on (M, g). If the corresponding vacuum constraint equations(2.2) belongs to the class $\mathfrak{K}_2$, then $K = \nabla df + (\Delta f - (n-1)Cf)g$ for the function $f \in C^{\infty}(M)$.*

6. The class $\mathfrak{K}_1 \oplus \mathfrak{K}_3$ and Killing tensors

The class $\mathfrak{K}_1 \oplus \mathfrak{K}_3$ of the vacuum constraint equations (2.2) is selected via condition (see [25])

$$(\nabla_X \bar{K})(Y, Z) + (\nabla_Y \bar{K})(Z, X) + (\nabla_Z \bar{K})(X, Y) = \frac{1}{n+2}\{\nabla_X(trace_g \bar{K})g(Y, Z) + \nabla_Y(trace_g \bar{K})g(Z, X) + \nabla_Z(trace_g \bar{K})g(X, Y)\} \tag{6.1}$$

for arbitrary tangent vector fields X, Y, Z on M. It is well-known from [24]) that a symmetric bilinear 2-form B on (M, g) is called a *symmetric Killing tensor* if

$$(\nabla_X B)(Y, Z) + (\nabla_Y B)(Z, X) + (\nabla_Z B)(X, Y) = 0 \tag{6.2}$$

for arbitrary vector fields X, Y, Z on M. In this case, $B = K - \frac{1}{n+2}(trace_g K)g$ is a symmetric Killing tensor. Following theorem was proved.

Theorem 6.1. *Let (M, g, K) be a data triple, where (M, g) is a Riemannian manifold and $K \in C^{\infty}(S^2M)$ be a symmetric bilinear differential form on (M, g). The corresponding vacuum constraint equations (2.2) belongs to the class $\mathfrak{K}_1 \oplus \mathfrak{K}_3$ if and only if $K - \frac{1}{n+2}(trace_g K)g$ is a symmetric Killing tensor.*

On the other hand, let (M, g) be a compact Riemannian manifold of dimension $n \geq 2$ of nonpositive sectional curvature, then every symmetric Killing tensor of rank k is parallel. In addition, if M is connected and there is a point $x_0 \in M$ such that all sectional curvatures at x_0 are negative then every symmetric Killing tensor is of the form Cg^k for some constant C (see [9]). From this statement we conclude that the following corollary holds.

Corollary 6.1. *Let (M, g, K) be a data triple, where (M, g) is a compact Riemannian manifold of nonpositive sectional curvature. If (M, g, K) belongs to the class $\mathfrak{K}_1 \oplus \mathfrak{K}_3$, then K is parallel. In addition, if M is connected and there is a point $x_0 \in M$ such that all sectional curvatures at x_0 are negative then K has the form Cg for some constant C.*

We recall the following result: if (M, g) is a Riemannian manifold of constant curvature, then there exist a local coordinate system $x^1, \dots, x^n$ in which the components $\varphi_i j$ of an arbitrary symmetric Killing tensor $\varphi \in C^\infty(S^2 M)$ can be expressed in the form (see [27])

$$\varphi_{ij} = e^2 f(A_{ijkl} x^k x^l + B_{ijk} x^k + C_{ij})$$

where $f = \frac{1}{2(n+1)} \ln(det\, g)$ and A_{ijkl}, B_{ijk} and C_{ij} are constants which satisfy the following identities:

$$A_{ijkl} = A_{jikl}, \quad A_{ijkl} = A_{ijlk}, \quad A_{ijkl} + A_{ikjl} = 0,$$

$$B_{ijk} = B_{jik}, \quad B_{ijk} + B_{ikj} = 0, \quad C_{ij} = C_{ji}.$$

Therefore, we can conclude that

$$K_{ij} = e^{2f} \left\{ (A_{ijkl} + \tfrac{1}{2} g^{mr} A_{mrkl}) x^k x^l + (B_{ijk} x^k + \tfrac{1}{2} g^{mr} B_{mrk}) x^k + (C_{ij} + \tfrac{1}{2} g^{mr} C_{mr}) \right\}$$

since $B = K - \frac{1}{n+2}\ (trace_g K) g$ is a symmetric Killing tensor. Therefore, the following corollary holds.

Corollary 6.2. *Let (M, g, K) be a data triple, where (M, g) is a Riemannian manifold of constant curvature and $K \in C^\infty(S^2 M)$ be a symmetric bilinear differential form on (M, g). If the corresponding vacuum constraint equations (2.2) belongs to the class $\mathfrak{K}_1$, then there exist a local coordinate system $x^1, \dots, x^n$ in which the local components K_{ij} of K have the form*

$$K_{ij} = e^{2f} \left\{ (A_{ijkl} + \tfrac{1}{2}\, g^{mr} A_{mrkl}) x^k x^l + (B_{ijk} x^k + \tfrac{1}{2}\, g^{mr} B_{mrk}) x^k + (C_{ij} + \tfrac{1}{2}\, g^{mr} C_{mr}) \right\}$$

where $f = \frac{1}{2(n+1)} \ln(\det g)$ and A_{ijkl}, B_{ijk} and C_{ij} are constant which satisfy the following identities:

$$A_{ijkl} = A_{jikl}, \quad A_{ijkl} = A_{ijlk}, \quad A_{ijkl} + A_{ikjl} = 0,$$

$$B_{ijk} = B_{jik}, \quad B_{ijk} + B_{ikj} = 0, \quad C_{ij} = C_{ji}.$$

for the contravariant components g^{ij} of the metric tensor g and some constant C.

7. The class $\mathfrak{K}_1$ and integrals of geodesic equations

The class $\mathfrak{K}_1$ of the vacuum constraint equations (2.2) is selected via conditions

$$(\nabla_X \bar{K})(Y, Z) + (\nabla_Y \bar{K})(Z, X) + (\nabla_Z \bar{K})(X, Y) = 0 \tag{7.1}$$

and

$$\sum_{k=1}^{n} (\nabla \bar{K})(X_k, X_k, Z) = 0, \tag{7.2}$$

where $\{X_1, \ldots, X_n\}$ is a local orthonormal basis of vector fields.

Using the identity $\bar{K} = K - (trace_g K)g$, we can rewrite (6.1) in the following form:

$$(7.3)\quad \begin{aligned} &(\nabla_X K)(Y,Z) + (\nabla_Y K)(Z,X) + (\nabla_Z K)(X,Y) = \\ &= \nabla_X(trace_g K)g(Y,Z) + \nabla_Y(trace_g K)g(Z,X) + \nabla_Z(trace_g K)g(X,Y). \end{aligned}$$

From condition (2.2),(7.2) and (7.3) can be deduced that $d(trace_g K) = 0$ and hence $trace_g K = \text{const}$. Then equations (7.3) can be rewritten in the form

$$(7.4)\qquad (\nabla_X K)(Y,Z) + (\nabla_Y K)(Z,X) + (\nabla_Z K)(X,Y) = 0$$

for arbitrary vector fields X, Y, Z on M. The reverse is also true. Namely, if $trace_g K = \text{const}$ and (8.6) is satisfied, then the second equation from (2.2) becomes an identity.

If we consider (M, g) as a spacelike hypersurface of a vacuum spacetime $(\bar{M}, \bar{g})$, then K is the second fundamental forms of (M, g). In our case (M, g) is a spacelike hypersurface of constant mean curvature (see [17]).

On the other hand, an arbitrary symmetric Killing tensor $\varphi \in C^\infty S^2 M$ along each geodesic line $\gamma = \gamma(t)$ satisfies the condition $\varphi(X, X) = \text{const}$, where $X = d\gamma/dt$ and $t \in J \subset \mathbb{R}$ is a canonical parameter such that $\nabla_X X = 0$. In this case, one says that the equations of geodesics admit a quadratic first integral (see [1, 14]). Therefore, the symmetric bilinear differential form K of (M, g, K) determines a quadratic first integral of the equations of geodesics.

Following theorem was proved:

Theorem 7.1. *Let (M, g, K) be a data triple, where (M, g) is a Riemannian manifold and $K \in C^\infty(S^2 M)$ be a symmetric bilinear differential form on (M, g). The corresponding vacuum constraint equations (2.2) belongs to the class $\mathfrak{K}_1$ if and only if $trace_g K$ =const and the equations of geodesic lines admit a first quadratic integral $K(X, X)$ =const, where $X = d\gamma/dt$ for an arbitrary geodesic line $\gamma = \gamma(t)$ of (M, g). Furthermore, if (M, g) is a spacelike hypersurface of a vacuum spacetime $(\bar{M}, \bar{g})$ with the second fundamental forms K, then its mean curvature is constant.*

We recall our local result: if (M, g) is a Riemannian manifold of constant curvature, then there exists a local coordinate system $x^1, \ldots, x^n$ in which the components K_{ij} of K can be expressed in the form

$$\varphi_{ij} = e^{2f}(A_{ijkl}x^k x^l + B_{ijk}x^k + C_{ij})$$

where $f = \frac{1}{2(n+1)} \ln(\det g)$ and A_{ijkl}, B_{ijk} and C_{ij} are constants which satisfy the well-known identities. Therefore, we can conclude that $K_{ij} = e^{2f}(A_{ijkl}x^k x^l + B_{ijk}x^k + C_{ij})$, where $g^{ij}(A_{ijkl}x^k x^l + B_{ijk}x^k + C_{ij}) = C \cdot e^{-2f}$ for the contravariant components g^{ij} of the metric tensor g and the constant $C = trace_g K$. Therefore, the following corollary holds.

Corollary 7.1. *Let (M, g, K) be a data triple, where (M, g) is a Riemannian manifold of constant curvature and $K \in C^\infty(S^2M)$ be a symmetric bilinear differential form on (M, g). If the corresponding vacuum constraint equations (2.2) belongs to the class $\mathfrak{K}_1$, then there exist a local coordinate system $x^1, \dots, x^n$ in which the local components K_{ij} of K have the form $K_{ij} = e^{2f}(A_{ijkl}x^k x^l + B_{ijk}x^k + C_{ij})$ where $f = \frac{1}{2(n+1)} \ln(\det g)$ and A_{ijkl}, B_{ijk} and C_{ij} are constant which satisfy the following identities:*

$$A_{ijkl} = A_{jikl}, \quad A_{ijkl} = A_{ijlk}, \quad A_{ijkl} + A_{ikjl} = 0,$$
$$B_{ijk} = B_{jik}, \quad B_{ijk} + B_{ikj} = 0, \quad C_{ij} = C_{ji}.$$
$$g^{ij}(A_{ijkl}x^k x^l + B_{ijk}x^k + C_{ij}) = C \cdot e^{-2f}$$

for the contravariant components g^{ij} of the metric tensor g and $C = trace_g K$.

Let $K \in C^\infty(S_0^pM)$ be a symmetric traceless Killing tensor on (M, g). In this case, we proved the following theorem (see [26]): On a simply connected complete Riemannian manifold (M, g) of nonpositive sectional curvature, any symmetric traceless Killing tensor $\varphi \in C^\infty(S_0^pM)$, $p \geq 2$, such that $\int_M \|\varphi\|^q dv_g < \infty$ for at least one $q \in (0, \infty)$ is a parallel tensor field. If, in addition, the volume of the manifold is infinite, then there exist no nonzero traceless Killing p-tensors on it for $p \geq 2$. From this theorem we conclude that the following corollary holds.

Corollary 7.2. *On a simply connected complete Riemannian manifold (M, g) with nonpositive sectional curvature and infinite volume there is not the data triple (M, g, K) of the class $\mathfrak{K}_1$ such that K is a traceless symmetric bilinear differential form and $\int_M \|K\|^q dv_g < \infty$ for at least one $q \in (0, \infty)$. On the other hand, if on a simply connected complete Riemannian manifold (M, g) with nonpositive sectional curvature there is the data triple (M, g, K) of the class $\mathfrak{K}_1$ such that $\int_M \|K\|^q dv_g < \infty$ for at least one $q \in (0, \infty)$, then K is a parallel tensor field.*

8. The class $\mathfrak{K}_2$ and bilinear symmetric harmonic forms

Class $\mathfrak{K}_2$ of the vacuum constraint equations (2.2) is selected via condition (see [23])

$$(\nabla_X \bar{K})(Y, Z) - (\nabla_Y \bar{K})(X, Z) = 0 \tag{8.1}$$

and

$$\sum_{k=1}^{n} (\nabla \bar{K})(X_k, X_k, Z) = 0, \tag{8.2}$$

where X, Y, Z are arbitrary vector fields and $\{X_1, \dots, X_n\}$ is a local orthonormal basis of vector fields on M. From (8.1) and (8.2) we deduce

$$\sum_{k=1}^{n} (\nabla \bar{K})(Z, X_k, X_k) = 0. \tag{8.3}$$

Equations (8.1) are known as Codazzi equations (see, for details, [3, pp. 436-440]). Using the identity $\bar{K} = K - (trace_g K)g$, we can rewrite (8.1) in the following form:

$$(\nabla_X K)(Y,Z) - (\nabla_Y K)(X,Z) = \\ = \nabla_X(trace_g K)g(Y,Z) - \nabla_Y(trace_g K)g(X,Z). \tag{8.4}$$

And furthermore, we can rewrite (8.6) as follows:

$$\sum_{k=1}^{n}(\nabla K)(Z,X_k,X_k) = 0 \tag{8.5}$$

and hence $trace_g K =$ const. In this case, equations (8.4) can be rewritten in the form

$$(\nabla_X K)(Y,Z) - (\nabla_Y K)(X,Z) = 0. \tag{8.6}$$

At the same time, from (8.6) we deduce that $\delta K = 0$ since $d(trace_g K) = 0$. In this case, equations (2.2) are satisfied automatically. It is well-known that a symmetric 2-tensor field B on (M,g) is called a *Codazzi tensor* if $(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) = 0$ for arbitrary tangent vectors X, Y, Z. In addition, B is called *harmonic* if B is a Codazzi tensor with constant trace (see [27, p. 350] and [28]).

Remark 1 (see [19, p. 350]). Simple examples of bilinear symmetric harmonic forms are the second fundamental form of a hypersurface with constant mean curvature of a Riemannian manifold of constant sectional curvature and the Ricci tensor of a locally conformal flat Riemannian manifold of constant scalar curvature.

Any one can be find in [21] other properties of bilinear symmetric harmonic forms. Following theorem was proved.

Theorem 8.1. *Let (M,g,K) be a data triple, where (M,g) is a Riemannian manifold and $K \in C^\infty(S^2M)$ be a bilinear symmetric differential form on (M,g). The corresponding vacuum constraint equations (2.2) belongs to the class $\mathfrak{K}_2$ if and only if K is harmonic. Furthermore, if (M,g) is a spacelike hypersurface of a vacuum spacetime $(\bar{M},\bar{g})$ with the second fundamental forms K, then its mean curvature is constant.*

In turn, we proved that every harmonic symmetric bilinear form $B \in C^\infty(S^2M)$ on a compact Riemannian manifold (M,g) with nonpositive sectional curvature is parallel. In addition, if M is connected and there is a point $x_0 \in M$ such that all sectional curvatures at x_0 are positive then every harmonic symmetric bilinear form is zero (see [21]). Therefore, we can formulate the corollary.

Corollary 8.1. *Let (M,g,K) be a data triple, where (M,g) is a compact Riemannian manifold of nonnegative sectional curvature. If (M,g,K) belongs to the class $\mathfrak{K}_2$, then K is parallel. In addition, if M is connected and there is a point $x_0 \in M$ such that all sectional curvatures at x_0 are positive, then K is zero.*

The following local result is well-known (see [3, p. 436]): an arbitrary Codazzi tensor $B \in C^\infty(S^2M)$ defined on a Riemannian manifold (M, g) of constant curvature C has the form $B = \nabla df + Cfg$ for an arbitrary function $f \in C^\infty(M)$. Therefore, on a Riemannian manifold (M, g) of constant curvature C, the symmetric bilinear differential form K from the data triple (M, g, K) of the class $\mathfrak{K}_2$ has the form $K = \nabla df + Cfg$ for the function $f \in C^\infty(M)$ that is a solution to the Poisson equation $\Delta f + nCfg = \text{const}$. Therefore, the following statement holds.

Corollary 8.2. *Let (M, g, K) be a data triple, where (M, g) is a Riemannian manifold of constant curvature C and $K \in C^\infty(S^2M)$ be a symmetric bilinear differential form on (M, g). If the corresponding vacuum constraint equations (2.2) belongs to the class $\mathfrak{K}_2$, then $K = \nabla df + Cfg$ for the function $f \in C^\infty(M)$ that is a solution of the Poisson equation $\Delta f + nCfg = const.$*

Remark 2 In turn, for a symmetric bilinear harmonic differential form $B \in C^\infty(S^2M)$ on (M, g) we can also conclude that $B = \nabla df + Cfg$ for the function $f \in C^\infty(M)$ that is a solution of the Poisson equation $\Delta f + nCfg = \text{const}$.

9. The class $\mathfrak{K}_3$ and geodesic mappings

Class $\mathfrak{K}_3$ of the vacuum constraint equations (2.2) is selected via condition

$$(9.1)\qquad \begin{aligned} (\nabla_X \bar{K})(Y,Z) = \tfrac{1}{(n+2)(n-1)}\,[(n+1)\nabla_X(trace_g \bar{K})g(Y,Z) - \\ \nabla_Y(trace_g \bar{K})g(Z,X) - \nabla_Z(trace_g \bar{K})g(X,Y)] \end{aligned}$$

for arbitrary vector fields X, Y, Z on M. Using the identity $\bar{K} = K - (trace_g K)g$, we can rewrite (9.1) in the following form:

$$(9.2)\qquad \begin{aligned} (\nabla_X K)(Y,Z) = \tfrac{1}{n+2}\,[\nabla_X(trace_g K)g(Y,Z) + \\ \nabla_Y(trace_g K)g(Z,X) + \nabla_Z(trace_g K)g(X,Y)]. \end{aligned}$$

Next, we in introduce the symmetric bilinear form $B = K - \frac{1}{n+2}\,(trace_g K)g$, which, as easily follows from (9.2), satisfy the differential equations

$$(9.3)\qquad (\nabla_X B)(Y,Z) = \theta(Y)g(Z,X) + \theta(Z)g(Y,X),$$

where $\theta(Y) = \frac{1}{2}\,\nabla_Y(trace_g B)$. At the same time, by well-known Sinyukov's theorem (see [22, p. 122], [17, p. 329]), a Riemannian manifold (M, g) bears a bilinear symmetric differential form B with components satisfying (9.3) if and only if (M, g) admits a projective diffeomorphism to some Riemannian manifold $(\tilde{M}, \tilde{g})$. We recall that, by definition, a projective diffeomorphism $f\colon (M, g) \to (\tilde{M}, \tilde{g})$ maps the geodesics of (M, g) to those of $(\tilde{M}, \tilde{g})$. Thus, we have proved the following result.

Theorem 9.1. *Let (M, g, K) be a data triple, where (M, g) is a Riemannian manifold and $K \in C^\infty(S^2M)$ be a bilinear symmetric differential form on (M, g). If corresponding vacuum constraint equations (2.2) belongs to the class $\mathfrak{K}_3$, then (M, g) admits a projective diffeomorphism to another Riemannian manifold $(\tilde{M}, \tilde{g})$.*

By (9.2) class $\mathfrak{K}_3$ of the vacuum constraint equations (2.2) is selected via condition $\nabla K \in C^\infty(S^3M)$ and hence K is a Codazzi tensor. At the same time, it is easy to check that $B = K - \frac{3}{n+2}(trace_g K)g$ is a symmetric Killing tensor. At the same time, every symmetric Killing two-tensor is parallel on a compact Riemannian manifold (M, g) of nonpositive sectional curvature. In addition, if M is connected and there is a point $x_0 \in M$ such that all sectional curvatures at x_0 are negative then every symmetric Killing two-tensor is of the form Cg for some constant C. Therefore, in our case from the second equations of (2.2) and $\nabla B = \nabla K - \frac{3}{n+2}\nabla(trace_g K)g = 0$ we obtain $\nabla K = 0$. In particular, $trace_g K =$ const and whence $K = \bar{C}g$ for some constant $\bar{C}$. Then we can formulate the following corollary.

Corollary 9.1. *Let (M, g, K) be a data triple, where (M, g) is a compact Riemannian manifold of nonpositive sectional curvature and $K \in C^\infty(S^2M)$ be a symmetric bilinear differential form on (M, g). If (M, g, K) belongs to the class $\mathfrak{K}_3$, then K is parallel. In addition, if M is connected and there is a point $x_0 \in M$ such that all sectional curvatures at x_0 are negative, then $K = Cg$ for some constant C*

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REFERENCES

1. L. J. Alas, A. Romero and M. Sanchez: *Spacelike hypersurfaces of constant mean curvature and Calabi-Bernstein type problems.* Thoku Math. J. **49** (1997), 337–345.
2. R. Avalos and J. H. Lira: *The Einstein Constraint Equations.* Brazil, Books in Bytes, (2021).
3. A. L. Besse: *Einstein Manifolds.* Springer-Verlag, Berlin & Heidelberg, (2008).
4. R. L. Bishop and S. I. Goldberg: *A characterization of the Euclidean sphere.* Bull. Amer. Math. Soc. **72** (1966), 122–124.
5. T. Branson: *Stein-Weiss operators and ellipticity.* J. Funct. Anal. **151** (1997), 334–383.
6. S. Capozziello, G. Lambiase and C. Stornaiolo: *Geometric classification of the torsion tensor in space-time.* Ann. Phys. **10** (2001), 713–727.
7. A. Carlotto: *The general relativistic constraint equations.* Living Reviews in Relativity **24** (2021), 1–170.
8. P. T. Chru and E. Delay: *On Mapping Properties of the General Relativistic Constraints Operator in Weighted Function Spaces, with Applications.* Mmoires de la Socit Mathmatique de France, (2003).
9. N. S. Dairbekov and V. A. Sharafutdinov: *On conformal Killing symmetric tensor fields on Riemannian manifolds.* Siberian Adv. Math. **21** (2011), 1–41.
10. R. Garattini: *Self sustained tranversable wormholes?.* Class. Quant. Grav. **22** (2005), 2673–2682.

11. R. Gicquaud and A. Ngo: *A new point of view on the solutions to the Einstein constraint equations with arbitrary mean curvature and small TT-tensor*. Class. Quant. Grav. **31** (2014), 195014.

12. A. Gray: *Einstein-like manifolds which are not Einstein*. Geom. Dedicate. **7** (1978), 259–280.

13. F. Helmut: *Cauchy problems for the conformal vacuum field equations in General Relativity*. Commun. Math. Phys. **91** (1983), 445–472.

14. D. Kramer, H. Stephani, M. Maccallum and E. Herl: *Exact Solutions of the Einstein Field Solutions*. Deutscher Verlag der Wissenschaften, Berlin, (1980).

15. C. A. Mantica, L. G. Molinary, Y. J. Suh and S. Shenawy: *Perfect-fluid, generalized Robertson-Walker space-times, and Gray's decomposition*. J. Math. Phys. **60** (2019), 052506.

16. J. E. Marsden and F. J. Tipler: *Maximal hypersurfaces and foliations of constant mean curvature in General Relativity*. Phys. Rep. **66** (1980), 109–139.

17. J. Mikeš, E. Stepanova, A. Vanžurová at al.: *Differential Geometry of Special Mappings*. 2th Ed. Palacky University Press, Olomouc, (2019).

18. A. M. Naveira: *A classification of Riemannian almost product manifolds*. Rend. Mat. Appl. **3** (1983), 577–592.

19. P. Peterson: *Riemannian Geometry (Graduate Texts in Mathematics Book 171)*. 3th Ed. Springer Int. Publishing AG, (2016).

20. R. Rani, S. B. Edgar and A. Barnes: *Killing tensors and conformal Killing tensors from conformal Killing vectors*. Thoku Math. J. **20** (2003), 1929–1942.

21. V. Rovenski, S. Stepanov and I. Tsyganok: *The Bourguignon Laplacian and harmonic symmetric bilinear forms*. Mathematics **8** (2020), Article number 83.

22. N. S. Sinyukov: *Geodesic Mappings of Riemannian Spaces*. Nauka, Moscow, (1979). (Russian).

23. S. E. Stepanov: *On a group approach to studying the Einstein and Maxwell equations*. Theor. Math. Phys. **111** (1997), 419–427.

24. S. E. Stepanov, I. A. Aleksandrova and I. I. Tsyganok: *On symmetric Killing tensors and Codazzi tensors of ranks $p \geq 2$*. Progress in Mathematics (J. Math. Sci. (NY)) **276** (2023), 443–468.

25. S. E. Stepanov and I. G. Shandra: *Harmonic diffeomorphisms of manifolds*. St. Petersburg Math. J. **16** (2005), 401-412.

26. S. E. Stepanov and I. I. Tsyganok: *Codazzi and Killing tensors on a complete Riemannian manifold*. Math. Notes **109** (2021), 932–939.

27. S. E. Stepanov, I. I. Tsyganok and M. B. Khripunova: *The Killing tensors on an n-dimensional manifold with $SL(n,\mathbb{R})$-structure*. Acta Univ. Palacki. Olomuc. Math. **54** (2015), 95–105.

28. F. Tricerri and L. Vanhecke: *Homogeneous structures*. Progress in Mathematics (Differential geometry) **32** (1983), 234–246.