FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 40, No 3 (2025), 697–715
https://doi.org/10.22190/FUMI240818048S
Original Scientific Paper

FIXED POINT RESULTS IN PARTIAL METRIC SPACES VIA INTEGRAL TYPE CONTRACTION WITH APPLICATION

Gurucharan Singh Saluja

H.N. 3/1005, Geeta Nagar, Raipur
Raipur-492001 (Chhattisgarh), India

ORCID ID: Gurucharan Singh Saluja https://orcid.org/0009-0006-1595-8133

Abstract. In this paper, we derive some fixed point results via integral-type contractive conditions having rational terms in the setting of complete partial metric spaces and provide some consequences of the established results. Also, we give some examples in support of the established results. An application to the Fredholm integral equation is also given. Our results generalize, extend and enrich several previously published well-known fixed point results from the existing literature (see, e.g. [10], [11], [26], and many others).
Keywords: fixed point, partial metric space, Fredholm integral equation.

1. Introduction and Preliminaries

The notion of partial metric space was introduced by *Matthews* [26, 27] in 1992. In fact, a partial metric space is a generalization of metric space in which each object does not necessarily have to have a zero distance from itself. Nonzero self-distance makes perfect sense in the setting of Computer Science, in particular, in the Domain Theory and Semantics (see, e.g., [15], [24], [32]-[34]). In the paper [27], *Matthews* proved an analog of the well-known Banach contraction mapping principle [10] in the context of complete partial metric space. After this result, many authors focused on partial metric spaces and its topological properties (see, e.g. [1]-[3], [5]-[8], [13], [14], [16]-[22], [28], [35]).

Received: August 18, 2024, revised: March 20, 2025, accepted: March 23, 2025
Communicated by Dijana Mosić
Corresponding Author: G. S. Saluja
E-mail addresses: saluja1963@gmail.com (G. S. Saluja)
2020 *Mathematics Subject Classification.* Primary 47H10; Secondary 54H25.

The purpose of the present paper is to prove some fixed point results via integral-type contractive conditions having rational terms in the framework of partial metric spaces. Moreover, we give some consequences of the established results. Also, we provide some illustrative examples to validate the results. An application to the Fredholm integral equation is also given. Our results extend, generalize and enrich several previously published results from the existing literature.

The definition of partial metric space is given by *Matthews* (see, e.g. [26, 27]) as follows.

Definition 1.1. Let $Y \neq \emptyset$ be a set and $p\colon Y \times Y \to [0, +\infty)$ be a self mapping satisfies the following conditions:

$(P1)$ $j = k \Leftrightarrow p(j, j) = p(k, k) = p(j, k)$,

$(P2)$ $p(j, j) \leq p(j, k)$,

$(P3)$ $p(j, k) = p(k, j)$,

$(P4)$ $p(j, k) \leq p(j, l) + p(l, k) - p(l, l)$,

for all $j, k, l \in Y$. Then p is called a partial metric on Y and the pair (Y, p) is a called partial metric space (in short PMS).

Remark 1.1. It is clear that if $p(j, k) = 0$, then from $(P1)$, $(P2)$, and $(P3)$, $j = k$. But if $j = k$, $p(j, k)$ may not be 0.

Example 1.1. ([8])

(1) Let $Y = [0, +\infty)$ and $p\colon Y \times Y \to [0, +\infty)$ be given by $p(j, k) = \max\{j, k\}$ for all $j, k \in Y$. Then (Y, p) is a partial metric space.

(2) Let I denote the set of all intervals $[u, v]$ for any real numbers $u \leq v$. Let $p\colon I \times I \to [0, +\infty)$ be a function such that

$$p\big([u, v], [p, q]\big) = \max\{v, q\} - \min\{u, p\}.$$

Then (I, p) is a partial metric space.

Example 1.2. ([12]) Let $Y = \mathbb{R}$ and $p\colon Y \times Y \to [0, +\infty)$ be given by $p(j, k) = e^{\max\{j,k\}}$ for all $j, k \in Y$. Then (Y, p) is a partial metric space.

Each partial metric p on Y generates a T_0 topology τ_p on Y with the family of open p-balls $\{B_p(x, \mu) : x \in Y, \mu > 0\}$ where $B_p(x, \mu) = \{z \in Y : p(x, z) < p(x, x) + \mu\}$ for all $x \in Y$ and $\mu > 0$. Similarly, closed p-ball is defined as $B_p[x, \mu] = \{z \in Y : p(x, z) \leq p(x, x) + \mu\}$ for all $x \in Y$ and $\mu > 0$.

Definition 1.2. (see, e.g. [26, 27]) Let (Y, p) be a partial metric space.

(A) A sequence $\{u_n\}$ converges to a point $u \in Y$ whenever $\lim_{n\to\infty} p(u, u_n) = p(u, u)$.

(B) A sequence $\{u_n\}$ in Y is called Cauchy whenever

$$\lim_{m,n\to\infty} p(u_m, u_n) \text{ exists (and is finite)}.$$

(C) A partial metric space (Y,p) is said to be complete if every Cauchy sequence $\{u_n\}$ in Y converges, with respect to τ_p, to a point $u \in Y$, such that

$$\lim_{m,n\to\infty} p(u_m,u_n) = p(u,u).$$

(D) A mapping $g\colon Y \to Y$ is said to be continuous at $u_0 \in Y$ if for every $\varepsilon > 0$, there exists $\alpha > 0$ such that $g\big(B_p(u_0,\alpha)\big) \subset B_p\big(g(u_0),\varepsilon\big)$.

There is a close relationship between metrics and partial metrics. Indeed, if p is a partial metric on Y, then the function $d_p\colon Y\times Y \to \mathbb{R}^+$ given by

$$d_p(j,k) = 2p(j,k) - p(j,j) - p(k,k), \tag{1.1}$$

is a (usual) metric on Y. Moreover,

$$\lim_{n\to\infty} d_p(u,u_n) = 0 \;\Leftrightarrow\; \lim_{n\to\infty} p(u,u_n) = \lim_{n,m\to\infty} p(u_n,u_m) = p(u,u). \tag{1.2}$$

The following lemmas play an important role in the proof of our main result.

Lemma 1.1. *(see, e.g. [26, 27]) Let (Y,p) be a partial metric space.*

(E) A sequence $\{u_n\}$ in (Y,p) is a Cauchy sequence $\Leftrightarrow$ $\{u_n\}$ is a Cauchy sequence in the metric space (Y,d_p),

(F) (Y,p) is complete $\Leftrightarrow$ the metric space (Y,d_p) is complete. Moreover,

$$\lim_{n\to\infty} d_p(u_n,u) = 0 \;\Leftrightarrow\; p(u,u) = \lim_{n\to\infty} p(u_n,u) = \lim_{n,m\to\infty} p(u_n,u_m).$$

Lemma 1.2. *([19]) Let (Y,p) be a partial metric space.*

(G) If $j,k \in Y$, $p(j,k)=0$, then $j=k$,

(H) If $j \neq k$, then $p(j,k) > 0$.

(I) If $u_n \to z$ as $n\to\infty$ in a partial metric space (Y,p) with $p(z,z)=0$, then $\lim_{n\to\infty} p(u_n,y) = p(z,y)$ for all $y\in Y$ (see [14]).

Definition 1.3. ([25]) Let $\{u_n\}_{n\in\mathbb{N}}$ be a non-negative sequence such that $\lim_{n\to\infty} u_n = a$. Then

$$\lim_{n\to\infty} \int_0^{u_n} \phi(t)\,dt = \int_0^a \phi(t)\,dt,$$

where $\phi\colon [0,+\infty) \to [0,+\infty)$ is a Lebesgue-integrable mapping which is summable on each compact subset of $[0,+\infty)$, and such that for each $\varepsilon > 0$, $\int_0^\varepsilon \phi(t)dt > 0$.

Definition 1.4. ([25]) Let $\{u_n\}_{n\in\mathbb{N}}$ be a non-negative sequence. Then

$$\lim_{n\to\infty} \int_0^{u_n} \phi(t)\,dt = 0,$$

if and only if $\lim_{n\to\infty} u_n = 0$, where $\phi\colon [0,+\infty) \to [0,+\infty)$ is a Lebesgue-integrable mapping which is summable on each compact subset of $[0,+\infty)$, and such that for each $\varepsilon > 0$, $\int_0^\varepsilon \phi(t)dt > 0$.

Definition 1.5. (altering distance function, [23]) The function $\psi\colon [0,+\infty) \to [0,+\infty)$ is called an altering distance function if the following properties are satisfied:

(1) ψ is continuous and monotone non-decreasing,

(2) $\psi(t) = 0$ if and only if $t = 0$.

In 2002, Branciari [11] obtained a fixed point result for a single mapping satisfying an integral type inequality. This celebrated result can be stated as follows.

Theorem 1.1. *([11]) Let (Y, d) be a complete metric space, $h \in [0, 1)$, and let $g\colon Y \to Y$ be a mapping such that for each $j, k \in Y$,*

$$\int_0^{d(gj,gk)} \phi(t)dt \leq h \int_0^{d(j,k)} \phi(t)dt,$$

where $\phi\colon [0,+\infty) \to [0,+\infty)$ is a Lebesgue-integrable mapping which is summable on each compact subset of $[0,+\infty)$, and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \phi(t)dt > 0$, then g has a unique fixed point $s \in Y$, such that for each $r \in Y$, $\lim_{n\to\infty} g^n(r) = s$.

After this remarkable result of Branciari [11], a lot of interesting research work on fixed point theorems involving more general contractive conditions of integral type was obtained in [4, 9, 29, 30, 31].

2. Main Results

In this section, we shall prove fixed point theorems for integral type contraction having rational terms and altering distance function in the setting of partial metric spaces.

Theorem 2.1. *Let (Y, p) be a complete partial metric space and $F\colon Y \to Y$ be a mapping satisfying the following contractive condition:*

$$\int_0^{\psi(p(Fu,Fv))} \phi(t)dt \leq \mu \int_0^{\Theta_p(u,v)} \phi(t)dt, \tag{2.1}$$

for all $u, v \in Y$, where $\Theta_p(u, v)$ is given by

$$\begin{aligned} \Theta_p(u,v) \quad = \quad & \max\Big\{\psi(p(u,v)), \psi(p(u,Fu)), \psi(p(v,Fv)), \\ & \psi\Big(\frac{p(v,Fv)(1+p(u,Fu))}{1+p(u,v)}\Big), \psi\Big(\frac{p(u,Fu)p(v,Fu)}{1+p(u,v)}\Big)\Big\}, \end{aligned}$$

$\mu \in [0, 1)$, ψ is an altering distance function and $\phi\colon [0,+\infty) \to [0,+\infty)$ is a Lebesgue-integrable mapping which is summable on each compact subset of $[0,+\infty)$, and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \phi(t)dt > 0$. Then F has a unique fixed point in Y. Moreover, $p(z, z) = 0$.

Proof. Let u_0 be an arbitrary point in Y. We construct the sequence $\{u_n\}$ in Y as follows

$$u_{n+1} = Fu_n, \quad n = 0, 1, 2, 3, \ldots.$$

If there exists n such that $u_n = u_{n+1} = Fu_n$, then u_n is a fixed point of F and the proof is finished. So, we assume that $u_n \neq u_{n+1}$ for all $n \geq 0$. Putting $u = u_{n-1}$ and $v = u_n$ in (2.1) and using condition $p(z, z) = 0$ and the property of ψ, we have

$$\begin{aligned} \int_0^{\psi(p(u_n,u_{n+1}))} \phi(t)dt &= \int_0^{\psi(p(Fu_{n-1},Fu_n))} \phi(t)dt \\ &\leq \mu \int_0^{\Theta_p(u_{n-1},u_n)} \phi(t)dt, \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} \Theta_p(u_{n-1}, u_n) &= \max\Big\{\psi(p(u_{n-1}, u_n)), \psi(p(u_{n-1}, Fu_{n-1})), \psi(p(u_n, Fu_n)), \\ &\qquad \psi\Big(\frac{p(u_n, Fu_n)(1 + p(u_{n-1}, Fu_{n-1}))}{1 + p(u_{n-1}, u_n)}\Big), \\ &\qquad \psi\Big(\frac{p(u_{n-1}, Fu_{n-1})p(u_n, Fu_{n-1})}{1 + p(u_{n-1}, u_n)}\Big)\Big\} \\ &= \max\Big\{\psi(p(u_{n-1}, u_n)), \psi(p(u_{n-1}, u_n)), \psi(p(u_n, u_{n+1})), \\ &\qquad \psi\Big(\frac{p(u_n, u_{n+1})(1 + p(u_{n-1}, u_n))}{1 + p(u_{n-1}, u_n)}\Big), \\ &\qquad \psi\Big(\frac{p(u_{n-1}, u_n)p(u_n, u_n)}{1 + p(u_{n-1}, u_n)}\Big)\Big\} \\ &= \max\Big\{\psi(p(u_{n-1}, u_n)), \psi(p(u_{n-1}, u_n)), \psi(p(u_n, u_{n+1})), \\ &\qquad \psi(p(u_n, u_{n+1})), 0\Big\} \\ &= \max\Big\{\psi(p(u_{n-1}, u_n)), \psi(p(u_n, u_{n+1}))\Big\}. \end{aligned} \tag{2.3}$$

The following cases arise.

If $\max\Big\{\psi(p(u_{n-1}, u_n)), \psi(p(u_n, u_{n+1}))\Big\} = \psi(p(u_n, u_{n+1}))$, then from equation (2.3), we obtain

$$\int_0^{\psi(p(u_n,u_{n+1}))} \phi(t)dt \leq \mu \int_0^{\psi(p(u_n,u_{n+1}))} \phi(t)dt,$$

which is a contradiction, since $0 < \mu < 1$. Thus, we conclude that

$$\int_0^{\psi(p(u_n,u_{n+1}))} \phi(t)dt \leq \mu \int_0^{\psi(p(u_{n-1},u_n))} \phi(t)dt. \tag{2.4}$$

Continuing the same process as above, we get

$$\begin{aligned}\int_0^{\psi(p(u_n,u_{n+1}))}\phi(t)dt &\le \mu\int_0^{\psi(p(u_{n-1},u_n))}\phi(t)dt \le \mu^2\int_0^{\psi(p(u_{n-2},u_{n-1}))}\phi(t)dt\\ &\le \cdots \le \mu^n\int_0^{\psi(p(u_0,u_1))}\phi(t)dt. \qquad (2.5)\end{aligned}$$

Passing to the limit as $n \to \infty$ in equation (2.5), we obtain

$$\lim_{n\to\infty}\int_0^{\psi(p(u_n,u_{n+1}))}\phi(t)dt = 0, \text{ since } 0<\mu<1. \qquad (2.6)$$

Hence, by the property of integral ϕ (by Definition 1.4), we obtain

$$\lim_{n\to\infty}\psi(p(u_n,u_{n+1})) = 0. \qquad (2.7)$$

Again by the properties of altering distance function ψ, we obtain

$$\lim_{n\to\infty}p(u_n,u_{n+1}) = 0. \qquad (2.8)$$

Due to equation (1.1), we have $d_p(u_n,u_{n+1}) \le 2p(u_n,u_{n+1})$. Therefore

$$\lim_{n\to\infty}d_p(u_n,u_{n+1}) = 0. \qquad (2.9)$$

Now, we prove that

$$\lim_{n,m\to\infty}p(u_n,u_m) = 0.$$

Suppose the contrary, that is,

$$\lim_{n\to\infty}p(u_n,u_m) \ne 0.$$

Then there exists $\varepsilon > 0$ for which we can find two subsequences $\{u_{m(s)}\}$ and $\{u_{n(s)}\}$ of $\{u_n\}$ such that $n(s)$ is the smallest integer for which

$$n(s) > m(s) > s, \;\; p(u_{m(s)},u_{n(s)}) \ge \varepsilon, \qquad (2.10)$$

and

$$p(u_{n(s)-1},u_{m(s)}) < \varepsilon. \qquad (2.11)$$

From equations (2.10) and (2.11), we have

$$\begin{aligned}\varepsilon &\le p(u_{n(s)},u_{m(s)})\\ &\le p(u_{n(s)},u_{n(s)-1}) + p(u_{n(s)-1},u_{m(s)}) - p(u_{n(s)-1},u_{n(s)-1})\\ &\le p(u_{n(s)},u_{n(s)-1}) + p(u_{n(s)-1},u_{m(s)})\\ &< \varepsilon + p(u_{n(s)},u_{n(s)-1}). \qquad (2.12)\end{aligned}$$

Taking the limit as $s \to \infty$ and using equation (2.9), we obtain

$$\lim_{s\to\infty} p(u_{n(s)}, u_{m(s)}) = \varepsilon. \tag{2.13}$$

By the triangle inequality, we have

$$\begin{aligned} p(u_{n(s)}, u_{m(s)}) &\leq p(u_{n(s)}, u_{n(s)-1}) + p(u_{n(s)-1}, u_{m(s)}) - p(u_{n(s)-1}, u_{n(s)-1}) \\ &\leq p(u_{n(s)}, u_{n(s)-1}) + p(u_{n(s)-1}, u_{m(s)}) \\ &\leq p(u_{n(s)}, u_{n(s)-1}) + p(u_{n(s)-1}, u_{m(s)-1}) + p(u_{m(s)-1}, u_{m(s)}) \\ &\quad -p(u_{m(s)-1}, u_{m(s)-1}) \\ &\leq p(u_{n(s)}, u_{n(s)-1}) + p(u_{n(s)-1}, u_{m(s)-1}) + p(u_{m(s)-1}, u_{m(s)}), \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} p(u_{n(s)-1}, u_{m(s)-1}) &\leq p(u_{n(s)-1}, u_{n(s)}) + p(u_{n(s)}, u_{m(s)-1}) - p(u_{n(s)}, u_{n(s)}) \\ &\leq p(u_{n(s)-1}, u_{n(s)}) + p(u_{n(s)}, u_{m(s)-1}) \\ &\leq p(u_{n(s)-1}, u_{n(s)}) + p(u_{n(s)}, u_{m(s)}) + p(u_{m(s)}, u_{m(s)-1}) \\ &\quad -p(u_{m(s)}, u_{m(s)}) \\ &\leq p(u_{n(s)-1}, u_{n(s)}) + p(u_{n(s)}, u_{m(s)}) + p(u_{m(s)}, u_{m(s)-1}). \end{aligned} \tag{2.15}$$

Taking the limit as $s \to \infty$ in equations (2.14) and (2.15) and using equations (2.9) and (2.13), we obtain

$$\lim_{s\to\infty} p(u_{n(s)-1}, u_{m(s)-1}) = \varepsilon. \tag{2.16}$$

Again, we have

$$\begin{aligned} p(u_{n(s)-1}, u_{m(s)-1}) &\leq p(u_{n(s)-1}, u_{m(s)}) + p(u_{m(s)}, u_{m(s)-1}) \\ &\quad -p(u_{m(s)}, u_{m(s)}) \\ &\leq p(u_{n(s)-1}, u_{m(s)}) + p(u_{m(s)}, u_{m(s)-1}), \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} p(u_{n(s)-1}, u_{m(s)}) &\leq p(u_{n(s)-1}, u_{n(s)}) + p(u_{n(s)}, u_{m(s)}) \\ &\quad -p(u_{n(s)}, u_{n(s)}) \\ &\leq p(u_{n(s)-1}, u_{n(s)}) + p(u_{n(s)}, u_{m(s)}). \end{aligned} \tag{2.18}$$

Taking the limit as $s \to \infty$ in equations (2.17), (2.18) and using equations (2.9), (2.13) and (2.16), we obtain

$$\lim_{s\to\infty} p(u_{n(s)-1}, u_{m(s)}) = \varepsilon. \tag{2.19}$$

Now from equation (2.1), we have

$$\begin{aligned} \int_0^{\psi(p(u_{m(s)}, u_{n(s)}))} \phi(t)dt &= \int_0^{\psi(p(Fu_{m(s)-1}, Fu_{n(s)-1}))} \phi(t)dt \\ &\leq \mu \int_0^{\Theta_p(u_{m(s)-1}, u_{n(s)-1})} \phi(t)dt, \end{aligned} \tag{2.20}$$

where

$$\begin{aligned}\Theta_p(u_{m(s)-1}, u_{n(s)-1}) &= \max\Big\{\psi(p(u_{m(s)-1}, u_{n(s)-1})), \psi(p(u_{m(s)-1}, Fu_{m(s)-1})),\\ &\quad \psi(p(u_{n(s)-1}, Fu_{n(s)-1})),\\ &\quad \psi\Big(\frac{p(u_{n(s)-1}, Fu_{n(s)-1})(1+p(u_{m(s)-1}, Fu_{m(s)-1}))}{1+p(u_{m(s)-1}, u_{n(s)-1})}\Big),\\ &\quad \psi\Big(\frac{p(u_{m(s)-1}, Fu_{m(s)-1})p(u_{n(s)-1}, Fu_{m(s)-1})}{1+p(u_{m(s)-1}, u_{n(s)-1})}\Big)\Big\}\\ &= \max\Big\{\psi(p(u_{m(s)-1}, u_{n(s)-1})), \psi(p(u_{m(s)-1}, u_{m(s)})),\\ &\quad \psi(p(u_{n(s)-1}, u_{n(s)})),\\ &\quad \psi\Big(\frac{p(u_{n(s)-1}, u_{n(s)})(1+p(u_{m(s)-1}, u_{m(s)}))}{1+p(u_{m(s)-1}, u_{n(s)-1})}\Big),\\ &\quad \psi\Big(\frac{p(u_{m(s)-1}, u_{m(s)})p(u_{n(s)-1}, u_{m(s)})}{1+p(u_{m(s)-1}, u_{n(s)-1})}\Big)\Big\}.\end{aligned}$$

By equations (2.9), (2.13), (2.16), (2.19) and using the properties of ψ, we have

$$\lim_{s\to\infty} \Theta_p(u_{m(s)-1}, u_{n(s)-1}) = \max\big\{\psi(\varepsilon), 0, 0, 0, 0\big\} = \psi(\varepsilon). \tag{2.21}$$

Now, passing to the limit as $s \to \infty$ in equation (2.20) and using equation (2.21), property of ψ and Definition 1.3, we obtain

$$\int_0^{\psi(\varepsilon)} \phi(t)dt \leq \mu \int_0^{\psi(\varepsilon)} \phi(t)dt,$$

which is a contradiction, since $0 < \mu < 1$. So, we have

$$\int_0^{\psi(\varepsilon)} \phi(t)dt = 0.$$

Again by the property of integral ϕ, we obtain $\psi(\varepsilon) = 0$ and so by the property of ψ, we have $\varepsilon = 0$, which is a contradiction, since $\varepsilon > 0$. Hence, we have

$$\lim_{n,m\to\infty} p(u_n, u_m) = 0.$$

Since $\lim_{n,m\to\infty} p(u_n, u_m)$ exists and is finite, we conclude that $\{u_n\}$ is a Cauchy sequence in partial metric space (Y, p).

Due to equation (1.1), we have $d_p(u_n, u_m) \leq 2p(u_n, u_m)$. Therefore

$$\lim_{n,m\to\infty} d_p(u_n, u_m) = 0. \tag{2.22}$$

Thus, by Lemma 1.1, $\{u_n\}$ is a Cauchy sequence in both (Y, d_p) and (Y, p).

Since (Y, p) is complete partial metric space, then there exists $t \in Y$ such that $\lim_{n\to\infty} p(u_n, t) = p(t, t)$. Since $\lim_{n,m\to\infty} p(u_n, u_m) = 0$, then again by Lemma 1.1, we have $p(t, t) = 0$. Now, we shall prove that t is a fixed point of F. Suppose that $Ft \neq t$. From equation (2.1) and Lemma 1.2 (I), we have

$$\begin{aligned} \int_0^{\psi(p(u_n,Ft))} \phi(t)dt &= \int_0^{\psi(p(Fu_{n-1},Ft))} \phi(t)dt \\ &\leq \mu \int_0^{\Theta_p(u_{n-1},t)} \phi(t)dt, \end{aligned} \tag{2.23}$$

where

$$\begin{aligned} \Theta_p(u_{n-1}, t) &= \max\Big\{\psi(p(u_{n-1}, t)), \psi(p(u_{n-1}, Fu_{n-1})), \psi(p(t, Ft)), \\ &\qquad \psi\Big(\frac{p(t, Ft)(1 + p(u_{n-1}, Fu_{n-1}))}{1 + p(u_{n-1}, t)}\Big), \\ &\qquad \psi\Big(\frac{p(u_{n-1}, Fu_{n-1})p(t, Fu_{n-1})}{1 + p(u_{n-1}, t)}\Big)\Big\} \\ &= \max\Big\{\psi(p(u_{n-1}, t)), \psi(p(u_{n-1}, u_n)), \psi(p(t, Ft)), \\ &\qquad \psi\Big(\frac{p(t, Ft)(1 + p(u_{n-1}, u_n))}{1 + p(u_{n-1}, t)}\Big), \\ &\qquad \psi\Big(\frac{p(u_{n-1}, u_n)p(t, u_n)}{1 + p(u_{n-1}, t)}\Big)\Big\}. \end{aligned}$$

Taking the limit as $n \to \infty$ in the above and using $p(t, t) = 0$ and the properties of ψ, we obtain

$$\lim_{n\to\infty} \Theta_p(u_{n-1}, t) = \max\big\{0, 0, \psi(p(t, Ft)), \psi(p(t, Ft)), 0\big\} = \psi(p(t, Ft)).$$

Taking the limit as $n \to \infty$ in equation (2.23) and using equation (2.24) and the property of ψ, we obtain

$$\int_0^{\psi(p(t,Ft))} \phi(t)dt \leq \mu \int_0^{\psi(p(t,Ft))} \phi(t)dt,$$

which is a contradiction, since $0 < \mu < 1$. Therefore, we have

$$\int_0^{\psi(p(t,Ft))} \phi(t)dt = 0.$$

Hence, by the property of integral ϕ, we conclude that $\psi(p(t, Ft)) = 0$. Now, by the property of ψ, we obtain $p(t, Ft) = 0$ and so $Ft = t$, that is, t is a fixed point of F. Now, we shall show that the fixed point of F is unique. Assume that t' is another fixed point of F such that $t \neq t'$. Then from equation (2.1) and using condition

$p(t,t) = 0$ and the property of ψ, we have

$$\begin{aligned}\int_0^{\psi(p(t,t'))} \phi(t)dt &= \int_0^{\psi(p(Ft,Ft'))} \phi(t)dt \\ &\leq \mu \int_0^{\Theta_p(t,t')} \phi(t)dt, \end{aligned} \tag{2.24}$$

where

$$\begin{aligned}\Theta_p(t,t') &= \max\Big\{\psi(p(t,t')), \psi(t,Ft)), \psi(p(t',Ft')), \\ &\qquad \psi\Big(\frac{p(t',Ft')(1+p(t,Ft))}{1+p(t,t')}\Big), \psi\Big(\frac{p(t,Ft)p(t',Ft)}{1+p(t,t')}\Big)\Big\} \\ &= \max\Big\{\psi(p(t,t')), \psi(p(t,t)), \psi(p(t',t')), \\ &\qquad \psi\Big(\frac{p(t',t')(1+p(t,t))}{1+p(t,t')}\Big), \psi\Big(\frac{p(t,t)p(t',t)}{1+p(t,t')}\Big)\Big\} \\ &= \max\big\{\psi(p(t,t')), 0, 0, 0, 0\big\} = \psi(p(t,t')).\end{aligned}$$

Using the above value in equation (2.24), we obtain

$$\int_0^{\psi(p(t,t'))} \phi(t)dt \leq \mu \int_0^{\psi(p(t,t'))} \phi(t)dt,$$

which is a contradiction, since $0 < \mu < 1$ and $p(t,t') > 0$. Therefore, we obtain

$$\int_0^{\psi(p(t,t'))} \phi(t)dt = 0.$$

Thus, regarding the property of integral ϕ, we conclude that $\psi(p(t,t')) = 0$. Again by the property of ψ, we obtain $p(t,t') = 0$ and so $t = t'$. This proves the uniqueness of fixed point. The proof is completed. □

3. Consequences of Theorem 2.1

If we take $\phi(t) = 1$ for all $t \geq 0$ in Theorem 2.1, then we have the following result.

Corollary 3.1. *Let (Y,p) be a complete partial metric space and $F\colon Y \to Y$ be a mapping satisfying the following contractive condition:*

$$\psi(p(Fu,Fv)) \leq \mu\, \Theta_p(u,v), \tag{3.1}$$

for all $u,v \in Y$, where $\mu \in [0,1)$, ψ is an altering distance function and $\Theta_p(u,v)$ is as in Theorem 2.1. Then F has a unique fixed point in Y. Moreover, $p(z,z) = 0$.

If we take $\phi(t) = 1$ for all $t \geq 0$ and $\psi(t) = t$ for all $t > 0$ in Theorem 2.1, then we have the following result.

Corollary 3.2. *Let (Y, p) be a complete partial metric space and $F\colon Y \to Y$ be a mapping satisfying the following contractive condition:*

$$p(Fu, Fv) \leq \mu\, \Delta_p(u, v), \tag{3.2}$$

where

$$\Delta_p(u,v) = \max\Big\{p(u,v), p(u,Fu), p(v,Fv), \frac{p(v,Fv)(1+p(u,Fu))}{1+p(u,v)}, \frac{p(u,Fu)p(v,Fu)}{1+p(u,v)}\Big\},$$

for all $u, v \in Y$ and $\mu \in [0,1)$ is a constant. Then F has a unique fixed point in Y. Moreover, $p(z,z) = 0$.

Corollary 3.3. *Let (Y, p) be a complete partial metric space and $F\colon Y \to Y$ be a mapping satisfying the following contractive condition:*

$$p(Fu, Fv) \leq \Lambda_p(u, v), \tag{3.3}$$

where

$$\Lambda_p(u,v) = A_1\, p(u,v) + A_2\, p(u,Fu) + A_3\, p(v,Fv) + A_4\, \frac{p(v,Fv)(1+p(u,Fu))}{1+p(u,v)} + A_5\, \frac{p(u,Fu)p(v,Fu)}{1+p(u,v)},$$

for all $u, v \in Y$ and A_1, A_2, A_3, A_4, A_5 are nonnegative reals such that $A_1 + A_2 + A_3 + A_4 + A_5 < 1$. Then F has a unique fixed point in Y. Moreover, $p(z,z) = 0$.

Proof. Follows from Corollary 3.2, by noting that

$$\begin{aligned} & A_1\, p(u,v) + A_2\, p(u,Fu) + A_3\, p(v,Fv) + A_4\, \frac{p(v,Fv)(1+p(u,Fu))}{1+p(u,v)} + A_5\, \frac{p(u,Fu)p(v,Fu)}{1+p(u,v)} \\ \leq\; & (A_1 + A_2 + A_3 + A_4 + A_5) \max\Big\{p(u,v), p(u,Fu), p(v,Fv), \frac{p(v,Fv)(1+p(u,Fu))}{1+p(u,v)}, \frac{p(u,Fu)p(v,Fu)}{1+p(u,v)}\Big\} \\ =\; & \mu \max\Big\{p(u,v), p(u,Fu), p(v,Fv), \frac{p(v,Fv)(1+p(u,Fu))}{1+p(u,v)}, \frac{p(u,Fu)p(v,Fu)}{1+p(u,v)}\Big\}, \end{aligned}$$

where $\mu = A_1 + A_2 + A_3 + A_4 + A_5 < 1$. □

If we take $\psi(t) = t$ for all $t > 0$ in Theorem 2.1, then we have the following result.

Corollary 3.4. *Let (Y, p) be a complete partial metric space and $F\colon Y \to Y$ be a mapping satisfying the following contractive condition:*

$$\int_0^{p(Fu,Fv)} \phi(t)dt \le \mu \int_0^{M_p(u,v)} \phi(t)dt, \tag{3.4}$$

for all $u, v \in Y$, where $M_p(u, v)$ is given by

$$\begin{aligned} M_p(u,v) &= \max\Big\{p(u,v), p(u,Fu), p(v,Fv), \\ &\qquad \frac{p(v,Fv)(1+p(u,Fu))}{1+p(u,v)}, \frac{p(u,Fu)p(v,Fu)}{1+p(u,v)}\Big\}, \end{aligned}$$

$\mu \in [0, 1)$, and $\phi\colon [0, +\infty) \to [0, +\infty)$ is a Lebesgue-integrable mapping which is summable on each compact subset of $[0, +\infty)$, and such that for each $\varepsilon > 0$, $\int_0^\varepsilon \phi(t)dt > 0$. Then F has a unique fixed point in Y. Moreover, $p(z, z) = 0$.

If we take

$$\max\Big\{p(u,v), p(u,Fu), p(v,Fv), \frac{p(v,Fv)(1+p(u,Fu))}{1+p(u,v)},$$

$$\frac{p(u,Fu)p(v,Fu)}{1+p(u,v)}\Big\} = p(u,v),$$

in Corollary 3.4, then we have the following result.

Corollary 3.5. *Let (Y, p) be a complete partial metric space and $F\colon Y \to Y$ be a mapping satisfying the following contractive condition:*

$$\int_0^{p(Fu,Fv)} \phi(t)dt \le \mu \int_0^{p(u,v)} \phi(t)dt, \tag{3.5}$$

for all $u, v \in Y$, where $\mu \in [0, 1)$ is a constant and $\phi\colon [0, +\infty) \to [0, +\infty)$ is a Lebesgue-integrable mapping which is summable on each compact subset of $[0, +\infty)$, and such that for each $\varepsilon > 0$, $\int_0^\varepsilon \phi(t)dt > 0$. Then F has a unique fixed point in Y.

Remark 3.1. Corollary 3.5 extends and generalizes Theorem 1.1 of Branciari [11] from complete metric spaces to complete partial metric spaces.

If we take $\phi(t) = 1$ for all $t \ge 0$ in Corollary 3.5, then we obtain the following result.

Corollary 3.6. *([26], Theorem 5.3) Let (Y,p) be a complete partial metric space and $F\colon Y\to Y$ be a mapping satisfying the following contractive condition:*

$$p(Fu,Fv)\le \mu\, p(u,v), \tag{3.6}$$

for all $u,v\in Y$, where $\mu\in[0,1)$ is a constant. Then F has a unique fixed point in Y.

Remark 3.2. Corollary 3.6 extends and generalizes well-known Banach contraction mapping principle [10] from complete metric spaces to complete partial metric spaces.

Corollary 3.7. *Let (Y,p) be a complete partial metric space and $F\colon Y\to Y$ be a mapping satisfying the following contractive condition:*

$$\begin{aligned}\int_0^{\psi(p(Fu,Fv))}\phi(t)dt \;\le\; & \nu_1\int_0^{\psi(p(u,v))}\phi(t)dt+\nu_2\int_0^{\psi(p(u,Fu))}\phi(t)dt\\ &+\nu_3\int_0^{\psi(p(v,Fv))}\phi(t)dt+\nu_4\int_0^{\psi\left(\frac{p(v,Fv)(1+p(u,Fu))}{1+p(u,v)}\right)}\phi(t)dt\\ &+\nu_5\int_0^{\psi\left(\frac{p(u,Fu)p(v,Fu)}{1+p(u,v)}\right)}\phi(t)dt,\end{aligned}$$

for all $u,v\in Y$, where $\nu_1,\nu_2,\nu_3,\nu_4,\nu_5$ are nonnegative reals such that $\nu_1+\nu_2+\nu_3+\nu_4+\nu_5<1$, ψ is an altering distance function and $\phi\colon[0,+\infty)\to[0,+\infty)$ is a Lebesgue-integrable mapping which is summable on each compact subset of $[0,+\infty)$, and such that for each $\varepsilon>0$, $\int_0^{\varepsilon}\phi(t)dt>0$. Then F has a unique fixed point in Y. Moreover, $p(z,z)=0$.

Proof. Follows from Theorem 2.1, by noting that

$$\begin{aligned}&\nu_1\int_0^{\psi(p(u,v))}\phi(t)dt+\nu_2\int_0^{\psi(p(u,Fu))}\phi(t)dt+\nu_3\int_0^{\psi(p(v,Fv))}\phi(t)dt\\ &+\nu_4\int_0^{\psi\left(\frac{p(v,Fv)(1+p(u,Fu))}{1+p(u,v)}\right)}\phi(t)dt+\nu_5\int_0^{\psi\left(\frac{p(u,Fu)p(v,Fu)}{1+p(u,v)}\right)}\phi(t)dt\\ &\le (\nu_1+\nu_2+\nu_3+\nu_4+\nu_5)\int_0^{\Theta_p(u,v)}\phi(t)dt\\ &= \mu\int_0^{\Theta_p(u,v)}\phi(t)dt,\end{aligned}$$

where $\mu=\nu_1+\nu_2+\nu_3+\nu_4+\nu_5<1$ and $\Theta_p(u,v)$ is given by

$$\begin{aligned}\Theta_p(u,v) = \max\Big\{&\psi(p(u,v)),\psi(p(u,Fu)),\psi(p(v,Fv)),\\ &\psi\Big(\frac{p(v,Fv)(1+p(u,Fu))}{1+p(u,v)}\Big),\psi\Big(\frac{p(u,Fu)p(v,Fu)}{1+p(u,v)}\Big)\Big\}.\end{aligned}$$

□

Example 3.1. Let $Y = [0,1]$ and $p(u,v) = \max\{u,v\}$ for all $u,v \in Y$, then (Y,p) is a complete partial metric space (PMS). Suppose $F\colon Y \to Y$ is defined by $F(u) = \frac{u}{4}$ for all $u \in Y$. Let $\phi\colon [0,+\infty) \to [0,+\infty)$ and $\psi\colon [0,+\infty) \to [0,+\infty)$ be such that $\phi(t) = 1$ for all $t \geq 0$ and $\psi(t) = t$ for all $t > 0$. Without loss of generality, we assume that $u \geq v$. Then we have

$$p(Fu,Fv) = \max\Big\{\frac{u}{4},\frac{v}{4}\Big\} = \frac{u}{4}. \tag{3.7}$$

On the other hand

$$\max\Big\{p(u,v), p(u,Fu), p(v,Fv), p(v,Fv)\frac{1+p(u,Fu)}{1+p(u,v)}, \frac{p(u,Fu)p(v,Fu)}{1+p(u,v)}\Big\}$$
$$= \max\Big\{u,u,v,v.\frac{1+u}{1+u}, \frac{uv}{1+u}\Big\} = u.$$

Combining the observations above, we obtain

$$p(Fu,Fv) = \frac{u}{4} \leq \mu\, u,$$

that is, $\mu \geq \frac{1}{4}$. If we take $0 < \mu < 1$, then all the conditions of Corollary 3.1 and Corollary 3.2 are satisfied. Hence, F has a unique fixed point, indeed $u = 0$ is the required point.

Example 3.2. Let $Y = [0,\infty)$. Define $F\colon Y \to Y$ by $F(u) = 2u$ for all $u \in Y$. Also, define $p\colon Y \times Y \to \mathbf{R}^+$ by $p(u,v) = \max\{u,v\}$ for all $u,v \in Y$, then (Y,p) is a complete partial metric space (PMS). It is clear that Matthew's Theorem (Theorem 5.3, [26]) (analog of BCP) does not work. Indeed, without loss of generality, we may assume that $u \leq v$. Then

$$p(Fu,Fv) = 2v > \mu\, v = \mu\, p(u,v),$$

for any $\mu \in [0,1)$.

However, for $\mu = \frac{1}{3}$, we have

$$\begin{aligned} p(Fu,Fv) &= 2v \leq \frac{1}{3}.2v.\frac{1+2u}{1+v} \\ &= \mu \max\Big\{p(u,v), p(u,Fu), p(v,Fv), p(v,Fv)\frac{1+p(u,Fu)}{1+p(u,v)}, \\ &\qquad \frac{p(u,Fu)p(v,Fu)}{1+p(u,v)}\Big\}. \end{aligned}$$

Thus by Corollary 3.2, F has a unique fixed point. Here 0 is the unique fixed point of F.

Example 3.3. Let $Y = \{1,2,3,4\}$ and $p\colon Y \times Y \to \mathbb{R}$ be defined by

$$p(u,v) = \begin{cases} |u-v| + \max\{u,v\}, & \text{if } u \neq v, \\ u, & \text{if } u = v \neq 1, \\ 0, & \text{if } u = v = 1, \end{cases}$$

for all $u,v \in Y$. Then (Y,p) is a complete partial metric space.

Now, we define a mapping $F\colon Y \to Y$ by

$$F(1) = 1,\ F(2) = 1,\ F(3) = 2,\ F(4) = 2.$$

Now, we have

$$
\begin{aligned}
p(F(1), F(2)) &= p(1,1) = 0 \leq \frac{3}{4}.3 = \frac{3}{4}p(1,2),\\
p(F(1), F(3)) &= p(1,2) = 3 \leq \frac{3}{4}.5 = \frac{3}{4}p(1,3),\\
p(F(1), F(4)) &= p(1,2) = 3 \leq \frac{3}{4}.7 = \frac{3}{4}p(1,4),\\
p(F(2), F(3)) &= p(1,2) = 3 \leq \frac{3}{4}.4 = \frac{3}{4}p(2,3),\\
p(F(2), F(4)) &= p(1,2) = 3 \leq \frac{3}{4}.6 = \frac{3}{4}p(2,4),\\
p(F(3), F(4)) &= p(2,2) = 2 \leq \frac{3}{4}.5 = \frac{3}{4}p(3,4).
\end{aligned}
$$

Thus, F satisfies all the conditions of Corollary 3.6 with $\mu = \frac{3}{4} < 1$. Now by applying Corollary 3.6, F has a unique fixed point. Here 1 is the unique fixed point of F.

4. An application to the Fredholm integral equation

In this section, we give an application of contraction condition (3.5) of Corollary 3.5 to the Fredholm integral equation:

$$
u(r) \quad = \quad w(r) + \lambda \int_a^b k(r,t)u(t)dt, \tag{4.1}
$$

where $u\colon [a,b] \to \mathbb{R}$ with $-\infty < a < b < +\infty$ and $k(r,t)$ is called the kernel of the integral equation (4.1) with $|k(r,t)| \leq M$ $(M > 0)$.

Let $Y = C[a,b]$ be the class of all real-valued continuous functions on $[a,b]$. Define $F\colon Y \to Y$ by

$$
F(u)(r) = w(r) + \lambda \int_a^b k(r,t)u(t)dt.
$$

Obviously, $u(r)$ is a solution of the Fredholm integral equation (4.1) if and only if $u(r)$ is a fixed point of F. Define $p\colon Y \times Y \to [0, +\infty)$ by

$$
p(u,v) = \sup_{r \in [a,b]} |u(r) - v(r)|,
$$

for all $u, v \in Y$. Then (Y,p) is a complete partial metric space. Now, we state and prove our result as follows.

Theorem 4.1. *Let (Y,p) be a complete partial metric space. Suppose that the following:*

(1) *The mappings $w\colon [a,b] \to \mathbb{R}$ and $k\colon [a,b] \to \mathbb{R}$ are continuous.*

(2) *There exists a nonnegative real number λ, such that*

$$
|\lambda| < \frac{1}{M(b-a)}.
$$

Then, the Fredholm integral equation (4.1) has a unique solution $u\colon [a,b] \to \mathbb{R}$ in Y.

Proof. Now, we show that F satisfies the contractive condition (3.5) of Corollary 3.5. Consider

$$p(Fu,Fv) = \sup_{r\in[a,b]} |Fu(r) - Fv(r)|$$

$$\begin{aligned}
&= \sup_{r\in[a,b]} \Big[\Big|w(r) + \lambda \int_a^b k(r,t)u(r)dt \\
&\qquad -\Big(w(r) + \lambda \int_a^b k(r,t)v(r)dt\Big)\Big|\Big] \\
&= \sup_{r\in[a,b]} \Big[\Big|\lambda \int_a^b k(r,t)[u(r) - v(r)]dt\Big|\Big] \\
&\leq \sup_{r\in[a,b]} |\lambda| \int_a^b |k(r,t)||u(r) - v(r)|dt \\
&\leq |\lambda| M \sup_{r\in[a,b]} |u(r) - v(r)| \int_a^b dt \\
&= |\lambda| M (b-a)\, p(u,v) \\
&< p(u,v),
\end{aligned}$$

which implies

$$\int_0^{p(Fu,Fv)} \phi(t)dt < \int_0^{p(u,v)} \phi(t)dt$$

for all $u, v \in Y$. Consequently, the contractive condition (3.5) is satisfied and the Fredholm integral equation (4.1) has a unique solution u in Y. □

Now, we give an example of Theorem 4.1.

Example 4.1. Let us consider the Fredholm integral equation defined as

$$u(r) = e + \lambda \int_1^e \frac{ln\, r}{t}\, u(t)dt. \tag{4.2}$$

Now we find a solution of the Fredholm integral equation (4.2) with initial condition $u_0(r) = 0$. We solve this equation for $|\lambda| < \frac{1}{e-1}$ since $\frac{1}{e-1} < 1$ for all $1 \leq r, t \leq e$. Thus, we obtain

$$u_1(r) = e,$$

$$u_2(r) = e + \lambda \int_1^e \frac{ln\, r}{t}\, e\, dt = e + \lambda\, e\, ln\, r,$$

$$u_3(r) = e + \lambda \int_1^e \frac{ln\, r}{t}\, (e + \lambda\, e\, ln\, t)\, dt = e + \lambda\, e\, ln\, r + \frac{\lambda^2}{2}\, e\, ln\, r,$$

$$
\begin{aligned}
u_4(r) &= e + \lambda \int_1^e \frac{ln\, r}{t} \, (e + \lambda\, e\, ln\, t + \frac{\lambda^2}{2}\, e\, ln\, t)\, dt \\
&= e + \lambda\, e\, ln\, r + \frac{\lambda^2}{2}\, e\, ln\, r + \frac{\lambda^3}{4}\, e\, ln\, r, \\
& \qquad \cdots \\
& \qquad \cdots \\
u_n(r) &= e + \lambda\, e\, ln\, r\Big[1 + \frac{\lambda}{2} + \frac{\lambda^2}{4} + \cdots + \frac{\lambda^n}{2^n}\Big] \\
&\to e + \frac{2\lambda\, e}{2-\lambda}\, ln\, r.
\end{aligned}
$$

Thus, this is a solution of the Fredholm integral equation (4.2) for $|\lambda| < \frac{1}{e-1} < 1$.

5. Conclusion

In the present paper, we prove some fixed point results via integral-type contractive conditions having rational terms and altering distance functions in the framework of partial metric spaces. Moreover, we give some consequences of the established results. Also, we provide some illustrative examples to validate the results. An application to the Fredholm integral equation is also given. Our results extend, generalize and enrich several previously published results from the existing literature (see, e.g. [10, 11, 26] and many others).

Acknowledgement

The author is grateful to the anonymous referees for their careful reading and valuable suggestions to improve this manuscript.

REFERENCES

1. M. Abbas, H. Aydi and S. Radenović: *Fixed point of T-Hardy-Rogers contractive mappings in ordered partial metric spaces.* Int. J. Math. Math. Sci. **2012**, Article ID 313675 (2012).
2. M. Abbas, T. Nazir and S. Romaguera: *Fixed point results for generalized cyclic contraction mappings in partial metric spaces.* Rev. R. Acad. Cienc. Exactas Fis. Nat., Ser. A Mat. RACSAM. **106(1)** (2012), 287–297.
3. T. Abdeljawad, E. Karapinar and K. Tas: *Existence and uniqueness of a common fixed point on partial metric spaces.* Appl. Math. Lett. **24(11)** (2011), 1894–1899.
4. I. Altun, D. Turkoglu and B. E. Rhoades: *Fixed points of weakly compatible maps satisfying a general contractive condition of integral type.* Fixed Point Theory Appl. **(2007)**, Art. ID 17301, 1-–9.
5. I. Altun, F. Sola and H. Simsek: *Generalized contractions on partial metric spaces.* Topology and its Appl. **157** (2010), 2778–2785.

6. I. Altun and A. Erduran: *Fixed point theorems for monotone mappings on partial metric spaces*. Fixed Point Theory Appl. **2011**, Article ID 508730 (2011). doi:10.1155/2011/508730
7. H. Aydi: *Fixed point results for weakly contractive mappings in ordered partial metric spaces*. J. Adv. Math. Stud. **4(2)** (2011), 1–12.
8. H. Aydi, M. Abbas and C. Vetro: *Partial Hausdorff metric and Nadler's fixed point theorem on partial metric spaces*. Topology and Its Appl. **159** (2012), No. 14, 3234–3242.
9. H. Aydi: *A common fixed point result by altering distances involving a contractive condition of integral type in partial metric spaces*. Demonstratio Math. **46(2)** (2013), 383–394.
10. S. Banach: *Sur les operation dans les ensembles abstraits et leur application aux equation integrals*. Fund. Math. **3**(1922), 133–181.
11. A. Branciari: *A fixed point theorem for mappings satisfying a general contractive condition of integral type*. Int. J. Math. Math. Sci. **29(9)** (2002), 531–536.
12. S. Chandok, D. Kumar and M. S. Khan: *Some results in partial metric space using auxiliary functions*. Applied Math. E-Notes **15** (2015), 233–242.
13. C. -M. Chen, E. Karapinar and D. O'Regan: *On $(\alpha-\phi)$-Meir-Keeler contractions on partial Hausdorff metric spaces*. U.P.B. Sci. Bull., Series A, Vol. **80**, No. 1, (2018), 101–110.
14. I. Erhan, E. Karapinar and D. Turkoglu: *Different types Meir-Keeler contractions on partial metric spaces*. J. Comput. Anal. Appl. **14(6)** (2012), 1000–1005.
15. R. Heckmann: *Approximation of metric spaces by partial metric spaces*. Appl. Categ. Structures, **7**, No. 1-2, (1999), 71–83.
16. E. Karapinar: *A note on common fixed point theorems in partial metric spaces*. Miskolc Math. Notes **12(2)** (2011), 185–191.
17. E. Karapinar and I. M. Erhan: *Fixed point theorems for operators on partial metric spaces*. Appl. Math. Lett. **24(11)** (2011), 1900–1904.
18. E. Karapinar and I. M. Erhan: *Cyclic contractions and fixed point theorems*. Filomat **26(4)** (2012), 777–782.
19. E. Karapinar, W. Shatanawi and K. Tas: *Fixed point theorems on partial metric spaces involving rational expressions*. Miskolc Math. Notes **14** (2013), 135–142.
20. E. Karapinar and S. Romaguera: *Nonunique fixed point theorems in partial metric spaces*: Filomat **27(7)** (2013), 1305–1314.
21. E. Karapinar: *Rational forms that imply the uniqueness of fixed points in partial metric spaces*. J. Nonlinear Convex Anal. **20(10)** (2019), 2171–2186.
22. E. Karapinar, R. P. Agarwal, S. S. Yesilkaya and C. Wang: *Fixed-point results for Meir-Keeler type contractions in partial metric spaces: A Survey*. Mathematics **2022**; 10(17), 3109. https://doi.org/10.3390/math10173109
23. M. S. Khan, M. Swaleh and S. Sessa: *Fixed point theorem by altering distances between the points*. Bull. Aus. Math. Soc. **30** (1984), 1–9.
24. H. P. A. Künzi, H. Pajoohesh and M. P. Schellekens: *Partial quasi-metrics*. Theoritical Comput. Sci. **365(3)** (2006), 237–246.
25. Z. Liu, J. Li and S. M. Kang: *Fixed point theorems of contractive mappings of integral type*. Fixed Point Theory Appl. 2013, **2013:300**, 17 pp.

26. S. G. Matthews: *Partial metric topology.* Research report 2012, Dept. Computer Science, University of Warwick, 1992.

27. S. G. Matthews: *Partial metric topology.* In: Proceedings of the 8th summer conference on topology and its applications, Annals of the New York Academy of Sciences, **728** (1994), 183–197.

28. S. Oltra and O. Valero: *Banach's fixed point theorem for partial metric spaces.* Rend. Ist. Mat. Univ. Trieste **36(1-2)** (2004), 17–26.

29. M. U. Rahman, M. Sarwar and M. U. Rahman: *Fixed point results of Altman integral type mappings in S-metric spaces.* Int. J. Anal. Appl. **10(1)** (2016), 58–63.

30. R. A. Rashwan and H. A. Hammad: *Common fixed point theorems for weakly compatible mappings satisfying a general contractive condition of integral type.* Palestine J. Math. **8(2)** (2019), 114–126.

31. B. E. Rhoades: *Two fixed point theorems for mappings satisfying a general contractive condition of integral type.* Int. J. Math. Sci. **3** (2003), 4007–4013.

32. S. Romaguera and M. P. Schellekens: *Duality and quasi-normability for complexity spaces.* Appl. General Topology **3** (2002), 91–112.

33. M. P. Schellekens: *Characterization of partial metrizability: domains are quantifiable.* Theoritical Comput. Sci. **305(1-3)** (2003), 409–432.

34. M. P. Schellekens: *The correspondence between partial metrics and semivaluations.* Theoritical Comput. Sci. **315(1)** (2004), 135–149.

35. O. Valero: *On Banach fixed point theorems for partial metric spaces.* Appl. Gen. Topl. **6(2)** (2005), 229–240.