

GRADIENT ρ -EINSTEIN SOLITONS AND APPLICATIONS

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Abstract. In this paper, we mainly study gradient ρ -Einstein solitons on doubly warped product manifolds. More explicitly, we obtain the necessary and sufficient conditions for a doubly warped product manifold to be a gradient ρ -Einstein soliton. We also apply our main result to warped product spacetime models such as generalized Robertson-Walker and standard static spacetimes as well as 3-dimensional Walker manifolds. We finally establish that there is no 3-dimensional essentially conformally symmetric gradient ρ -Einstein soliton.

Keywords: manifolds, ρ -Einstein soliton, Ricci solution.

1. Introduction

The theory of Ricci solitons is a topic that has been studied extensively for the last twenty years. This theory, in addition to being famous for solving the Poincaré Conjecture proposed by Perelman, is a subject having a wide range of uses in differential geometry and physical applications. One of the most important reasons why the Ricci flow theory is important and worth studying is that it has self-similar solutions called Ricci solitons and are often used to classify the singularity models.

The motivation for the Ricci soliton and Ricci flow theory has paved the way for the study of different flow equations and their special solutions in the literature. In recent years, combinations of different flow equations have attracted much attention.

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For example, the Ricci-Yamabe map, defined as a scalar combination of Ricci and Yamabe flow equations, has been introduced by Güler and Crasmareanu [27] in 2019 as one of the most recent examples. Another of these new flow equations is first introduced by Bourguignon [9, 13], defined by the equation

$$\partial_t g = -2(\text{Ric} - \rho \tau g)$$

and named as Ricci-Bourguignon flow. Also, it is known that with appropriate rescaling over time, when ρ is not positive, the Ricci-Bourguignon flow is an interpolation between the Ricci flow and the Yamabe flow, (for more, we refer [30, 44, 12]).

As a self-similar solution to the Ricci-Bourguignon flow, the gradient ρ -Einstein soliton is defined as follows: Let (M^n, g) be an n -dimensional semi-Riemannian manifold and ρ is a real number. Then (M^n, g) is said to be a gradient ρ -Einstein soliton if there exist a smooth function φ on M and a real number λ such that

$$(1.1) \quad \text{Ric} + \text{Hess}(\varphi) = (\rho\tau + \lambda)g,$$

where τ is the scalar curvature of M . Here, the underlying gradient ρ -Einstein soliton is denoted by $(M, g, \varphi, \rho, \lambda)$.

Notice that if $\rho = 0$, then (1.1) is reduced to the gradient Ricci soliton equation. The gradient ρ -Einstein solitons are said to be steady if $\lambda = 0$, shrinking if $\lambda > 0$, and expanding if $\lambda < 0$. Moreover, corresponding to special values of the parameter ρ , we refer to the ρ -Einstein solitons with different names: The ρ -Einstein soliton is called an Einstein soliton if $\rho = 1/2$, a traceless soliton if $\rho = 1/n$, and a Schouten soliton if $\rho = 1/2(n-1)$.

Now, we will refer to some major studies in the field of gradient ρ -Einstein solitons. In [14] the Catino, Mazzieri and Mongodi consider shrinking gradient ρ -Einstein solitons and they classify non-compact gradient shrinkers with bounded non-negative sectional curvature. In [13], they also prove that every compact gradient Einstein, traceless Ricci, or Schouten soliton is trivial. Furthermore, they establish that every complete steady gradient Schouten soliton is trivial, and every complete 3-dimensional shrinking gradient Schouten soliton is isometric to a finite quotient of either $\mathbb{S}^3$, $\mathbb{R}^3$ or $\mathbb{R} \times \mathbb{S}^2$. By using algebraic curvature estimates and the Yamabe-Sobolev inequality, integral pinching rigidity results for compact gradient shrinking gradient ρ -Einstein solitons are proved in [32]. In [35], Mondal, Kumar and Shaikh obtain some sufficient conditions for a Riemannian manifold admitting an almost η -Ricci soliton to be compact. In [31], Ho considers Ricci-Bourguignon flows of 3-dimensional locally homogeneous geometries on a closed 3-dimensional manifold endowed with evolution of Yamabe constants and then he studies the complete Bach-flat shrinking gradient solitons as well. In [37], Pina and Menezes study the gradient ρ -Einstein solitons that are conformal to a pseudo-Euclidean space and invariant under the action of the pseudo-orthogonal group and then they provide all the gradient Schouten solitons for this type. In [7], another version of gradient Ricci-Bourguignon soliton is studied by means of the potential vector field. In [22], the author showed that a compact gradient Ricci-Bourguignon almost

soliton is isometric to a Euclidean sphere if it has constant scalar curvature or its associated vector field is conformal.

Also, Blaga and Taştan study many types of gradient solitons including Riemann, Ricci, Yamabe and conformal solitons on doubly warped product manifolds, [8]. Moreover, in [6], Blaga describes some of the different geometrical aspects of a perfect fluid spacetime in terms of different curvature tensors and some special type of Ricci solitons.

In [28], quasi-Yamabe gradient solitons on warped product manifolds are studied and the authors prove the existence of the non-trivial gradient Yamabe soliton on generalized Robertson-Walker spacetimes, standard static spacetimes, Walker manifolds and pp-wave spacetimes. In [19], Ricci solitons, gradient Ricci solitons and gradient Yamabe solitons in a perfect fluid spacetime are characterized in $f(\mathcal{R})$ -gravity. In [41], Turki et al. study gradient ρ -Einstein solitons on warped product manifolds and investigate Einstein solitons on warped product manifolds admitting conformal vector fields.

Although the gradient ρ -Einstein soliton structure has been studied mainly on the warped product manifolds in the literature, it can be said that this soliton structure has not yet been studied in depth for double warped product manifolds which is a much more general product metric and, even for Walker manifolds. Therefore, this paper is aimed to fill this gap in the literature and it is organized as follows. In Section 2, after we provide basic curvature-related formulas for doubly warped products, we derive our main results, that is we obtain the necessary and sufficient conditions for a doubly warped product manifold to be a gradient ρ -Einstein soliton. In Section 3, we consider gradient ρ -Einstein solitons on warped product spacetime models and 3-dimensional Walker manifolds. We finally establish that there exists no 3-dimensional essentially conformally symmetric gradient ρ -Einstein soliton.

2. Doubly warped product gradient ρ -Einstein solitons

Let (M_1, g_1) and (M_2, g_2) be two semi-Riemannian manifolds of dimensions m_i , $i = 1, 2$. Let $\pi_1 : M_1 \times M_2 \rightarrow M_1$ and $\pi_2 : M_1 \times M_2 \rightarrow M_2$ be the canonical projections. Then doubly warped product manifold $(f_2 M_1 \times_{f_1} M_2, g)$ of (M_1, g_1) and (M_2, g_2) is the product manifold $M_1 \times M_2$ equipped with the metric

$$(2.1) \quad g = (f_2 \circ \pi_2)^2 \pi_1^* g_1 + (f_1 \circ \pi_1)^2 \pi_2^* g_2$$

where $f_1 : M_1 \rightarrow (0, \infty)$ and $f_2 : M_2 \rightarrow (0, \infty)$ are called the warping functions of the doubly warped product [5, 23, 17, 43]. Doubly warped product manifolds studied by different aspects. In [24, 25], Gebarowski studies the divergence-free and conformally recurrent doubly warped products. Also, in [29], the non-existence of compact Einstein doubly warped product with non-positive scalar curvature is proved. We also refer to the recent article [26] about the doubly warped products of smooth measure spaces to establish Bakry-Émery Ricci curvature (lower) bounds thereof in terms of those of the factors.

If one of the functions f_1 or f_2 is constant, then $(_{f_2}M_1 \times_{f_1} M_2, g)$ reduces to a warped product. If both f_1 and f_2 are constant, then we obtain a direct product manifold.

First, we will give the lemmas we need to prove our main results. We skip their complete proof, but for the detailed calculations we used the references [23, 43].

Lemma 2.1. *On a doubly warped product of the form $(_{f_2}M_1 \times_{f_1} M_2, g)$, for any $X, Y \in \mathfrak{L}(M_1)$ and $U, V \in \mathfrak{L}(M_2)$, we have:*

1. $\nabla_X Y = \nabla_X^1 Y - g(X, Y) \nabla(\ln(f_2 \circ \pi_2))$,
2. $\nabla_X V = \nabla_V X = V(\ln(f_2 \circ \pi_2))X + X(\ln(f_1 \circ \pi_1))V$,
3. $\nabla_U V = \nabla_U^2 V - g(U, V) \nabla(\ln(f_1 \circ \pi_1))$.

Remark 2.1. From now on, we will denote $k = \ln f_1$ (respectively, $l = \ln f_2$) and use the same symbol for the function k (respectively, l) and its pullback $k \circ \pi_1$ (respectively, $l \circ \pi_2$).

Lemma 2.2. *Let φ be a smooth function on a doubly warped product of the form $(_{f_2}M_1 \times_{f_1} M_2, g)$. For any $X, Y \in \mathfrak{L}(M_1)$ and $U, V \in \mathfrak{L}(M_2)$, the Hessian of φ satisfies:*

1. $(\text{Hess}\varphi)(X, Y) = (\text{Hess}_1\varphi)(X, Y) + g(\nabla l, \nabla\varphi)g(X, Y)$,
2. $(\text{Hess}\varphi)(U, V) = (\text{Hess}_2\varphi)(U, V) + g(\nabla k, \nabla\varphi)g(U, V)$,
3. $(\text{Hess}\varphi)(X, U) = -X(k)U(\varphi) - X(\varphi)U(l)$,

where $k = \ln f_1$ and $l = \ln f_2$.

Thus, by using Lemma 2.2, we have:

Lemma 2.3. *On a doubly warped product of the form $(_{f_2}M_1 \times_{f_1} M_2, g)$, for any $X, Y \in \mathfrak{L}(M_1)$ and $U, V \in \mathfrak{L}(M_2)$, we have*

1. $(\text{Hess}k)(X, Y) = (\text{Hess}_1k)(X, Y)$,
2. $(\text{Hess}l)(X, Y) = g(\nabla l, \nabla l)g(X, Y)$,
3. $(\text{Hess}k)(U, V) = g(\nabla k, \nabla k)g(U, V)$,
4. $(\text{Hess}l)(U, V) = (\text{Hess}_2l)(U, V)$.

where $k = \ln f_1$ and $l = \ln f_2$.

Lemma 2.4. *For any $X, Y \in \mathfrak{L}(M_1)$ and $U, V \in \mathfrak{L}(M_2)$, the components of the Ricci tensor of a doubly warped product of the form $(_{f_2}M_1 \times_{f_1} M_2, g)$ are given by:*

1. $\text{Ric}(X, Y) = \text{Ric}_1(X, Y) - \frac{m_2}{f_1} \text{Hess}_1 f_1(X, Y) - (\Delta l)g(X, Y),$
2. $\text{Ric}(X, V) = (m_1 + m_2 - 2)X(k)V(l),$
3. $\text{Ric}(U, V) = \text{Ric}_2(U, V) - \frac{m_1}{f_2} \text{Hess}_2 f_2(U, V) - (\Delta k)g(U, V),$

where $k = \ln f_1$, $l = \ln f_2$ and Δ denotes the Laplacian on $(f_2 M_1 \times_{f_1} M_2, g)$ and $m_i = \dim(M_i)$, for $(i = 1, 2)$.

Lemma 2.5. *The scalar curvatures of a doubly warped product manifold $(f_2 M_1 \times_{f_1} M_2, g)$, its base (M_1, g_1) and fiber (M_2, g_2) are related by:*

$$\tau = \frac{\tau_1}{f_2^2} + \frac{\tau_2}{f_1^2} - \frac{m_2}{f_1 f_2^2} \Delta_1 f_1 - \frac{m_1}{f_2 f_1^2} \Delta_2 f_2 - m_1 \Delta l - m_2 \Delta k,$$

where τ_i denotes the scalar curvatures of (M_i, g_i) ($i = 1, 2$), Δ_i denotes the Laplacian on M_i , ($i = 1, 2$).

Theorem 2.1. *Let $(M =_{f_2} M_1 \times_{f_1} M_2, g)$ be a doubly warped product manifold admitting a gradient ρ -Einstein soliton with the potential function φ . Then:*

- (1) *If $\varphi \in \mathcal{F}(M_1)$, then (M, g) reduces to the warped product $(M_1 \times_{f_1} M_2, \tilde{g}_1 + f_1^2 g_2)$ or $\varphi = (m_1 + m_2 - 2) \ln f_1$.*
- (2) *If $\varphi \in \mathcal{F}(M_2)$, then (M, g) reduces to the warped product $(f_2 M_1 \times M_2, f_2^2 g_1 + \tilde{g}_2)$ or $\varphi = -(m_1 + m_2 - 2) \ln f_2$.*
- (3) *If $(M =_{f_2} M_1 \times_{f_1} M_2, g)$ is a non-trivial doubly warped product, then the potential function has to depend on both M_1 and M_2 and satisfies*

$$(2.2) \quad (m_1 + m_2 - 2)X(k)U(l) - X(k)U(\varphi) - X(\varphi)U(l) = 0,$$

for all $X \in \mathfrak{L}(M_1)$ and $U \in \mathfrak{L}(M_2)$, where $\mathcal{F}(M_i)$ denote the set of all smooth functions on M_i ($i = 1, 2$).

Proof. First note that if (M, g) is a gradient ρ -Einstein soliton doubly warped product manifold, then

$$\text{Ric} + \text{Hess}(\varphi) = (\rho\tau + \lambda)g, \quad \text{where } \rho, \lambda \in \mathbb{R}.$$

Substituting $X \in \mathfrak{L}(M_1)$, and $U \in \mathfrak{L}(M_2)$, the above equation yields:

$$(2.3) \quad (m_1 + m_2 - 2)X(k)U(l) - X(k)U(\varphi) - X(\varphi)U(l) = 0.$$

(1) If the potential function φ only depends on M_1 , then $U(l) = 0$. By equation (2.3),

$$U(l)[(m_1 + m_2 - 2)X(k) - X(\varphi)] = 0.$$

Thus,

- If $U(l) = 0$ for all $U \in \mathfrak{L}(M_2)$, then $l = \ln f_2$ is constant as $l \in \mathcal{F}(M_2)$. So, g can be simplified to a singly warped product metric.
- If $X(\varphi) = (m_1 + m_2 - 2)X(k)$, for all $X \in \mathfrak{L}(M_1)$, then φ and $k = \ln f_1$ are proportional.

(2) If the potential function φ depends only on M_2 , then $X(\varphi) = 0$, for any $X \in \mathfrak{L}(M_1)$ and the equation (2.3) yields,

$$X(k)[(m_1 + m_2 - 2)U(l) - U(\varphi)] = 0.$$

Thus, $X(k) = 0$ or $U(\varphi) = (m_1 + m_2 - 2)U(l)$.

- If $X(k) = 0$, for any $X \in \mathfrak{L}(M_1)$, then $l = \ln f_2$ is constant as $l \in \mathcal{F}(M_2)$. So, g can be simplified to a singly warped product metric. $\square$

Definition 2.1. For the potential function φ , if the equation

$$(2.4) \quad \text{Ric} + \text{Hess}(\varphi) = \gamma g + \mu \eta \otimes \eta$$

holds, where η is the 1-form, γ and μ are smooth functions, the manifold is called gradient almost η -Ricci soliton [18] and briefly denoted by $(M, g, \varphi, \eta, \mu, \gamma)$.

Theorem 2.2. Let $(M =_{f_2} M_1 \times_{f_1} M_2, g)$ be a doubly warped product manifold. Then $(M =_{f_2} M_1 \times_{f_1} M_2, g, \varphi, \rho, \lambda)$ is a gradient ρ -Einstein soliton if and only if the following conditions hold:

- (1) $(M_1, g_1, \varphi_1, \eta_1, \mu_1, \gamma_1)$ is a gradient almost η -Ricci soliton, where $\varphi_1 = \varphi - m_2 k$, $\eta_1 = \nabla k$, $\mu_1 = -m_2$ and $\gamma_1 = f_2^2[\rho\tau + \lambda + \Delta l - g(\nabla l, \nabla \varphi)]$.
- (2) $(M_2, g_2, \varphi_2, \eta_2, \mu_2, \gamma_2)$ is a gradient almost η -Ricci soliton, where $\varphi_2 = \varphi - m_1 l$, $\eta_2 = \nabla l$, $\mu_2 = -m_1$ and $\gamma_2 = f_1^2[\rho\tau + \lambda + \Delta k - g(\nabla k, \nabla \varphi)]$.
- (3) The potential function φ satisfies

$$(m_1 + m_2 - 2)X(k)U(l) - X(k)U(\varphi) - X(\varphi)U(l) = 0,$$

for all $X \in \mathfrak{L}(M_1)$ and $U \in \mathfrak{L}(M_2)$, where $\mathcal{F}(M_i)$ denote the set of all smooth functions on M_i ($i = 1, 2$).

Proof. Assume that (M, g) is a gradient ρ -Einstein soliton doubly warped product manifold with potential function φ . Then by using the fundamental formulas stated in previous lemmas, equation (1.1) takes the following form:

$$(2.5) \quad \begin{aligned} \text{Ric}^1(X, Y) &= \frac{m_2}{f_1} h_1^f(X, Y) - f_2^2 \Delta(l) g_1(X, Y) + h_1^\varphi(X, Y) \\ &+ f_2^2 g(\nabla l, \nabla \varphi) g_1(X, Y) = f_2^2 [\rho\tau + \lambda] g_1(X, Y). \end{aligned}$$

Note that for any smooth positive function f , we have the following well-known identity:

$$h^{\ln f} = \frac{1}{f} h^f - \frac{1}{f^2} df \otimes df.$$

Hence,

$$\text{Ric}^1 - m_2(h^k + dk \otimes dk) + h_1^\varphi = \Lambda_1 g_1,$$

where $\Lambda_1 = f_2^2[(\rho\tau + \lambda) + \Delta l - g(\nabla l, \nabla\varphi)]$. Then

$$\text{Ric}^1 + h_1^{\varphi - m_2 k} - m_2 dk \otimes dk = \Lambda_1 g_1,$$

i.e, (M_1, g_1) is a gradient η_1 -Ricci soliton where $\psi_1 = \varphi - m_2 k$ is a potential function, $\mu_1 = -m_2$ and $\eta_1 = \nabla k$. Similarly, the fundamental formulas stated in previous lemmas, equation (1.1) takes the following form:

$$\begin{aligned} (2.6) \quad \text{Ric}^2(V, U) &= \frac{m_1}{f_2} h_2^f(V, U) - f_1^2 \Delta(k) g_2(V, U) \\ &+ h_2^\varphi(V, U) + f_1^2 g(\nabla k, \nabla\varphi) g_2(V, U) = f_1^2 [\rho\tau + \lambda] g_2(V, U). \end{aligned}$$

Thus,

$$\text{Ric}^2 - m_2(h^l + dl \otimes dl) + h_2^\varphi = \Lambda_2 g_2,$$

where $\Lambda_2 = f_1^2[(\rho\tau + \lambda) + \Delta k - g(\nabla k, \nabla\varphi)]$. Then

$$\text{Ric}^2 + h_2^{\varphi - m_1 l} - m_1 dl \otimes dl = \Lambda_2 g_2,$$

i.e, (M_2, g_2) is also a gradient η_2 -Ricci soliton where $\psi_2 = \varphi - m_1 l$ is a potential function, $\mu_2 = -m_1$ and $\eta_2 = \nabla l$. $\square$

If one of the functions f_1 and f_2 is constant, then $(f_2 M_1 \times_{f_1} M_2, g)$ reduces to a warped product. Thus, we may consider the warped product $M = B \times_b F$ endowed with the metric

$$(2.7) \quad g = \pi^*(g_B) \oplus (b \circ \pi)^2 \sigma^*(g_F),$$

where $*$ denotes the pull-back operator on tensors [5, 36, 40]. The function b is called the warping function of the warped product manifold $B \times_b F$, and the manifolds B and F are called base and fiber, respectively. In particular, if $b = 1$, then $B \times_1 F = B \times F$ is the usual Cartesian product manifold. For the sake of simplicity, throughout this paper, all relations will be written, without involving the projection maps from $B \times F$ to each component B and F as in $g = g_B \oplus b^2 g_F$.

Proposition 2.1. *Let $M = B \times_b F$ be a singly warped product manifold with the metric tensor $g = g_B \oplus b^2 g_F$.*

$$\tau = \tau_B + \frac{\tau_F}{b^2} - 2s \frac{\Delta_B(b)}{b} - s(s-1) \frac{g_B(\nabla^B(b), \nabla^B(b))}{b^2},$$

where $r = \dim(B)$ and $s = \dim(F)$.

Notation 1. Suppose that $M = B \times_b F$ is a warped product manifold with the metric tensor $g = g_B \oplus b^2 g_F$. Then

$$b^\sharp = b\Delta^B(b) + (s-1)g_B(\nabla^B(b), \nabla^B(b)).$$

Theorem 2.3. Let $M = B \times_b F$ be a warped product manifold equipped with the metric $g = g_B \oplus b^2 g_F$. Then $(M, g, \varphi, \rho, \lambda)$ is a gradient ρ -Einstein soliton if and only if the following hold:

1. the potential function φ depends only on the base manifold B ,
2. the scalar curvature τ_F of the fiber manifold (F, g_F) is constant,
3. $\text{Ric}_B + \text{Hess}_B^\varphi = (\rho\tau + \lambda)g_B + \frac{s}{b}\text{Hess}_B^b$
4. $\text{Ric}_F = [b^\sharp - bg_B(\nabla^B(b), \nabla^B(b)) + (\rho\tau + \lambda)b^2]g_F$.

Proof. Assume that $(M, g, \varphi, \rho, \lambda)$ is a gradient ρ -Einstein soliton which is also a warped product. If X, Y are vector fields on B and V, W are vector fields on F , then:

- By evaluating the equation (1.1) at (X, Y) , we obtain:

$$\text{Ric}_B(X, Y) - \frac{s}{b}\text{H}_B^b(X, Y) + \text{H}_B^\varphi(X, Y) = (\rho\tau + \lambda)g_B(X, Y).$$

The scalar curvature of warped product manifolds implies that the scalar curvature of (F, g_F) , denoted by τ_F is constant.

- By evaluating the equation (1.1) at (X, V) , we also obtain:
 $\text{H}^\varphi(X, V) = 0$. So, $g_F(d\sigma(\nabla\varphi), V) = 0$ for any vector field V . Thus φ depends on B , i.e., $\varphi \in \mathcal{C}^\infty(B)$.
- By evaluating the equation (1.1) at (V, W) , we finally obtain (4). $\square$

Remark 2.2. If $s = \dim(F) \geq 3$, then (4) of Theorem 2.3 can be stated as (F, g_F) is Einstein with λ_F , i.e., $\text{Ric}_F = \lambda_F g_F$ where

$$\lambda_F = b^\sharp - bg_B(\nabla^B(b), \nabla^B(b)) + (\rho\tau + \lambda)b^2.$$

3. Applications

3.1. Gradient ρ -Einstein soliton on generalized Robertson-Walker spacetimes

We first define generalized Robertson-Walker spacetimes. Assume that (F, g_F) is an s -dimensional Riemannian manifold and $b : I \rightarrow (0, \infty)$ is a smooth function. Then the $(s+1)$ -dimensional product manifold $I \times_b F$ equipped with the metric tensor

$$(3.1) \quad g = -dt^2 \oplus b^2 g_F$$

is called a generalized Robertson-Walker spacetime and is denoted by $M = I \times_b F$ where I is an open and connected interval in $\mathbb{R}$ and dt^2 is the usual Euclidean metric tensor on I . This structure was introduced to extend Robertson-Walker spacetimes [38, 39] and have been studied by many authors, such as [33, 34, 16]. From now on, we will denote $\frac{\partial}{\partial t} \in \mathfrak{X}(I)$ by ∂_t to state our results in compact forms.

We will apply our main result Theorem 2.3. Assume that $\varphi \in \mathcal{C}^\infty(I)$ is a potential function for a generalized Robertson-Walker spacetime of the form $M = I \times_b F$.

Hence, we can state that:

Theorem 3.1. *Suppose that $M = I \times_b F$ is a generalized Robertson-Walker spacetime with the metric tensor $g = -dt^2 \oplus b^2 g_F$. Then $(M, g, \varphi, \rho, \lambda)$ is a gradient ρ -Einstein soliton if and only if the following is satisfied:*

1. *the potential function φ depends only on the base manifold I ,*
2. *the scalar curvature τ_F of the fiber manifold (F, g_F) is constant,*
3. $\varphi'' = -(\rho\tau + \lambda) + s \frac{b''}{b^2},$
4. $\text{Ric}_F = [-bb'' - (s-1)(b')^2 + b(b')^2 + (\rho\tau + \lambda)b^2]g_F,$

where $\tau = \frac{\tau_F}{b^2} + 2s \frac{b''}{b} + s(s-1) \frac{(b')^2}{b^2}.$

3.2. Gradient ρ -Einstein soliton on standard static spacetimes

We begin by defining standard static spacetimes. Let (F, g_F) be an s -dimensional Riemannian manifold and $f : F \rightarrow (0, \infty)$ be a smooth function. Then the $(s+1)$ -dimensional product manifold ${}_f I \times F$ furnished with the metric tensor

$$(3.2) \quad g = -f^2 dt^2 \oplus g_F$$

is called a standard static spacetime and is denoted by $M = {}_f I \times F$ where I is an open and connected subinterval in $\mathbb{R}$ and dt^2 is the usual Euclidean metric tensor on I .

Note that standard static spacetimes can be considered as a generalization of the Einstein static universe [1, 2, 3, 4] and many spacetime models that characterize the universe and the solutions of Einstein's field equations are known to have this structure.

Again we apply Theorem 2.3. Suppose that $\varphi \in \mathcal{C}^\infty(F)$ is a potential function for a standard static spacetime of the form $M = {}_f I \times F$.

Theorem 3.2. *Let $M = {}_f I \times F$ be standard static spacetime with the metric tensor $g = -f^2 dt^2 \oplus g_F$. Then $(M, g, \varphi, \rho, \lambda)$ is a gradient ρ -Einstein soliton if and only if the following is satisfied:*

1. the potential function φ depends only on the fiber manifold F ,
2. $\text{Ric}_F + \text{Hess}_F^\varphi = [(\rho\tau_F + \lambda) - 2\rho\frac{\Delta_F(f)}{f}]g_F + \frac{1}{f}\text{Hess}_F^f$,
3. $-\nabla^F(f) + \varphi(f) + 2\rho\Delta_F(f) = [\rho\tau_F + \lambda]f$.

Remark 3.1. $f\text{Ric}_F - \text{Hess}_F^f + f\text{Hess}_F^\varphi = [-\nabla^F(f) + \varphi(f)]g_F$.

3.3. Gradient ρ -Einstein soliton on 3-dimensional Walker manifolds

In general, a 3-dimensional manifold admitting a parallel degenerate line field is said to be a Walker manifold, [42, 15]. Suppose that (M, g) is a 3-dimensional Walker Manifold then there exist local coordinates (t, x, y) such that the Lorentzian metric tensor with respect to the local frame fields $\{\partial_t, \partial_x, \partial_y\}$ takes the form given as:

$$(3.3) \quad g = 2dt dy + dx^2 + \phi(t, x, y)dy^2,$$

for some function $\phi(t, x, y)$. The restricted case of Walker manifolds where ϕ described as a function of only x and y is called a strictly Walker manifold and in particular strictly Walker manifolds are geodesically complete. Also, it is known that a Walker manifold is Einstein if and only if it is flat, [42].

Now, we will investigate conditions on this particular class of manifolds to have gradient Yamabe solitons, that is,

$$(3.4) \quad \text{Ric} + \text{Hess}(\varphi)_{ij} = (\rho\tau + \lambda)g_{ij},$$

where φ is a potential function.

By using the metric (3.3) and straightforward computations, we have:

$$(3.5) \quad \begin{cases} \text{Hess}(\varphi)_{tt} = \varphi_{tt}, \\ \text{Hess}(\varphi)_{tx} = \varphi_{tx}, \\ \text{Hess}(\varphi)_{ty} = \varphi_{ty} - \frac{1}{2}\phi_t\varphi_t, \\ \text{Hess}(\varphi)_{xx} = \varphi_{xx}, \\ \text{Hess}(\varphi)_{xy} = \varphi_{xy} - \frac{1}{2}\phi_x\varphi_t, \\ \text{Hess}(\varphi)_{yy} = \varphi_{yy} - \frac{1}{2}(\phi\phi_t + \phi_y)\varphi_t + \frac{1}{2}\phi_x\varphi_x + \frac{1}{2}\phi_t\varphi_y. \end{cases}$$

Moreover,

$$(3.6) \quad \text{Ric} = \begin{bmatrix} 0 & 0 & \frac{1}{2}\phi_{tt} \\ 0 & 0 & \frac{1}{2}\phi_{tx} \\ \frac{1}{2}\phi_{tt} & \frac{1}{2}\phi_{tx} & \frac{1}{2}(\phi\phi_{tt} - \phi_{xx}) \end{bmatrix}$$

By applying equations (3.4), (3.5) and (3.6) we obtain the following system of PDEs:

$$(3.7) \quad \begin{cases} \varphi_{tt} = 0, \\ \varphi_{tx} = 0, \\ \frac{1}{2}\phi_{tt} + \varphi_{ty} - \frac{1}{2}\phi_t\varphi_t = (\rho\tau + \lambda), \\ \varphi_{xx} = (\rho\tau + \lambda), \\ \frac{1}{2}\phi_{tx} + \varphi_{xy} - \frac{1}{2}\phi_x\varphi_t = 0, \\ \frac{1}{2}(\phi\phi_{tt} - \phi_{xx}) + \varphi_{yy} - \frac{1}{2}(\phi\phi_t + \phi_y)\varphi_t + \frac{1}{2}\phi_x\varphi_x + \frac{1}{2}\phi_t\varphi_y = (\rho\tau + \lambda)\phi. \end{cases}$$

The first two equations of (3.7) imply that

$$(3.8) \quad \varphi(t, x, y) = tB(y) + C(x, y) \quad \text{for some functions } B, C.$$

Combining (3.7) and (3.8), we obtain:

$$(3.9) \quad \begin{cases} \frac{1}{2}\phi_{tt} + B'(y) - \frac{1}{2}\phi_t B(y) = (\rho\tau + \lambda), \\ C_{xx} = (\rho\tau + \lambda), \\ \frac{1}{2}\phi_{tx} + C_{xy}(x, y) - \frac{1}{2}\phi_x B(y) = 0, \\ \frac{1}{2}(\phi\phi_{tt} - \phi_{xx}) + tB'' + C_{yy}(x, y) - \frac{1}{2}(\phi\phi_t + \phi_y)B(y) \\ + \frac{1}{2}(tB'(y) + C_y(x, y))\phi_t + \frac{1}{2}C_x(x, y)\phi_x = (\rho\tau + \lambda)\phi. \end{cases}$$

Differentiating the first and third equations of (3.9), with respect to x and t , respectively, we have:

$$C_x(x, y) = xD(t, y) + H(y).$$

$$(3.10) \quad C(x, y) = \frac{x^2}{2}D(y) + xE(y) + F(y) \quad \text{for some functions } D, E, F.$$

Thus,

$$(3.11) \quad f(t, x, y) = tB(y) + \frac{x^2}{2}D(y) + xE(y) + F(y).$$

Now, putting (3.10) into the system (3.9), to get a concrete solution we may impose the condition: " ϕ depends only on x ". Then by the last equation of the system (3.9), we get $\phi_{xx}B(y) = 0$. Thus, the following two cases are obtained:

Case I: $\phi_{xx} = 0$, then from (3.9), the potential function and the metric function are found by

$$(3.12) \quad f(t, x, y) = \gamma t + (\alpha x + \beta)x + F(y), \quad \phi = ax + b,$$

for some scalars $a, b, \alpha, \beta, \gamma \in \mathbb{R}$.

Case II: On the other hand, if $B(y) = 0$, then the potential function and the metric function is found by

$$(3.13) \quad f(x, y) = mx + n\frac{y^2}{2} + py + r, \quad \phi = \frac{k}{m^2}e^{mx} + lx + s,$$

for some scalars $k, l, m, n, p, r, s \in \mathbb{R}$.

Theorem 3.3. *Let (M, g) be a 3-dimensional Lorentzian Walker manifold equipped with metric:*

$$g = 2dt dy + dx^2 + \phi(x)dy^2.$$

Then (M, g) is a gradient ρ -Einstein soliton if and only if the potential function of the soliton and the metric function are given by one of the following 2-cases:

1.

$$f(t, x, y) = \gamma t + (\alpha x + \beta)x + F(y), \quad \phi = ax + b,$$

for some scalars $a, b, \alpha, \beta, \gamma \in \mathbb{R}$.

2.

$$f(x, y) = mx + n\frac{y^2}{2} + py + r, \quad \phi = \frac{k}{m^2}e^{mx} + lx + s,$$

for some scalars $k, l, m, n, p, r, s \in \mathbb{R}$.

Moreover, a pseudo-Riemannian manifold is said to be conformally symmetric if its Weyl tensor is parallel, i.e., $\nabla W = 0$, [20, 21]. It is known that any conformally symmetric Riemannian manifold is either locally symmetric (i.e., $\nabla R = 0$) or locally conformally flat (i.e., $W = 0$). In the nontrivial case ($\nabla W = 0$ and $\nabla R \neq 0$, $W \neq 0$), the manifold (M, g) is said to be essentially conformally symmetric.

Theorem 3.4. [11] *A three-dimensional pseudo-Riemannian manifold is essentially conformally symmetric if and only if it is a strict Lorentzian Walker manifold, locally isometric to $(\mathbb{R}^3, (t, x, y), g_a)$, where*

$$g_a = 2dt dy + dx^2 + (x^3 + a(y)x)dy^2,$$

for an arbitrary smooth function $a(y)$.

Therefore, by applying the above theorem for gradient ρ -Einstein solitons, we terminate our study with the following result:

Theorem 3.5. *There exists no 3-dimensional essentially conformally symmetric gradient ρ -Einstein soliton.*

Proof. By using the metric g_a given in Theorem [11], the system (3.7) yields:

$$(3.14) \quad \begin{cases} \phi_{tt} = 0, \\ \phi_{tx} = 0, \\ \phi_{ty} = \lambda = \phi_{xx}, \\ \phi_{xy} - \frac{1}{2}[3x^2 + a(y)]\phi_t = 0, \\ \phi_{yy} - \frac{1}{2}a'(y)x\phi_t + \frac{1}{2}[3x^2 + a(y)]\phi_x - \frac{x}{3} = \lambda[x^3 + a(y)x] \end{cases}$$

First, $\phi_{tx} = 0$ implies that $\phi_t = A(t, y)$. Also, $\phi_{tt} = A_t(t, y) = 0$. So, $A(t, y) = B(y)$. Then $\phi_t = B(y)$. Thus, $\phi = tB(y) + C(x, y)$ and $\lambda = B'(y)$.

Also, $\phi_{xx} = \lambda = B'(y)$. So, $C_{xx}(x, y) = B'(y)$. Then $C_x(x, y) = xB'(y) + D(y)$. Hence, $C(x, y) = \frac{x^2}{2}B'(y) + xD(y) + E(y)$. Thus, we get $\phi = tB(y) + \frac{x^2}{2}B'(y) + xD(y) + E(y)$.

- $\phi_{xy} = \frac{1}{2}[3x^2 + a(y)]\phi_t$ implies that

$$(3.15) \quad xB''(y) + D'(y) = \frac{1}{2}[3x^2 + a(y)]B(y).$$

- $\phi_{yy} - \frac{1}{2}a'(y)x\phi_t + \frac{1}{2}[3x^2 + a(y)]\phi_x - \frac{x}{3} = \lambda[x^3 + a(y)x]$ implies that

$$(3.16) \quad \begin{aligned} tB''(y) + \frac{x^2}{2}B'''(y) + xD''(y) + E''(y) &= \frac{x}{2}a'(y)B(y) \\ - \frac{1}{2}[3x^2 + a(y)][xB'(y) + D(y)] + \frac{x}{3} &+ [x^3 + xa(y)]B'(y). \end{aligned}$$

Differentiating the last two equations of the system (3.14) with respect to x and then combining the resulting equations, we obtain:

$$(3.17) \quad \frac{3x^2}{2}B'(y) + 3xD(y) - \frac{1}{3} = \frac{a(y)}{2}B'(y).$$

By differentiating equation (3.17) with respect to x , we have $D'(y) = -xB''(y)$. The last equation and the equation (3.15) imply that

$$(3.18) \quad [3x^2 + a(y)]B(y) = 0.$$

Thus we have two cases:

- $3x^2 + a(y) = 0$ cannot happen since a is a function of y .
- If $B(y) = 0$, then in this case, we get $D(y) = 0$ and thus λ vanishes, which is a contradiction.

Therefore, the proof is completed. $\square$

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