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GEOMETRIC STRUCTURES ON THE CROSS SECTION IN THE TANGENT BUNDLE

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Abstract. Our goal is to investigate the Lie derivative in the tangent bundle TM with regard to the complete and vertical lifts of almost para-contact structures by using partial differential equations. We study certain theorems about vector fields inTM that are almost analytic. Additionally, the complete lift of an almost product structure in TM is examined along the cross-section.
Keywords: Tangent bundle, vertical lift, complete lift, partial differential equations, Lie derivative, almost analytic vector field, mathematical operators, cross-section.

1. Introduction

As a source of many new issues in the study of contemporary differential geometry, the differential geometry of tangent bundles holds an important place in the field. The differential geometry of tangent bundles has been greatly advanced by a number of researchers, including Yano and Ishihara [1], Davies [2], Yano and Davies [3], and Innus and Udriste [4]. Yano and Ishihara [5], Tanno [18] have explored the complete, vertical and horizontal lifts of tensor fields and connections on any manifold M to tangent bundle TM. Furthermore, Khan [8] has determined geometric properties of tensor fields by means of liftings from a para-Sasakian manifold to TM. Khan et. al. ([11], [16]) has studied tangent bundles endowed with

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a semi-symmetric non-metric connection on a Riemannian manifold and a quarter-symmetric metric connection from a Sasakian manifold, respectively. Numerous investigators have studied lifts on the various structures and connections including ([10], [12], [13], [14], [15], [20]).

In the following article, we review the fundamental definitions of lifts, almost product structure and almost para-contact structure in the general theory portion. The final section of our study focuses on the Lie derivative and almost analytic vector fields in the tangent bundle. Additionally, the complete lift of an almost product structure in TM is examined along the cross-section.

2. General theory

Let M be an n-dimensional differentiable manifold and let $\mathsf{TM} = \bigcup_{p\in \mathsf{M}} \mathsf{T}_p\mathsf{M}$ be its tangent bundle. Therefore, TM is a differentiable manifold as well. Let $\mathsf{X} = \sum_{i=1}^{n} x^i(\frac{\partial}{\partial x^i})$ and $\eta = \sum_{i=1}^{n} \eta^i dx^i$ be the expressions in the term of partial differential equations with local coordinates for the vector field X and a 1-form η in M. Let (a^i, b^i) be local coordinates of point in TM induced from (U, a^i) in M. Notations: the set of functions, vector fields, 1-forms, and tensor fields of type (1, 1) on M are denoted, respectively, by symbols $\wp_0^0(\mathsf{M}), \wp_0^1(\mathsf{M})$, and $\wp_1^1(\mathsf{M})$ ([5],[17]).

If f is a function in M, then f^V denotes the function in TM that may be derived by composing $\pi : \mathsf{TM} \longrightarrow \mathsf{M}$ and $f : \mathsf{M} \longrightarrow \mathsf{R}$, such that

$$\mathsf{f}^V = \mathsf{f} o \pi \tag{2.1}$$

is called the vertical lift of the function f.

The general properties of the vertical lift is given by

$$(G \otimes H)^V = G^V \otimes H^V, (G + K)^V = G^V + K^V,$$

where G, H and K are arbitrary elements of $\wp(M)$, $\wp(M)$ represents set of all tensor fields.

Let f^C denotes the function in TM given as

$$\mathsf{f}^C = i(d\mathsf{f}) \tag{2.2}$$

is called the complete lift of f and its local expression in the term of partial differential equations is given by

$$\mathsf{f}^C = b^i \partial_i \mathsf{f} = \partial \mathsf{f} \tag{2.3}$$

in TM such that $\partial\mathsf{f}$ represents $b^i\partial_i\mathsf{f}$. Let $\mathsf{X} \in \wp_0^1(M)$ and $\mathsf{X}^C \in \wp_0^1(\mathsf{TM})$ is given by

$$\mathsf{X}^C \mathsf{f}^C = (\mathsf{X}\mathsf{f})^C, \tag{2.4}$$

then X^C is called the complete lift of X in TM and in the term of x^h is given as

$$\mathsf{X}^C : \begin{bmatrix} x^h \\ \partial x^h \end{bmatrix} \tag{2.5}$$

with respect to (in short, wrt) the induced coordinates in TM [9].

Let $\mathsf{X} \in \wp_0^1(M)$ and a 1-form Υ^C in TM is given by

$$\Upsilon^C(\mathsf{X}^C) = (\Upsilon(\mathsf{X}))^C, \tag{2.6}$$

then Υ^C is called the complete lift of 1-form Υ.

The general properties of the complete lift are given by

$$(G \otimes H)^C = G^C \otimes H^V + G^V \otimes H^C, (G+K)^C = G^C + K^C,$$

where G, H and K are arbitrary elements of $\wp(M)$.

3. Propossed theorems on complete lifts of almost para-contact structures in the tangent bundle

Consider M, a n-dimensional differentiable manifold belonging to the class C^∞. We state that P provides an almost product structure on M if there exists on M a tensor field P of type (1,1) such that

$$\mathsf{P}^2 = \mathsf{I},$$

where I indicates the unit tensor field [6].

Let TM be the tangent bundle of M. Assume that a tensor field F, vector field ζ and 1-form Υ in M, fulfilling

$$\begin{aligned} \mathsf{F}^2 &= \mathsf{I} - \zeta \otimes \Upsilon \\ \mathsf{F}\zeta &= 0, \quad \Upsilon \circ \mathsf{F} = 0, \quad \Upsilon(\zeta) = 1. \end{aligned} \tag{3.1}$$

Consequently, the structure $(\mathsf{F}, \zeta, \Upsilon)$ on M is referred to as an almost para-contact structure and M is termed an almost para-contact manifold ([7], [21], [22]) .

Acquiring the complete lifts by means of mathematical operators on (3.1), we get

$$\begin{aligned} (\mathsf{F}^C)^2 &= \mathsf{I} - (\zeta^V \otimes \Upsilon^C + \zeta^C \otimes \Upsilon^V), \\ \mathsf{F}^C\zeta^V &= 0, \quad \mathsf{F}^C\zeta^C = 0, \\ \Upsilon^V \circ \mathsf{F}^C &= 0,\ \Upsilon^C \circ \mathsf{F}^V = 0,\ \Upsilon^C \circ \mathsf{F}^C = 0,\ \Upsilon^V \circ \mathsf{F}^V = 0, \\ \Upsilon^V(\zeta^V) &= 0,\ \Upsilon^V(\zeta^C) = 1,\ \Upsilon^C(\zeta^V) = 1,\ \Upsilon^C(\zeta^C) = 0. \end{aligned} \tag{3.2}$$

We introduced a tensor field $\tilde{J}$ of (1,1)-type as

$$\tilde{J} = -\mathsf{F}^C - (\zeta^V \otimes \Upsilon^V - \zeta^C \otimes \Upsilon^C). \tag{3.3}$$

Afterwards, it is clear from (3.2) and (3.3) that

$$\tilde{J}^2 = \mathsf{I},$$

indicating that $\tilde{J}$ is an almost product structure in TM. As a result, we have

Theorem 3.1. *Let* $(\mathsf{F}, \zeta, \Upsilon)$ *an almost para-contact structure on* M*. Then the tensor field* $\tilde{J}$*, given by (3.3) provides an almost product structure on* TM*.*

Corollary 3.1. *Let* X *be a vector field on* M *and a tensor field* $\tilde{J}$ *on* TM*, then*

$$\begin{aligned} \tilde{J}\mathsf{X}^V &= -(\mathsf{FX})^V + (\Upsilon(\mathsf{X}))^V \zeta^C, \\ \tilde{J}\mathsf{X}^C &= -(\mathsf{FX})^C - (\Upsilon(\mathsf{X}))^V \zeta^C - (\Upsilon(\mathsf{X}))^C \zeta^C. \end{aligned} \tag{3.4}$$

Proof. Proof follows in view of (3.2) and (3.3). □

Corollary 3.2. *Let* X *be a vector field on* M *and a tensor field* $\tilde{J}$ *on* TM*, then*

$$\begin{aligned} \tilde{J}\mathsf{X}^V &= -(\mathsf{FX})^V, \quad \tilde{J}\mathsf{X}^C = -(\mathsf{FX})^C, \\ \tilde{J}\zeta^V &= -\zeta^C, \quad \tilde{J}\zeta^C = -\zeta^V, \end{aligned} \tag{3.5}$$

such that $\Upsilon = 0$.

Proof. Proof follows in view of (3.2) and (3.3). □

Formulas on Lie Derivatives

Let X be a vector field in M. The differential transformation $\pounds_{\mathsf{X}}$ is said to be Lie derivative wrt X if [5]

$$\begin{aligned} &(a) \qquad \pounds_{\mathsf{X}}\mathsf{f} = \mathsf{Xf}, \quad \forall \mathsf{f} \in \wp_0^0(\mathsf{M}), \\ &(b) \qquad \pounds_{\mathsf{X}}\mathsf{Y} = [\mathsf{X}, \mathsf{Y}], \quad \forall \mathsf{X}, \mathsf{Y} \in \wp_0^1(\mathsf{M}), \end{aligned} \tag{3.6}$$

where $[,]$ indicates Lie bracket.

The Lie derivative $\pounds_{\mathsf{X}}\mathsf{F}$ is given by

$$(\pounds_{\mathsf{X}}\mathsf{F}) = [\mathsf{X}, \mathsf{FY}] - \mathsf{F}[\mathsf{X}, \mathsf{Y}], \tag{3.7}$$

$\forall \mathsf{F} \in \wp_1^1(\mathsf{M})$.

The following properties of Lie derivative $\pounds_{\mathsf{X}}$ wrt vector fields X^C and X^V are given as

$$\begin{aligned} &(a) \qquad \pounds_{\mathsf{X}^V}\mathsf{Y}^V = 0, \\ &(b) \qquad \pounds_{\mathsf{X}^V}\mathsf{Y}^C = (\pounds_{\mathsf{X}}\mathsf{Y})^V, \\ &(c) \qquad \pounds_{\mathsf{X}^C}\mathsf{Y}^V = (\pounds_{\mathsf{X}}\mathsf{Y})^V, \\ &(d) \qquad \pounds_{\mathsf{X}^C}\mathsf{Y}^C = (\pounds_{\mathsf{X}}\mathsf{Y})^C. \end{aligned} \tag{3.8}$$

Definition 1. Let F be an almost para-contact structure on M. Then a vector field X is said to be almost analytic vector field if $\pounds_{\mathsf{X}}\mathsf{F} = 0$ ([5], [23]).

Theorem 3.2. *Assume that $\tilde{J}$ be a (1,1) tensor field in* TM *of* M *described in (3.3), then* $\forall \mathsf{X}, \mathsf{Y} \in \wp_0^1(\mathsf{M})$ *such that* $\Upsilon(\mathsf{Y}) = 0$, *we infer*

$$
\begin{aligned}
&(a) && (\pounds_{\mathsf{X}^V}\tilde{J})\mathsf{Y}^V = 0,\\
&(b) && (\pounds_{\mathsf{X}^V}\tilde{J})\mathsf{Y}^C = -((\pounds_{\mathsf{X}}\mathsf{F})\mathsf{Y})^V + ((\pounds_{\mathsf{X}}\Upsilon)\mathsf{Y})^V\zeta^C,\\
&(c) && (\pounds_{\mathsf{X}^V}\tilde{J})\zeta^V = (\pounds_{\mathsf{X}}\zeta)^V,\\
&(d) && (\pounds_{\mathsf{X}^V}\tilde{J})\zeta^C = -((\pounds_{\mathsf{X}}\mathsf{F})\zeta)^V + ((\pounds_{\mathsf{X}}\Upsilon(\zeta))^V\zeta^C.
\end{aligned}
\tag{3.9}
$$

and

$$
\begin{aligned}
&(a) && (\pounds_{\mathsf{X}^C}\tilde{J})\mathsf{Y}^V = -((\pounds_{\mathsf{X}}\mathsf{F})\mathsf{Y})^V + (\pounds_{\mathsf{X}}\Upsilon(\mathsf{Y}))^V\zeta^C,\\
&(b) && (\pounds_{\mathsf{X}^C}\tilde{J})\mathsf{Y}^C = -((\pounds_{\mathsf{X}}\mathsf{F})\mathsf{Y})^C - ((\pounds_{\mathsf{X}}\Upsilon)(\mathsf{Y}))^V\zeta^C\\
&&& \qquad +((\pounds_{\mathsf{X}}\Upsilon)(\mathsf{Y}))^C\zeta^C,\\
&(c) && (\pounds_{\mathsf{X}^C}\tilde{J})\zeta^V = ((\pounds_{\mathsf{X}}\mathsf{F})\zeta)^C + [\mathsf{X},\zeta]^C + ((\pounds_{\mathsf{X}}\Upsilon)U)^V\zeta^C,\\
&(d) && (\pounds_{\mathsf{X}^C}\tilde{J})\zeta^C = -((\pounds_{\mathsf{X}}\mathsf{F})\zeta)^C - [\mathsf{X},\zeta]^V\\
&&& \qquad -((\pounds_{\mathsf{X}}\Upsilon)(\zeta))^V\zeta^C + ((\pounds_{\mathsf{X}}\Upsilon)(\zeta))^C\zeta^C.
\end{aligned}
\tag{3.10}
$$

Proof. The following proof uses (3.2), (3.4) and (3.5) in a clear way. □

Theorem 3.3. *Assume that* $(\mathsf{F}, U, \Upsilon)$ *be an almost para-contact structure on* M. *Then* X^V *in* TM *of* X *in* M *is almost analytic wrt an almost product structure* $\tilde{J}$ *described by (3.3) in* TM *if and only if*

$$\pounds_{\mathsf{X}}\mathsf{F} = 0, \qquad \pounds_{\mathsf{X}}\Upsilon_p = 0, \qquad \pounds_{\mathsf{X}}\zeta = 0 \tag{3.11}$$

are satisfied in M.

Proof. As a result of (3.9a), (3.9b), (3.10a) and (3.10b), the condition

$$\pounds_{\mathsf{X}^V}\tilde{J} = 0$$

is identical to conditions

$$\pounds_{\mathsf{X}}\mathsf{F} = 0, \qquad \pounds_{\mathsf{X}}\Upsilon = 0, \qquad \pounds_{\mathsf{X}}\zeta = 0.$$

□

Theorem 3.4. *Let* $(\mathsf{F}, \zeta, \Upsilon)$ *be an almost para-contact structure on* M. *Then* X^C *in* TM *of* X *in* M *is almost analytic wrt an almost product structure* $\tilde{J}$ *described by (3.3) in* TM *if and only if*

$$\pounds_{\mathsf{X}}\mathsf{F} = 0, \qquad \pounds_{\mathsf{X}}\zeta = c\zeta, \qquad \pounds_{\mathsf{X}}\Upsilon = -c\Upsilon, \tag{3.12}$$

where c is a non-zero constant, are satisfied in M.

Proof. As a result of (3.9c), (3.9d), (3.10c) and (3.10d), the condition

$$\pounds_{\mathsf{X}^C}\tilde{J} = 0$$

is identical to conditions

$$\pounds_{\mathsf{X}}\mathsf{F} = 0, \qquad \pounds_{\mathsf{X}}\zeta = c\zeta, \qquad \pounds_{\mathsf{X}}\Upsilon = -c\Upsilon.$$

□

4. Cross section in the tangent bundle

Suppose that TM be the tangent bundle of M and $\beta_{\mathsf{V}}(\mathsf{M})$ of TM be submanifold (dim n) is said to be the cross-section given by a vector field V in M, where β_{V} is a mapping $\beta_{\mathsf{V}} : \mathsf{M} \to \mathsf{TM}$. Let $\mathsf{V}^h(a)$ be the local components of V, then the cross-section $\beta_{\mathsf{V}}(\mathsf{M})$ is given by ([5], [19])

$$a^h = a^h, b^h = \mathsf{V}^h(a) \tag{4.1}$$

wrt the induced coordinates $(a^{\mathsf{A}}) = (a^h, b^h)$ in TM.

Let X be a vector field in M with local components a^h and local components of vector fields BX and CX in TM in the term of partial differential equations are

$$\mathsf{BX} : (\mathsf{B}_i^{\mathsf{A}}\mathsf{X}^i) = \begin{bmatrix} a^h \\ a^i \partial_i \mathsf{V}^h \end{bmatrix} \tag{4.2}$$

and

$$\mathsf{CX} : (\mathsf{C}_i^{\mathsf{A}}\mathsf{X}^i) = \begin{bmatrix} 0 \\ a^h \end{bmatrix}, \tag{4.3}$$

where BX is tangential to $\beta_{\mathsf{V}}(\mathsf{M})$ and CX is tangential to the fibre, given that for fibre a^h =constant, $b^h = b^h, b^h$ is parameters.

Applying complete and vertical lifts on (4.2) and (4.3), we infer along $\beta_V(\mathsf{M})$

$$\mathsf{X}^C = \mathsf{BX} + \mathsf{C}(\pounds_{\mathsf{V}}\mathsf{X}), \quad \mathsf{X}^V = \mathsf{CX}, \ \forall \mathsf{X} \in \wp_0^1(\mathsf{M}), \tag{4.4}$$

where $\pounds_{\mathsf{V}}\mathsf{X}$ represents the Lie derivative of X wrt V.

In the local components, X^C and X^V along $\beta_{\mathsf{V}}(\mathsf{M})$ are given by

$$\mathsf{X}^C : \begin{bmatrix} a^h \\ \pounds_{\mathsf{V}} a^h \end{bmatrix}, \quad \mathsf{X}^V : \begin{bmatrix} 0 \\ a^h \end{bmatrix}. \tag{4.5}$$

For any $\mathsf{F} \in \wp_1^1(\mathsf{M})$ and $\mathsf{F}^C \in \wp_1^1(\mathsf{TM})$ along $\beta_{\mathsf{V}}(\mathsf{M})$ having local components F_i^h is given by

$$\mathsf{F}^C : \begin{bmatrix} \mathsf{F}_i^h & 0 \\ \pounds_{\mathsf{V}}\mathsf{F}_i^h & \mathsf{F}_i^h \end{bmatrix}, \tag{4.6}$$

therefore, we infer along the cross section $\beta_{\mathsf{V}}(\mathsf{M})$

$$\mathsf{F}^C(B\mathsf{X}) = \mathsf{B}(\mathsf{FX}) + \mathsf{C}((\pounds_{\mathsf{V}}\mathsf{F})\mathsf{X}), \ \forall \mathsf{X} \in \wp_0^1(\mathsf{M}). \tag{4.7}$$

If $\mathsf{F}^C(\mathsf{BX})$ is tangential to $\beta_{\mathsf{V}}(\mathsf{M})$, then F^C is called leave $\beta_{\mathsf{V}}(\mathsf{M})$ invariant. Therefore, we infer

Proposition 4.1. $\forall \mathsf{F} \in \wp_1^1(\mathsf{M})$, *the element* $\mathsf{F}^C \in \wp_1^1(\mathsf{TM})$ *leave the cross section* $\beta_{\mathsf{V}}(\mathsf{M})$ *invariant if and only if* $\pounds_{\mathsf{V}}\mathsf{F} = 0$.

Theorem 4.1. *Let* F *be an almost product structure in* M *with* $\pounds_{\mathsf{V}}\mathsf{F} = 0, \mathsf{V} \in \wp_0^1(\mathsf{M})$, *then* $\mathsf{F}^{C\#}$ *is an almost product structure on the cross section in* TM *given by* V.

Proof. $\forall \mathsf{F} \in \wp_1^1(\mathsf{M})$, the element $\mathsf{F}^C \in \wp_1^1(\mathsf{TM})$ leave the cross section $\beta_{\mathsf{V}}(\mathsf{M})$ invariant. From [5], $\mathsf{F}^{C\#} \in \wp_1^1(\beta_{\mathsf{V}}(\mathsf{M}))$ given as

$$\mathsf{F}^{\mathsf{C}\#}(\mathsf{BX}) = \mathsf{F}^C(\mathsf{BX}) = \mathsf{F}(\mathsf{BX}). \tag{4.8}$$

Therefore, $\mathsf{F}^{C\#}$ is said to be induced on $\beta_{\mathsf{V}}(\mathsf{M})$ from F^C. Since F is an almost product structure in M and $\pounds_{\mathsf{V}}\mathsf{F} = 0$ i.e. from equation (4.8), we have

$$\mathsf{F}^2 = \mathsf{I} \quad \textit{and} \quad \pounds_{\mathsf{V}}\mathsf{F} = 0, \tag{4.9}$$

from equation (4.8), we have

$$(\mathsf{F}^{C\#})^2 = \mathsf{I} \tag{4.10}$$

Hence, $\mathsf{F}^{C\#}$ is an almost product structure in $\beta_{\mathsf{V}}(\mathsf{M})$.

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