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WEAK ALMOST CONTACT STRUCTURES: A SURVEY

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Abstract. Weak almost contact structures, defined by the author and R. Wolak, allowed us to take a new look at the theory of contact manifolds. The paper surveys recent results (concerning geodesic and Killing fields, rigidity and splitting theorems, Ricci-type solitons and Einstein-type metrics, etc.) in this new field of Riemannian geometry.
Keywords: weak almost contact structures, contact manifolds, geodesic fields, Killing fields, Ricci-type solitons, Einstein-type metrics, Riemannian geometry.

1. Introduction

Contact and Kähler Riemannian geometry is of growing interest due to its important role in mechanics, [2, 20]. Sasakian manifolds (normal contact metric manifolds) and, more generally, K-contact manifolds have become an important and active subject, especially after the appearance of the fundamental treatise [4] of C. Boyer and K. Galicki. On a K-contact manifold $M(f,\xi,\eta,g)$ the vector field ξ is Killing (the Lie derivative $\pounds_{\xi}\, g=0$), and geodesic ($\nabla_{\xi}\,\xi=0$). The restriction of f to $f(TM)$ determines a complex structure, and the structure group of TM reduces to $U(n)\times 1$. If a plane in TM contains ξ, then its sectional curvature is called ξ-*sectional curvature*. The ξ-sectional curvature of a K-contact manifold is equal to one. Two important subclasses of K-contact manifolds are

$$(\nabla_X f)Y = \begin{cases} g(X,Y)\,\xi-\eta(Y)X\,, & \text{Sasakian,}\\ 0\,, & \text{cosymplectic.}\end{cases} \tag{1.1}$$

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Any cosymplectic manifold is locally the product of a Kähler manifold and $\mathbb{R}$. A Riemannian manifold (M^{2n+1}, g) with a contact 1-form η (i.e., $\eta \wedge (d\eta)^n \neq 0$) is Sasakian, if its cone $M \times \mathbb{R}^{>0}$ with the metric $t^2 g + dt^2$ is a Kähler manifold. We get $\nabla_X\, \xi = -\frac{1}{2}\, fX$ on Sasakian manifolds, and $\nabla_X\, \xi = 0$ on cosymplectic manifolds.

Nearly Kähler manifolds [12] are defined by the condition that the symmetric part of ∇J vanishes, in contrast to the Kähler case where $\nabla J = 0$. Nearly Sasakian/cosymplectic manifolds are defined similarly – by a constraint on the symmetric part of the tensor f – starting from Sasakian/cosymplectic manifolds:

$$(1.2)\quad (\nabla_X f)Y + (\nabla_Y f)X = \begin{cases} 2\, g(X,Y)\xi - \eta(Y)X - \eta(X)Y\,, & \text{nearly Sasakian,} \\ 0\,, & \text{nearly cosymplectic.} \end{cases}$$

In dimensions greater than 5, every nearly Sasakian manifold is Sasakian, and a nearly cosymplectic manifold M^{2n+1} splits into $\mathbb{R} \times F^{2n}$ or $B^5 \times F^{2n-4}$, where F is a nearly Kähler manifold and B is a 5-dimensional nearly cosymplectic manifold. These structures appeared in the study of harmonic almost contact manifolds. For nearly Sasakian/cosymplectic manifolds we get $g(R_{\xi,Z}\, fX, fY) = 0$, see [8, 17], where $R_{X,Y}Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$ is the Riemann tensor; thus,

$$(1.3)\quad R_{X,Y} Z \perp \xi \quad (X, Y, Z \perp \xi) – \text{the distribution } \ker \eta \text{ is } \textit{curvature invariant}.$$

For example, the distribution $\ker \eta$ of any 1-form η on $\mathbb{R}^m$ satisfies (1.3).

The Ricci tensor is given by $\mathrm{Ric}(X, Y) = \mathrm{trace}\,(Z \to R_{Z,X} Y) = \sum_i g(R_{e_i,X} Y, e_i)$, where (e_i) is any local orthonormal basis of TM. The Ricci operator $\mathrm{Ric}^\sharp$ associated with the Ricci tensor is given by $\mathrm{Ric}(X, Y) = g(\mathrm{Ric}^\sharp X, Y)$. The scalar curvature of (M, g) is given by $r = \mathrm{trace}_g \mathrm{Ric}$. On some compact manifolds there are no Einstein metrics, which motivates the study of generalizations of such metrics. The *generalized Ricci soliton* is given for some smooth vector field V and real c_1, c_2 and λ by

$$(1.4)\qquad (1/2)\, \pounds_V\, g = -c_1 V^\flat \otimes V^\flat + c_2 \,\mathrm{Ric} + \lambda\, g.$$

If $V = \nabla f$ in (1.4) for some $f \in C^\infty(M)$, then by the definition $\mathrm{Hess}_f(X, Y) = \frac{1}{2}\, (\pounds_{\nabla f}\, g)(X, Y)$, we get the *generalized gradient Ricci soliton* equation, see [5]:

$$(1.5)\qquad \mathrm{Hess}_{f_1} = -c_1 df_2 \otimes df_2 + c_2 \,\mathrm{Ric} + \lambda\, g$$

for some $f_1, f_2 \in C^\infty(M)$ and real c_1, c_2 and λ. For different values of c_1, c_2, λ, equation (1.4) is a generalization of Einstein metric, $\mathrm{Ric} + \lambda\, g = 0$ ($c_1 = 0$, $c_2 = -1$, $V = 0$), Killing equation ($c_1 = c_2 = \lambda = 0$), Ricci soliton equation ($c_1 = 0$, $c_2 = -1$), etc., see [11]. In [6], Cho-Kimura defined *η-Ricci soliton* on a contact metric manifold $M^{2n+1}(f, \xi, \eta, g)$ by the following equation:

$$(1.6)\qquad (1/2)\, \pounds_V\, g + \mathrm{Ric} + \lambda\, g + \mu\, \eta \otimes \eta = 0,$$

where $\lambda, \mu \in \mathbb{R}$. For $\mu = 0$, (1.6) gives a Ricci soliton; moreover, if $V = 0$, then (1.6) gives an Einstein manifold. A contact metric manifold is called *η-Einstein*, if

$\mathrm{Ric} = a\,g + b\,\eta\otimes\eta$ is true, where $a, b \in C^\infty(M)$, see [31, p. 285] for $a, b \in \mathbb{R}$. Note that a and b can be non-constant for an η-Einstein Kenmotsu manifold, see [15].

Many articles study the questions: How interesting Ricci-type solitons are for contact metric manifolds? When an almost contact metric manifold equipped with a Ricci-type soliton carries Einstein-type metrics?

In [27,30], we introduced metric structures on a smooth manifold that generalize the almost contact, Sasakian, cosymplectic, etc. metric structures. Such so-called "weak" structures (the complex structure on the contact distribution $f(TM)$ is replaced by a nonsingular skew-symmetric tensor) allow us to take a fresh look at the theory of classical structures and find new applications. A *weak almost contact structure* on a smooth manifold M^{2n+1} is defined by a $(1,1)$-tensor field f of rank $2n$, a vector field ξ, a 1-form η and a nonsingular $(1,1)$-tensor field Q satisfying, see [30],

$$f^2 = -Q + \eta\otimes\xi, \qquad \eta(\xi) = 1, \qquad Q\,\xi = \xi. \tag{1.7}$$

A "small" (1,1)-tensor $\widetilde{Q} = Q - \mathrm{id}$ measures the difference between a weak almost contact structure and an almost contact one. We assume that a $2n$-dimensional distribution $\ker\eta$ is f-invariant (as in the classical theory [2], where $Q = \mathrm{id}$).

If there exists a Riemannian metric g such that

$$g(fX, fY) = g(X, Q\,Y) - \eta(X)\,\eta(Y), \quad X, Y \in \mathfrak{X}_M, \tag{1.8}$$

then (f, Q, ξ, η, g) is called a *weak almost contact metric structure*, and g is a *compatible metric*. Putting $Y = \xi$ in (1.8) and using $Q\,\xi = \xi$, we get $g(X, \xi) = \eta(X)$; moreover, the tensor f is skew-symmetric and the tensor Q is self-adjoint.

Remark 1.1. (i) The concept of an almost para-contact structure is closely related to an almost contact structure and an almost product structure, see [9]. Similarly to (1.7), we define a *weak almost para-contact structure* by $f^2 = Q - \eta\otimes\xi$, $Q\,\xi = \xi$, see [30]. (ii) A weak almost contact structure admits a compatible Riemannian metric if f admits a skew-symmetric representation, i.e., for any $x \in M$ there exist a neighborhood $U_x \subset M$ and a frame $\{e_k\}$ on U_x, for which f has a skew-symmetric matrix.

A *weak contact metric structure* is a weak almost contact metric structure satisfying $d\eta = \Phi$, where the 2-form Φ is defined by $\Phi(X, Y) = g(X, fY)$, $X, Y \in \mathfrak{X}_M$. For a weak contact metric structure (f, Q, ξ, η, g), the 1-form η is contact. A weak almost contact structure is *normal* if the tensor $\mathcal{N}^{(1)} = [f, f] + 2\,d\eta\otimes\,\xi$ vanishes.

A weak almost contact metric structure is called a *weak K-structure* if it is normal and $d\Phi = 0$. We define two subclasses of weak K-manifolds as follows: *weak cosymplectic manifolds* if $d\eta = 0$, and *weak Sasakian manifolds* if $d\eta = \Phi$. Omitting the normality condition, we get the following: a weak almost contact metric structure is called a *weak almost cosymplectic structure* if $d\eta = \Phi$ is valid; and a *weak almost Sasakian structure* if Φ and η are closed forms.

In [27], we proved rigidity results: a weak Sasakian structure is the Sasakian structure and a weak almost contact metric structure satisfying $\nabla f = 0$ is a weak cosymplectic structure. For this, we calculated the derivative ∇f, using a new tensor $\mathcal{N}^{(5)}$, see (2.1), in addition to classical tensors $\mathcal{N}^{(i)}$ $(i = 1, 2, 3, 4)$.

Example 1.1. Let $M^{2n+1}(\varphi, Q, \xi, \eta)$ be a weak almost contact manifold. Consider the product manifold $\bar M = M^{2n+1}\times\mathbb{R}$, and define tensor fields $\bar f$ and $\bar Q$ on $\bar M$ putting

$$\bar f(X,\, a\partial_1) = (fX - a\,\xi,\, \eta(X)\partial_1), \quad \bar Q(X,\, a\,\partial_1) = (QX,\, a\partial_1), \quad a\in C^\infty(M).$$

Hence, $\bar f(X,0) = (fX,0)$, $\bar Q(X,0) = (QX,0)$ for $X\in\ker f$, $\bar f(\xi,0) = (0,\partial_1)$, $\bar Q(\xi,0) = (\xi,0)$ and $\bar f(0,\partial_1) = (-\xi,0)$, $\bar Q(0,\partial_1) = (0,\partial_1)$. Then it is easy to verify that $\bar f^{\,2} = -\bar Q$. The tensors $\mathcal{N}^{\,(i)}$ $(i = 1,2,3,4)$ appear when we use the integrability condition $[\bar f,\bar f] = 0$ of $\bar f$ to express the normality condition $\mathcal{N}^{\,(1)} = 0$ of a weak almost contact structure.

The relationships between some classes of weak structures is summarized in the following diagram (known for classical structures):

$$\left|\begin{array}{c}\text{weak almost}\\ \text{contact metric}\end{array}\right| \overset{\Phi=d\eta}{\longrightarrow} \left|\begin{array}{c}\text{weak contact}\\ \text{metric}\end{array}\right| \overset{\xi\text{-Killing}}{\longrightarrow} \left|\begin{array}{c}\text{weak}\\ \text{K-contact}\end{array}\right| \overset{\mathcal{N}^{\,(1)}=0}{\longrightarrow} \left|\begin{array}{c}\text{weak}\\ \text{Sasakian}\end{array}\right|.$$

In [30], we prove that in the case of a weak structure, the partial Ricci flow, $\partial g/\partial t = -2\,(\mathrm{Ric}^\perp(g) - \Phi\,\mathrm{id}^\perp)$ where $\Phi : M\to\mathbb{R}$, reduces to ODE's, $(d/dt)\,\mathrm{Ric}^\perp = 4\,\mathrm{Ric}^\perp(\mathrm{Ric}^\perp - \Phi\,\mathrm{id}^\perp)$, and that weak structures with positive partial Ricci curvature retract to the classical structures with positive constant partial Ricci curvature.

We define a weak almost Hermitian structure on a Riemannian manifold of even dimension, (M^{2n}, g), equipped with a skew-symmetric (1,1)-tensor f by condition $f^{\,2} < 0$. Such (g, f) is a weak Kählerian structure, if $\nabla f = 0$, where ∇ is the Levi-Civita connection, and a weak nearly Kählerian structure, if $(\nabla_X f)X = 0$. Weak nearly Sasakian/cosymplectic manifolds are defined by a similar condition (1.2) together with (1.8) in the same spirit starting from Sasakian/cosymplectic manifolds, see [25].

A distribution $\mathcal{D}\subset TM$ is *totally geodesic* if and only if $\nabla_X Y + \nabla_Y X \in \mathcal{D}$ $(X, Y \in \mathcal{D})$ – this is the case when any geodesic of M that is tangent to $\mathcal{D}$ at one point is tangent to $\mathcal{D}$ at all its points, e.g., [29, Section 1.3.1]. Any such integrable distribution defines a totally geodesic foliation. A foliation, whose orthogonal distribution is totally geodesic, is said to be a Riemannian foliation.

At first glance, the results in [27,30] suggest that weak structures are not "far" from classical ones. But it turns out that weak nearly Sasakian/cosymplectic manifolds are relatively "far" from classical ones. In [22, 24, 25], we gave examples of $(4n+1)$-dimensional proper weak nearly Sasakian manifolds and found conditions:

$$(\nabla_X\,\widetilde Q)\,Y = 0 \quad (X\in TM,\ Y\in\ker\eta), \tag{1.9}$$

$$R_{\widetilde Q X, Y} Z\in\ker\eta \quad (X, Y, Z\in\ker\eta), \tag{1.10}$$

(trivial for $\widetilde Q = 0$) under which weak nearly cosymplectic manifolds split and weak nearly Sasakian manifolds are Sasakian – which generalizes the results in [17]. We do not extend (1.9) for $Y = \xi$, since then at any point either $\nabla\xi = 0$ or $\widetilde Q = 0$. By (1.10) and the first Bianchi identity, we obtain $R_{X,Y}\,\widetilde Q Z\in\ker\eta$ $(X, Y, Z\in\ker\eta)$.

The notion of a warped product is popular in differential geometry as well as in general relativity. Some solutions of Einstein field equations are warped products and some spacetime models are warped products. Z. Olszak [19] characterized in terms of warped products a class of almost contact manifolds, known as β-Kenmotsu manifolds for $\beta = const > 0$ (defined by K. Kenmotsu [15], when $\beta = 1$), given by

$$(\nabla_X f)Y = \beta\{g(fX, Y)\,\xi - \eta(Y)\,fX\}. \tag{1.11}$$

In [28] we extended results on Ricci-type solitons and Einstein-type metrics from Kenmotsu to weak β-Kenmotsu case.

Several authors studied the problem of finding skew-symmetric parallel 2-tensors (different from almost complex structures) on a Riemannian manifold and classified them, e.g., [13], or proved that some spaces do not admit them, e.g., [16]. The idea of considering the bundle of almost-complex structures compatible with a given metric led to the twistor construction and then to twistor string theory. Thus, we delegate further research into the study of the bundles of weak Hermitian and weak almost contact structures that are compatible with a given metric.

This article surveys our recent results in [21–28, 30].

2. Preliminaries

Here, we discuss the basic properties of weak structures.

By (1.7), $\ker\eta$ is Q-invariant and the following equalities are true:

$$f\,\xi = 0, \quad \eta\circ f = 0, \quad \eta\circ Q = \eta, \quad [Q,\, f] := Q\circ f - f\circ Q = 0.$$

Recall the following formulas with the Lie derivative $\pounds$ in the Z-direction:

$$(\pounds_Z f)X = [Z, fX] - f[Z, X], \quad (\pounds_Z\,\eta)X = Z(\eta(X)) - \eta([Z, X]),$$
$$(\pounds_Z\, g)(X, Y) = g(\nabla_X\, Z, Y) + g(\nabla_Y\, Z, X).$$

The Nijenhuis torsion of f and the exterior derivative of η and Φ are given by

$$[f, f](X, Y) = f^2[X, Y] + [fX, fY] - f[fX, Y] - f[X, fY],$$
$$d\eta(X, Y) = (1/2)\,\{X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y])\},$$
$$d\Phi(X, Y, Z) = (1/3)\,\big\{X\,\Phi(Y, Z) + Y\,\Phi(Z, X) + Z\,\Phi(X, Y)$$
$$- \Phi([X, Y], Z) - \Phi([Z, X], Y) - \Phi([Y, Z], X)\big\}.$$

The following tensors on almost contact manifolds are well known, see [2]:

$$\mathcal{N}^{(2)}(X, Y) = (\pounds_{fX}\,\eta)Y - (\pounds_{fY}\,\eta)X = 2\,d\eta(fX, Y) - 2\,d\eta(fY, X),$$
$$\mathcal{N}^{(3)}(X) = (\pounds_\xi f)X = [\xi, fX] - f[\xi, X],$$
$$\mathcal{N}^{(4)}(X) = (\pounds_\xi\,\eta)X = \xi(\eta(X)) - \eta([\xi, X]) = 2\,d\eta(\xi, X).$$

The following statement is based on [2, Theorem 6.1], i.e., $Q = \mathrm{id}_{TM}$.

Proposition 2.1. *Let a weak almost contact structure be normal. Then* $\mathcal{N}^{(3)} = \mathcal{N}^{(4)} = 0$ *and* $\mathcal{N}^{(2)}(X,Y) = \eta([\widetilde{Q}X,\, fY])$*; moreover, the vector field* ξ *is geodesic.*

Proposition 2.2. *For a metric weak almost contact structure we get*

$$\begin{aligned}2\,g((\nabla_X f)Y,Z) &= 3\,d\Phi(X,fY,fZ) - 3\,d\Phi(X,Y,Z) + g(\mathcal{N}^{(1)}(Y,Z),fX)\\ &+\mathcal{N}^{(2)}(Y,Z)\,\eta(X) + 2\,d\eta(fY,X)\,\eta(Z) - 2\,d\eta(fZ,X)\,\eta(Y) + \mathcal{N}^{(5)}(X,Y,Z),\end{aligned}$$

where a skew-symmetric with respect to Y *and* Z *(0,3)-tensor* $\mathcal{N}^{(5)}$ *is defined by*

$$\begin{aligned}\mathcal{N}^{(5)}(X,Y,Z) &= (fZ)\,(g(X,\widetilde{Q}Y)) - (fY)\,(g(X,\widetilde{Q}Z)) + g([X,fZ],\widetilde{Q}Y)\\ &- g([X,fY],\widetilde{Q}Z) + g([Y,fZ]-[Z,fY]-f[Y,Z],\ \widetilde{Q}X).\end{aligned} \tag{2.1}$$

For particular values of the tensor $\mathcal{N}^{(5)}$ *we get*

$$\begin{aligned}&\mathcal{N}^{(5)}(X,\xi,Z) = -\mathcal{N}^{(5)}(X,Z,\xi) = g(\mathcal{N}^{(3)}(Z),\,\widetilde{Q}X),\\ &\mathcal{N}^{(5)}(\xi,Y,Z) = g([\xi,fZ],\widetilde{Q}Y) - g([\xi,fY],\widetilde{Q}Z),\\ &\mathcal{N}^{(5)}(\xi,\xi,Y) = \mathcal{N}^{(5)}(\xi,Y,\xi) = 0.\end{aligned}$$

Theorem 2.1. (i) *On a weak K-contact manifold the vector field* ξ *is Killing and geodesic,* $\mathcal{N}^{(1)}(\xi,\cdot) = \mathcal{N}^{(5)}(\xi,\cdot,\cdot) = \mathcal{N}^{(5)}(\cdot,\xi,\cdot) = \mathcal{N}^{(5)}(\cdot,\xi,\cdot) = 0,$ $\pounds_\xi Q = \nabla_\xi Q = 0,$ $\nabla_\xi f = 0,$ $\nabla_X\,\xi = -fX,$ *and*

$$\begin{aligned}g((\nabla_X f)Y,Z) &= d\eta(fY,X)\,\eta(Z) - d\eta(fZ,X)\,\eta(Y)\\ &+\frac{1}{2}\,\eta([\widetilde{Q}Y,\,fZ])\,\eta(X) + \frac{1}{2}\,\mathcal{N}^{(5)}(X,Y,Z).\end{aligned}$$

(ii) *For a weak almost Sasakian structure, the tensors* $\mathcal{N}^{(2)}$ *and* $\mathcal{N}^{(4)}$ *vanish; moreover,* $\mathcal{N}^{(3)}$ *vanishes if and only if* ξ *is a Killing vector field, and*

$$\begin{aligned}g((\nabla_X f)Y,Z) &= \frac{1}{2}\,g(\mathcal{N}^{(1)}(Y,Z),fX)\\ &+ g(fX,fY)\,\eta(Z) - g(fX,fZ)\,\eta(Y) + \frac{1}{2}\,\mathcal{N}^{(5)}(X,Y,Z).\end{aligned} \tag{2.2}$$

In particular, ξ *is a geodesic vector field and* $g((\nabla_\xi f)Y,Z) = \frac{1}{2}\,\mathcal{N}^{(5)}(\xi,Y,Z)$.

(iii) *For a weak almost cosymplectic manifold, we get* $\mathcal{N}^{(2)} = \mathcal{N}^{(4)} = 0$, $\mathcal{N}^{(1)} = [f,f]$*, and* ξ *is geodesic. Moreover,* $\mathcal{N}^{(3)} = 0$ *if and only if* ξ *is a Killing vector field.*

3. The rigidity of Sasakian structure and characteristic of a cosymplectic structure

Here, we present rigidity results for weak Sasakian manifolds and characterize weak cosymplectic manifolds in the class of weak almost contact metric manifolds.

Proposition 3.1. *For a weak Sasakian structure,* (2.2) *reduces to*

$$g((\nabla_X f)Y, Z) = g(QX, Y)\,\eta(Z) - g(QX, Z)\,\eta(Y) + \eta(X)\big(\eta(Y)\,\eta(Z) - \eta(Y)\,\eta(Z)\big) + \frac{1}{2}\,\mathcal{N}^{(5)}(X, Y, Z).$$

The equality $\mathcal{N}^{(3)} = 0$ is valid for weak Sasakian manifolds, since it is true for Sasakian manifolds, see Theorem 3.1. We get the rigidity of the Sasakian structure.

Theorem 3.1. *A weak almost contact metric structure is a weak Sasakian structure if and only if it is a Sasakian structure.*

Proposition 3.2. *Let* (f, Q, ξ, η, g) *be a weak cosymplectic structure. Then*

$$2\,g((\nabla_X f)Y, Z) = \mathcal{N}^{(5)}(X, Y, Z). \tag{3.1}$$

In particular, using (3.1) *and* (1.7)*, we get* $g(\nabla_X\,\xi,\, QZ) = -\frac{1}{2}\,\mathcal{N}^{(5)}(X, \xi, fZ)$.

Recall that a K-structure is a cosymplectic structure if and only if f is parallel. Our following theorem generalizes this result.

Theorem 3.2. *A weak almost contact metric structure with* $\nabla f = 0$ *is a weak cosymplectic structure with* $\mathcal{N}^{(5)} = 0$.

Example 3.1. Let M be a $2n$-dimensional smooth manifold and $\bar f : TM \to TM$ an endomorphism of rank $2n$ such that $\nabla \bar f = 0$. To construct a weak cosymplectic structure on $M \times \mathbb{R}$ or $M \times \mathbb{S}^1$, take any point (x, t) of either space and set $\xi = (0, d/dt)$, $\eta = (0, dt)$ and

$$f(X, Y) = (\bar f X, 0), \quad Q(X, Y) = (-\bar f^{\,2} X,\, Y).$$

where $X \in M_x$ and $Y \in \{\mathbb{R}_t, \mathbb{S}^1_t\}$. Then (1.7) holds and Theorem 3.2 can be used.

4. Weak K-contact structure

Here, we generalize some properties of K-contact manifolds to weak K-contact case. We characterize weak K-contact manifolds among all weak contact metric manifolds by the following well known property of K-contact manifolds, see [2]:

$$\nabla\,\xi = -f. \tag{4.1}$$

Theorem 4.1. *A weak contact metric manifold is weak K-contact* (*that is ξ is a Killing vector field*) *if and only if* (4.1) *is valid.*

A Riemannian manifold with a unit Killing vector field and the property $R_{X,\xi}\,\xi = X$ $(X \perp \xi)$ is a K-contact manifold, e.g., [2]. We generalize this in the following

Theorem 4.2. *A Riemannian manifold* (M^{2n+1},g) *admitting a unit Killing field* ξ *with positive* ξ*-sectional curvature is a weak* K*-contact manifold* $M(f,Q,\xi,\eta,g)$ *with the tensors:* $\eta=g(\cdot,\,\xi)$*,* $f=-\nabla\,\xi$*, see* (4.1)*, and* $QX=R_{X,\,\xi}\,\xi$ $(X\in\ker\eta)$*.*

Example 4.1. By Theorem 4.2, we can search for examples of weak K-contact (not K-contact) manifolds among Riemannian manifolds of positive sectional curvature that admit unit Killing vector fields. Indeed, let M be an ellipsoid of $\mathbb{R}^{2n+2}$,

$$M=\Big\{(u_1,\ldots,u_{2n+2})\in\mathbb{R}^{2n+2}:\sum\nolimits_{i=1}^{n+1}u_i^2+a\sum\nolimits_{i=n+2}^{2n+1}u_i^2=1\Big\},\quad 0<a=const\neq 1,$$

where $n\geqslant 1$ is odd. The sectional curvature of (M,g) is positive. It follows that

$$\xi=(-u_2,u_1,\ldots,-u_{n+1},u_n,-\sqrt{a}\,u_{n+3},\sqrt{a}\,u_{n+2},\ldots,-\sqrt{a}\,u_{2\,n+2},\sqrt{a}\,u_{2\,n+1})$$

is a Killing vector field on $\mathbb{R}^{2\,n+2}$, whose restriction to M has unit length. Since ξ is tangent to M, so ξ is a unit Killing vector field on (M,g), see [7, p. 5]. For $n=1$, we get a weak K-contact manifold $M^3=\big\{u_1^2+u_2^2+au_3^2+au_4^2=1\big\}\subset\mathbb{R}^4$ with $\xi=(-u_2,u_1,-\sqrt{a}\,u_4,\sqrt{a}\,u_3)$.

Proposition 4.1. *For a weak* K*-contact manifold, the following equalities hold:*

$$R_{\xi,X}=\nabla_X f,\quad R_{\xi,X}\,\xi=f^2X,\quad \operatorname{Ric}(\xi,\xi)=\operatorname{trace}\,Q=2\,n+\operatorname{trace}\,\tilde Q.$$

If a Riemannian manifold admits a unit Killing vector field ξ, then $K(\xi,X)\geqslant 0$ $(X\perp\xi,\ X\neq 0)$, thus $\operatorname{Ric}(\xi,\xi)\geqslant 0$; moreover, $\operatorname{Ric}(\xi,\xi)\equiv 0$ if and only if $\nabla\xi\equiv 0$. In the case of K-contact manifolds, $K(\xi,X)=1$, see [2, Theorem 7.2].

Corollary 4.1. *For a weak* K*-contact manifold, the* ξ*-sectional curvature is*

$$K(\xi,X)=g(QX,X)>0\quad (X\in\mathcal{D},\ \|X\|=1);$$

therefore, for the Ricci curvature we get $\operatorname{Ric}(\xi,\xi)>0$*.*

Using Theorem 4.2 and Corollary 4.1, we show that a weak K-contact structure can be deformed (by the partial Ricci flow) to a K-contact structure, see [30].

Corollary 4.2. *A weak* K*-contact manifold* $M(f,Q,\xi,\eta,g_0)$ *admits a smooth family of metrics* g_t $(t\in\mathbb{R})$*, such that* $M(f_t,Q_t,\xi,\eta,g_t)$ *are weak* K*-contact manifolds with certainly defined* f_t *and* Q_t*; moreover,* g_t *converges exponentially fast, as* $t\to-\infty$*, to a limit metric* $\hat g$ *that gives a* K*-contact structure.*

The following theorem generalizes a well known result, e.g., [31, Proposition 5.1].

Theorem 4.3. *A weak* K*-contact manifold with conditions* $(\nabla\operatorname{Ric})(\xi,\,\cdot)=0$ *and* $\operatorname{trace}\,Q=const$ *is an Einstein manifold of scalar curvature* $r=(2\,n+1)\operatorname{trace}\,Q$*.*

Remark 4.1. For a weak K-contact manifold, by $\nabla_\xi f=\frac12\mathcal{N}^{(5)}(\xi,Y,Z)=0$ and Proposition 4.1, we get the equality (well known for K-contact manifolds, e.g., [2]): $\operatorname{Ric}^\sharp(\xi)=\sum_{i=1}^{2n}(\nabla_{e_i}f)\,e_i$, where (e_i) is any local orthonormal basis of $\ker\eta$; and for contact manifolds we have $\sum_{i=1}^{2n}(\nabla_{e_i}f)\,e_i=2\,n\,\xi$. For K-contact manifolds, this gives $\operatorname{Ric}^\sharp(\xi)=2\,n\,\xi$, and $\operatorname{Ric}(\xi,\xi)=2\,n$; moreover, the last condition characterizes K-contact manifolds among all contact metric manifolds.

The Ricci curvature of any K-contact manifold satisfies the condition

$$\mathrm{Ric}(\xi, X) = 0 \quad (X \in \mathcal{D}). \tag{4.2}$$

Quasi Einstein manifolds are defined by the condition $\mathrm{Ric}(X,Y) = a\,g(X,Y) + b\,\mu(X)\,\mu(Y)$, where a and $b \neq 0$ are real scalars, and μ is a 1-form of unit norm.

The next our theorem generalizes [11, Theorem 3.1] on Ricci type solitons.

Theorem 4.4. *Let a weak K-contact manifold with* trace $Q = const$ *satisfy* (1.5) *with $c_1 a \neq -1$ for $a = \lambda + c_2$* trace Q. *If* (4.2) *is true, then $f = const$. Furthermore,*

– if $c_1 a \neq 0$, then $\mathrm{Hess}_{f_2} = \frac{1}{a}\, df_2 \otimes df_2 - \frac{c_2}{c_1 a}\,\mathrm{Ric} - \frac{\lambda}{c_1 a}\, g$; if $c_1 a \neq -1$, then $f_2 = const$; and if $c_2 \neq 0$, then (M, g) is an Einstein manifold.

– if $a = 0$ and $c_1 \neq 0$, then $0 = c_2\,\mathrm{Ric} - c_1\, df_2 \otimes df_2 + \lambda\, g$. If $c_2 \neq 0$ and $f_2 \neq const$, then we get a gradient quasi Einstein manifold.

– for $c_1 = 0$, then $c_2\,\mathrm{Ric} + \lambda\, g = 0$, and for $c_2 \neq 0$ we get an Einstein manifold.

Quasi contact metric manifolds (introduced by Y. Tashiro) are an extension of contact metric manifolds. In [26], we study a weak analogue of quasi contact metric manifolds and provide new criterions for K-contact and Sasakian manifolds.

We pose the following questions. Is the condition "the ξ-sectional curvature is positive" sufficient for a weak almost contact metric manifold to be weak K-contact? Does a weak almost contact metric manifold of dimension > 3 have some positive ξ-sectional curvature? Is a compact weak K-contact Einstein manifold a Sasakian manifold? When a weak almost contact manifold equipped with a Ricci-type soliton structure, carries a canonical (for example, with constant sectional curvature or Einstein-type) metric? Is a compact weak K-contact Einstein manifold a Sasakian manifold? Is a compact weak K-contact manifold admitting a generalized Ricci soliton structure a Sasakian manifold? To answer these questions, we need to generalize some deep results about contact manifolds to weak contact manifolds.

5. Weak Nearly Sasakian/Cosymplectic Manifolds

The following result generalizes Proposition 3.1 in [3] and Theorem 5.2 in [1].

Theorem 5.1. *(i) Both on weak nearly Sasakian and weak nearly cosymplectic manifolds the vector field ξ is geodesic; moreover, if the condition* (1.9) *is valid, then the vector field ξ is Killing. (ii) There are no weak nearly cosymplectic structures with the condition* (1.9)*, which are weak contact metric structures.*

Theorem 5.2. *For a weak nearly cosymplectic manifold, $\nabla\xi = 0$ if and only if the manifold is locally a metric product of $\mathbb{R}$ and a weak nearly Kähler manifold.*

Proposition 5.1. *Let a weak almost contact manifold $M(f,Q,\xi,\eta)$ satisfy* (1.9) *and $Q\,|_{\,\ker\eta}=\lambda\,\mathrm{id}|_{\,\ker\eta}$ for a positive function $\lambda\in C^\infty(M)$. Then, $\lambda=const$ and $(\tilde f,\xi,\eta)$ is an almost-contact structure on M, where $\tilde f$ is given by*

$$f=\sqrt{\lambda}\,\tilde f. \tag{5.1}$$

Moreover, if a weak nearly Sasakian/cosymplectic structure (f,Q,ξ,η,g) satisfies (5.1) *and*

$$g|_{\,\ker\eta}=\lambda^{-\frac12}\,\tilde g|_{\,\ker\eta},\quad g(\xi,\,\cdot)=\tilde g(\xi,\,\cdot), \tag{5.2}$$

then $(\tilde f,\xi,\eta,\tilde g)$ is a nearly Sasakian/cosymplectic structure.

Example 5.1. Let $M(f,Q,\xi,\eta,g)$ be a three-dimensional weak almost-contact metric manifold. The tensor Q has on the plane field $\ker\eta$ in the form $\lambda\,\mathrm{id}_{\,\ker\eta}$ for some positive function $\lambda\in C^\infty(M)$. Suppose that (1.9) is true, then $\lambda=$ *const* and this structure reduces to the almost-contact metric structure $(\tilde f,\xi,\eta,\tilde g)$ satisfying (5.1) and (5.2).

Let (1.2) hold for $M(f,Q,\xi,\eta,g)$. By Proposition 5.1(ii), $M(\tilde f,\xi,\eta,\tilde g)$ is nearly cosymplectic or nearly Sasakian, respectively. Since $\dim M=3$, we obtain Sasakian (Theorem 5.1 in [18]) or cosymplectic (see [14]) structures $(\tilde f,\xi,\eta,\tilde g)$, respectively.

We generalize rigidity Theorem 3.2 in [3].

Theorem 5.3. *For a weak nearly Sasakian structure satisfying* (1.9)*, normality* ($N^{(1)}=0$) *is equivalent to a weak contact metric property* ($d\eta=\Phi$)*. Therefore, a normal weak nearly Sasakian structure satisfying* (1.9) *is Sasakian.*

Proposition 5.2. *A 3-dimensional weak nearly cosymplectic structure satisfying* (1.9) *reduces to cosymplectic one.*

Example 5.2. Let a 3-dimensional weak nearly Sasakian manifold $M(f,Q,\xi,\eta,g)$ satisfy (1.9). By (1.7), Q has on the plane field $\ker\eta$ the form $\lambda\,\mathrm{id}_{\,\ker\eta}$ for some positive $\lambda\in\mathbb{R}$. This structure reduces to the nearly Sasakian structure $(\tilde f,\xi,\eta,\tilde g)$, where $\tilde f=\lambda^{-\frac12}f$, $\tilde g|_{\,\ker\eta}=\lambda^{\frac12}\,g|_{\,\ker\eta}$, $\tilde g(\xi,\,\cdot)=g(\xi,\,\cdot)$. Since $\dim M=3$, the structure $(\tilde f,\xi,\eta,\tilde g)$ is Sasakian.

Next, we will study weak nearly Sasakian/cosymplectic hypersurfaces in weak nearly Kähler manifolds (generalizing nearly Kähler manifolds).

Example 5.3. Let $(\bar M,\bar f,\bar g)$ be a weak nearly Kähler manifold:$(\bar\nabla_X\bar f)X=0$ ($X\in T\bar M$). To build a weak nearly cosymplectic structure (f,Q,ξ,η,g) on the product $M=\bar M\times\mathbb{R}$ of $(\bar M,\bar g)$ and a Euclidean line $(\mathbb{R},\partial_t)$, we take any point (x,t) of M and set

$$\xi=(0,\partial_t),\quad \eta=(0,dt),\quad f(X,\partial_t)=(\bar fX,0),\quad Q(X,\partial_t)=(-\bar f^{\,2}X,\partial_t),\quad X\in T_x\bar M$$

(similarly to Example 3.1). Note that if $\bar\nabla_X\bar f^2=0$ ($X\in T\bar M$), then (1.9) holds.

The scalar second fundamental form b of a hypersurface $M \subset (\bar M, \bar g)$ with a unit normal N is related with $\overline\nabla$ and the Levi-Civita connection ∇ induced on the M metric g via the Gauss equation $\overline\nabla_X Y = \nabla_X Y + b(X,Y)\,N$ $(X,Y \in TM)$. The Weingarten operator $A_N : X \mapsto -\overline\nabla_X N$ is related with b via the equality $\bar g(b(X,Y),N) = g(A_N(X),Y)$ $(X,Y \in TM)$. A hypersurface is called *totally geodesic* if $b = 0$. A hypersurface is called *quasi-umbilical* if $b(X,Y) = c_1\, g(X,Y) + c_2\, \mu(X)\, \mu(Y)$, where $c_1, c_2 \in C^\infty(M)$ and $\mu \neq 0$ is a one-form.

Lemma 5.1. *A hypersurface (M,g) with a unit normal N in a weak Hermitian manifold $(\bar M, \bar f, \bar g)$ inherits a weak almost-contact structure (f,Q,ξ,η,g) given by*

$$\xi = \bar f N, \quad \eta = \bar g(\bar f N, \cdot), \quad f = \bar f + \bar g(\bar f N, \cdot)\, N, \quad Q = -\bar f^{\,2} + \bar g(\bar f^{\,2} N, \cdot)\, N.$$

The following theorem generalizes the fact (see [3]) that a hypersurface of a nearly Kähler manifold is nearly Sasakian or nearly cosymplectic if and only if it is quasi-umbilical with respect to the (almost) contact form.

Theorem 5.4. *Let M be a hypersurface with a unit normal N of a weak nearly Kähler manifold $(\bar M^{2n+2}, \bar f, \bar g)$. Then, the induced structure (f,Q,ξ,η,g) on M is (i) weak nearly Sasakian; (ii) weak nearly cosymplectic. This is true if and only if M is quasi-umbilical with the scalar second fundamental form*

(i) $b(X,Y) = g(X,Y) + (b(\xi,\xi) - 1)\, \eta(X)\, \eta(Y)$; *(ii)* $b(X,Y) = b(\xi,\xi)\, \eta(X)\, \eta(Y)$.

In both cases, $A_N f + f A_N = 2f$ is true, and (1.9) *follows from the condition* $((\overline\nabla_X \bar f^2) Y)^\top = 0$ $(X, Y \in TM,\ Y \perp \xi)$.

6. Splitting of weak nearly cosymplectic manifolds

Here, we show that a weak nearly cosymplectic manifold satisfies (1.3), if we assume a weaker condition (1.10). Then, we show the splitting of weak nearly cosymplectic manifolds satisfying (1.9) and (1.10) and generalize some well known results. We also characterize 5-dimensional weak nearly cosymplectic manifolds.

We define a (1,1)-tensor $h = \nabla\xi$ on M as in the classical case, e.g., [8]. Note that $h = 0$ if and only if $\ker\eta$ is integrable, i.e., $[X,Y] \in \ker\eta$ $(X,Y \in \ker\eta)$. Since ξ is a geodesic vector field, we get $h\,\xi = 0$ and $h(\ker\eta) \subset \ker\eta$. Since ξ is a Killing vector field, the tensor h is skew-symmetric: $g(hX,\, X) = g(\nabla_X\, \xi, X) = \frac{1}{2}\, (\pounds_\xi\, g)(X,X) = 0$. We also get $\eta \circ h = 0$ and $d\,\eta(X, \cdot) = \nabla_X\, \eta = g(hX, \cdot)$.

Lemma 6.1. *For a weak nearly cosymplectic manifold $M(f,Q,\xi,\eta,g)$ we obtain*

$$(\nabla_X\, h)\,\xi = -h^2 X, \qquad (\nabla_X\, f)\,\xi = -f\,hX.$$

Moreover, if the condition (1.9) *is true, then*

$$h\,f + f\,h = 0 \quad (h \text{ anticommutes with } f), \tag{6.1}$$

$$h\,Q = Q\,h \quad (h \text{ commutes with } Q). \tag{6.2}$$

Lemma 6.2. *For a weak nearly cosymplectic manifold we get the equality*

$$g(R_{fX,Y}Z,V)+g(R_{X,fY}Z,V)+g(R_{X,Y}fZ,V)+g(R_{X,Y}Z,fV)=0.$$

Moreover, if the conditions (1.9) *and* (1.10) *are true, then* $g(R_{\xi,Z}\,fX,fY)=0$.

Lemma 6.3. *Let a weak nearly cosymplectic manifold satisfy* (1.9)–(1.10)*, then*

$$g((\nabla_X\, f)Y, fhZ)=\eta(X)\,g(hY,hQZ)-\eta(Y)\,g(hX,hQZ).$$

Lemma 6.4. *For a weak nearly cosymplectic manifold satisfying* (1.10)*, we get*

$$(\nabla_X\, h)Y=g(h^2X,Y)\,\xi-\eta(Y)\,h^2X, \tag{6.3}$$

$$R_{\xi,X}Y=-(\nabla_X\, h)Y,\quad \mathrm{Ric}\,(\xi,Z)=-\eta(Z)\,\mathrm{tr}\,h^2. \tag{6.4}$$

In particular, $\nabla_\xi\, h=0$ *and* $\mathrm{tr}(h^2)=const$*. By* (6.3)–(6.4)*, we get*

$$g(R_{\xi,X}Y,Z)=-g((\nabla_X\, h)Y,Z)=\eta(Y)\,g(h^2X,Z)-\eta(Z)\,g(h^2X,Y).$$

The following proposition generalizes [17, Proposition 4.2].

Proposition 6.1. *For a weak nearly cosymplectic manifold satisfying* (1.9) *and* (1.10)*, the eigenvalues and their multiplicities of the operator* h^2 *are constant.*

By Proposition 6.1, the spectrum of the self-adjoint operator h^2 has the form

$$Spec(h^2)=\{0,-\lambda_1^2,\ldots-\lambda_r^2\}, \tag{6.5}$$

where λ_i is a positive real number and $\lambda_i\neq\lambda_j$ for $i\neq j$. If $X\neq 0$ is an eigenvector of h^2 with eigenvalue $-\lambda_i^2$, then X, fX, hX and $h\,fX$ are orthogonal nonzero eigenvectors of h^2 with eigenvalue $-\lambda_i^2$. Since $h(\xi)=0$, the eigenvalue 0 has multiplicity $2p+1$ for some $p\geqslant 0$. Denote by D_0 the smooth distribution of the eigenvectors with eigenvalue 0 orthogonal to ξ. Let D_i be the smooth distribution of the eigenvectors with eigenvalue $-\lambda_i^2$. Thus, D_0 and D_i belong to $\ker\eta$ and are f-invariant and h-invariant. The following proposition generalizes [17, Proposition 4.3].

Proposition 6.2. *Let a weak nearly cosymplectic manifold satisfy* (1.9)–(1.10)*, and let the spectrum of the self-adjoint operator* h^2 *have the form* (6.5)*. Then,*

(*a*) *each distribution* $[\xi]\oplus D_i$ $(i=1,\ldots,r)$ *is integrable with totally geodesic leaves.*

Moreover, if the eigenvalue 0 *of* h^2 *is not simple, then*

(*b*) *the distribution* D_0 *is integrable with totally geodesic leaves, and each leaf of* D_0 *is endowed with a weak nearly Kähler structure* $(\bar f,\bar g)$ *satisfying* $\bar\nabla(\bar f^{\,2})=0$*;*

(*c*) *the distribution* $[\xi]\oplus D_1\oplus\ldots\oplus D_r$ *is integrable with totally geodesic leaves.*

Proposition 6.3. *For a weak nearly cosymplectic (non-weak-cosymplectic) manifold,* $h\equiv 0$ *if and only if the manifold is locally isometric to the Riemannian product of a real line and a weak nearly Kähler (non-weak-Kähler) manifold.*

We generalize Theorem 4.5 in [17] on splitting of nearly cosymplectic manifolds.

Theorem 6.1. *Let $M^{2n+1}(f,Q,\xi,\eta,g)$ be a weak nearly cosymplectic (non-weak-cosymplectic) manifold of dimension $2n+1>5$ with conditions* (1.9) *and* (1.10). *Then M is locally isometric to one of the Riemannian products: $\mathbb{R}\times\bar M^{2n}$ or $B^5\times\bar M^{2n-4}$, where $\bar M(\bar f,\bar g)$ is a weak nearly Kähler manifold satisfying $\bar\nabla(\bar f^2)=0$, and B^5 is a weak nearly cosymplectic (non-weak-cosymplectic) manifold satisfying* (1.9) *and* (1.10). *If M is complete and simply connected, then the isometry is global.*

The following theorem generalizes Theorem 4.4 in [17].

Theorem 6.2. *Let $M(f,Q,\xi,\eta,g)$ be a weak nearly cosymplectic manifold with conditions* (1.9)–(1.10) *such that* 0 *is a simple eigenvalue of h^2. Then* $\dim M=5$.

7. Characterization of Sasakian manifolds

Here, we give examples of proper weak nearly Sasakian manifolds, present two theorems characterizing Sasakian manifolds in the class of weak almost contact metric manifolds satisfying conditions (1.9)–(1.10). On a weak nearly Sasakian manifold satisfying (1.9), the unit vector field ξ is Killing ($\pounds_\xi\, g=0$). Therefore, ξ-curves determine a Riemannian geodesic foliation.

Example 7.1. We construct proper weak nearly Sasakian manifolds from a pair of classical structures with the same ξ,η and g. Assume $0\leqslant i<n$ and define a manifold

$$M=\{(x_0,x_1,\dots,x_{4n})\in\mathbb{R}^{4n+1}: x_{4i+2}x_{4i+4}\neq 0\}$$

with standard coordinates $(x_0,x_1,\dots,x_{4n})$. The vector fields $X_0=\xi=-\partial_0$, $X_{4i+1}=2\big(x_{4i+2}\partial_0-\partial_{4i+1}\big)$, $X_{4i+2}=\partial_{4i+2}$, $X_{4i+3}=2\big(x_{4i+4}\partial_0-\partial_{4i+3}\big)$, $X_{4i+4}=\partial_{4i+4}$ are pointwise linearly independent. The non-vanishing Lie brackets are $[X_{4i+1},X_{4i+2}]=[X_{4i+3},X_{4i+4}]=2\,\xi$. Define a Riemannian metric of M by $g(X_i,X_j)=\delta_{ij}$, or,

$$g=dx_0^2+\sum\nolimits_i\big\{(1/4-x_{4i+2}^2)dx_{4i+1}^2+dx_{4i+2}^2+(1/4-x_{4i+4}^2)dx_{4i+3}^2+dx_{4i+4}^2\big\}.$$

Set $\eta=-dx_0$ and define a (1,1)-tensor f_1 on M by $f_1X_0=0$, $f_1X_{4i+1}=X_{4i+2}$, $f_1X_{4i+2}=-X_{4i+1}$, $f_1X_{4i+3}=X_{4i+4}$, $f_1X_{4i+4}=-X_{4i+3}$. Thus, (f_1,ξ,η,g) is an almost contact metric structure on M. The non-zero derivatives $\nabla_{X_a}X_b$ are

$$\begin{aligned}&\nabla_{X_{4i+1}}X_{4i+2}=-\nabla_{X_{4i+2}}X_{4i+1}=\xi,\quad \nabla_{X_{4i+1}}\xi=\nabla_\xi X_{4i+1}=-X_{4i+2},\\ &\nabla_{X_{4i+2}}\xi=\nabla_\xi X_{4i+2}=X_{4i+1},\quad \nabla_{X_{4i+3}}\xi=\nabla_\xi X_{4i+3}=-X_{4i+4},\\ &\nabla_{X_{4i+4}}\xi=\nabla_\xi X_{4i+4}=X_{4i+3},\quad \nabla_{X_{4i+3}}X_{4i+4}=-\nabla_{X_{4i+4}}X_{4i+3}=\xi.\end{aligned}\tag{7.1}$$

Thus (f_1,ξ,η,g) is a Sasakian structure. In particular, the distribution $\ker\eta$ is curvature invariant. Define a tensor f_2 on M by $f_2X_{4i+1}=X_{4i+4}$, $f_2X_{4i+4}=-X_{4i+1}$, $f_2X_0=0$, $f_2X_{4i+3}=X_{4i+2}$, $f_2X_{4i+2}=-X_{4i+3}$. It is easy to check that (f_2,ξ,η,g) is an almost contact metric structure on M. Using (7.1), we find that $(\nabla_Y f_2)Y=0$ and $(\nabla f_2)(X_{4i+1},\xi)=-X_{4i+3}\neq 0$, i.e., (f_2,ξ,η,g) is a proper nearly cosymplectic structure.

To construct a weak nearly Sasakian structure using the two above structures, we define a tensor $f := \cos(t)\, f_1 + \sin(t)\, f_2$ for small $t > 0$. A tensor $\psi := f_1\, f_2 + f_2\, f_1$ is self-adjoint with the following nonzero components:

$$\psi X_{4i+1} = -2X_{4i+3}, \quad \psi X_{4i+2} = -2X_{4i+4}, \quad \psi X_{4i+3} = -2X_{4i+1}, \quad \psi X_{4i+4} = -2X_{4i+2}.$$

Thus $Q = \mathrm{id} - \sin(t)\cos(t)\,\psi \ne \mathrm{id}$ and $(f, Q, \tilde\xi, \tilde\eta, \tilde g)$ is a weak nearly Sasakian structure, where $\tilde g = \mu^2 g$, $\tilde\xi = \mu^{-1}\xi$, $\tilde\eta = \mu\,\eta$ and $\mu = \cos(t)$. Since the distribution $\ker\eta$ is curvature invariant and $\tilde R = \mu^2 R$, then $(f, Q, \tilde\xi, \tilde\eta, \tilde g)$ satisfies (1.10); but $(f, Q, \tilde\xi, \tilde\eta, \tilde g)$ does not satisfy (1.9); for example, $(\tilde\nabla_{X_{4i+3}} Q)X_{4i+2} = \sin(2\,t)\,\xi \ne 0$.

Here, we generalize some properties of nearly Sasakian manifolds to the case of weak nearly Sasakian manifolds satisfying (1.9) and (1.10). Define a (1,1)-tensor field h on M, as in the classical case, by $h = \nabla\xi + f$. We get $\eta\circ h = 0$ and $h(\ker\eta) \subset \ker\eta$. Since ξ is a geodesic field, we also get $h\,\xi = 0$. Since ξ is a Killing and f is skew-symmetric, the tensor h is skew-symmetric:

$$g(hX,\, X) = g(\nabla_X\,\xi, X) + g(fX, X) = (1/2)\,(\pounds_\xi\, g)(X, X) = 0,$$

and $\nabla_X\,\eta = g((h-f)X,\,\cdot)$ holds. The $\ker\eta$ is integrable if and only if $h = f$, and in this case, our manifold is locally the metric product (splits along ξ and $\ker\eta$).

Lemma 7.1. *For a weak nearly Sasakian manifold $M^{2n+1}(f, Q, \xi, \eta, g)$ we obtain*

$$(\nabla_X h)\,\xi = -h(h-f)X, \qquad (\nabla_X f)\,\xi = -f(h-f)X.$$

Moreover, if (1.9) *is true, then*

$$h\,f + f\,h = -2\,\widetilde Q, \quad h\,Q = Q\,h \quad (h \text{ commutes with } Q),$$
$$h^2 f = fh^2, \quad hf^2 = f^2 h, \quad h^2 f^2 = f^2 h^2.$$

Proposition 7.1. *Let a weak nearly Sasakian manifold satisfy* (1.9)–(1.10)*, then*

$$g(R_{\xi,Z}\, fX, fY) = 0, \quad \text{hence, } \ker\eta \text{ is a curvature invariant distribution.}$$

Lemma 7.2. *For a weak nearly Sasakian manifold $M^{2n+1}(f, Q, \xi, \eta, g)$ with conditions* (1.9) *and* (1.10)*, we obtain*

$$R_{\xi,X} Y = -(\nabla_X\,(h-f)\,)\,Y, \tag{7.2}$$
$$(\nabla_X\,(h-f))Y = g((h-f)^2 X, Y)\,\xi - \eta(Y)\,(h-f)^2 X, \tag{7.3}$$
$$\mathrm{Ric}\,(\xi, Z) = -\eta(Z)\,(\mathrm{tr}\,(h^2 + \widetilde Q) - 2\,n). \tag{7.4}$$

In particular, $\mathrm{tr}(h^2 + \widetilde Q) = const$, $\mathrm{Ric}\,(\xi,\xi) = const \geqslant 0$ *and* $\nabla_\xi\, h = \nabla_\xi\, f = fh + \widetilde Q$. *By* (7.2)–(7.3)*, we get* $g(R_{\xi,X}Y, Z) = \eta(Y)\,g((h-f)^2X, Z) - \eta(Z)\,g((h-f)^2X, Y)$.

Proposition 7.2. *For a weak nearly Sasakian manifold with the property* (1.9)*, the equality $h = 0$ holds if and only if the manifold is Sasakian.*

Proposition 7.3. *For a weak nearly Sasakian manifold with conditions* (1.9) *and* (1.10)*, the eigenvalues (and their multiplicities) of the self-adjoint operator h^2 are constant. The spectrum of h^2 has the form* (6.5)*: $Spec(h^2) = \{0, -\lambda_1^2, \dots -\lambda_r^2\}$.*

In particular, $\operatorname{tr}(h^2) = const \leqslant 0$, and by Lemma 7.2, $\operatorname{tr} Q = const > 0$.

Denote by $[\xi]$ the 1-dimensional distribution generated by ξ, and by D_0 a smooth distribution of the eigenvectors of h^2 with eigenvalue 0 orthogonal to ξ. Denote by D_i a smooth distribution of the eigenvectors of h^2 with eigenvalue $-\lambda_i^2$. Note that the distributions D_0 and D_i $(i = 1, \dots, r)$ belong to $\ker \eta$ and are f-invariant and h-invariant. In particular, the eigenvalue 0 has multiplicity $2p+1$ for some $p \geqslant 0$. If X is a unit eigenvector of h^2 with eigenvalue $-\lambda_i^2$, then by (6.1) and (6.2), X, fX, hX and $h\,fX$ are nonzero eigenvectors of h^2 with eigenvalue $-\lambda_i^2$. First, we show that weak nearly Sasakian manifolds with (1.9)–(1.10) have a foliated structure.

Theorem 7.1. *Let $M^{2n+1}(f, Q, \xi, \eta, g)$ be a weak nearly Sasakian manifold with conditions* (1.9) *and* (1.10)*, and let the spectrum of the self-adjoint operator h^2 have the form* (6.5)*, where the eigenvalue* 0 *has multiplicity* $2p + 1$ *for some integer* $p \geqslant 0$*. Then, the distribution $[\xi] \oplus D_0$ and each distribution $[\xi] \oplus D_i$ $(i = 1, \dots, r)$ are integrable with totally geodesic leaves. If $p > 0$, then*

(*a*) *the distribution $[\xi] \oplus D_1 \oplus \dots \oplus D_r$ is integrable and defines a (2n-2p+1)-dimensional Riemannian foliation with totally geodesic leaves;*

(*b*) *the leaves of $[\xi] \oplus D_0$ are $(2p + 1)$-dimensional Sasakian manifolds.*

Next, we give some properties of the tensors f and h, and two theorems characterizing Sasakian manifolds in the class of weak almost contact metric manifolds. First, we consider weak almost contact metric manifolds with the condition (1.9) and characterize Sasakian manifolds in this class using the property (1.1).

Theorem 7.2. *Let $M(f, Q, \xi, \eta, g)$ be a weak almost contact metric manifold with conditions* (1.1) *and* (1.9)*. Then $Q = \mathrm{id}_{TM}$ and $M(f, \xi, \eta, g)$ is Sasakian.*

Next, we generalize Proposition 3.1 in [17].

Proposition 7.4. *Let a weak nearly Sasakian manifold satisfy* (1.9)–(1.10)*, then*

$$
\begin{aligned}
(\nabla_X f)Y &= \eta(X)\,(fhY + \widetilde{Q}Y) - \eta(Y)\,(fhX + QX) + g(fhX + QX, Y)\,\xi,\\
(\nabla_X h)Y &= \eta(X)\,(fhY + \widetilde{Q}Y) - \eta(Y)\,h(h - f)X + g(h(h - f)X, Y)\,\xi,\\
(\nabla_X fh)Y &= \eta(X)\,(fh^2Y - hY + \widetilde{Q}fY) - \eta(Y)\,g(fh^2X - QhX + 2\,\widetilde{Q}fX)\\
&\quad + g(fh^2X - hX + \widetilde{Q}hX, Y)\,\xi,\\
g((\nabla_X f)Y, hZ) &= -\eta(X)\,g((fh^2 + \widetilde{Q}h)Z, Y) + \eta(Y)\,g((fh^2 - h + \widetilde{Q}h)Z, X).
\end{aligned}
$$

Recall, that for any 2-form β and 1-form η we have

$$
3\,(\eta \wedge \beta)(X, Y, Z) = \eta(X)\,\beta(Y, Z) + \eta(Y)\,\beta(Z, X) + \eta(Z)\,\beta(X, Y).
$$

Proposition 7.5. *Let η be a contact 1-form on a smooth manifold M of dimension $2n+1>5$ and $\Lambda^p(M)$ the vector bundle of differential p-forms on M. Then, the operator $\Upsilon_{d\eta}:\beta\in\Lambda^2(M)\to d\eta\wedge\beta\in\Lambda^4(M)$ is injective.*

Using the above, we generalize Theorem 3.3 in [17].

Theorem 7.3. *Let a weak nearly Sasakian manifold $M(f,Q,\xi,\eta,g)$ $(\dim M>5)$ satisfy (1.9)–(1.10). Then $Q=\mathrm{id}_{TM}$ and the structure (f,ξ,η,g) is Sasakian.*

8. Weak β-Kenmotsu manifolds

Recall [19] that the warped product $\mathbb{R}\times_\sigma\bar M$ (of $\mathbb{R}$ and a Kähler manifold $(\bar M,\bar g)$) with the metric $g=dt^2\oplus\sigma^2\,\bar g$ and the function $\sigma>0$ given on $(-\varepsilon,\varepsilon)$ and satisfying

$$(\partial_t\,\sigma)/\sigma=\beta, \tag{8.1}$$

admits a β-Kenmotsu structure (ξ,η,f), see (1.11); conversely, any point of a β-Kenmotsu manifold has a neighbourhood, which is a warped product $(-\varepsilon,\varepsilon)\times_\sigma\bar M$ of an interval and a Kähler manifold, where σ satisfies (8.1).

8.1. Geometry of weak β-Kenmotsu manifolds

The following formulas are true for a weak β-Kenmotsu manifold:

$$\nabla_X\,\xi=\beta\{X-\eta(X)\,\xi\},\qquad d\eta(\xi,\,X)=0\qquad(X\in\mathfrak{X}_M).$$

Proposition 8.1. *A weak β-Kenmotsu manifold $M^{2n+1}(f,Q,\xi,\eta,g)$ with $n>1$ is a weak almost contact metric manifold satisfying $N^{(1)}{=}d\eta{=}0$ and $3\,d\Phi=2\beta\eta\wedge\Phi$.*

The condition $d\beta\wedge\eta=0$ follows from (1.11) if $\dim M>3$, and it does not hold if $\dim M=3$. Indeed, by Proposition 8.1, we get $0=3\,d^2\Phi=2\,d\beta\wedge\eta\wedge\Phi$.

Example 8.1. Let $(\bar M,\bar g)$ be a Riemannian manifold. A *warped product* $\mathbb{R}\times_\sigma\bar M$ is the product $M=\mathbb{R}\times\bar M$ with the metric $g=dt^2\oplus\sigma^2(t)\,\bar g$, where $\sigma>0$ is a smooth function on $\mathbb{R}$. Set $\xi=\partial_t$ and denote by $\mathcal{D}$ the distribution on M orthogonal to ξ. The Levi-Civita connections, ∇ of g and $\bar\nabla$ of $\bar g$, are related as follows:

(i) $\nabla_\xi\,\xi=0$, $\nabla_X\,\xi=\nabla_\xi\,X=\xi(\log\sigma)X$ for $X\in\mathcal{D}$.

(ii) $\pi_{1*}(\nabla_XY)=-g(X,Y)\,\xi(\log\sigma)$, where $\pi_1:M\to\mathbb{R}$ is the orthoprojector.

(iii) $\pi_{2*}(\nabla_XY)$ is the lift of $\bar\nabla_XY$, where $\pi_2:M\to\bar M$ is the orthoprojector.

Submanifolds $\{t\}\times\bar M$ (called the *fibers*) are totally umbilical, i.e., the Weingarten operator $A_\xi=-\nabla\xi$ on $\mathcal{D}$ is conformal with the factor $-\xi(\log\sigma)$, see (iii). Note that σ is constant along the fibers; thus, $X(\sigma)=\xi(\sigma)\,\eta(X)$ for all $X\in\mathfrak{X}_M$.

Let $(\bar M, \bar g, \bar f)$ be a weak Kählerian manifold, and $\partial_t\sigma \ne 0$. Then the warped product $\mathbb{R}\times_\sigma \bar M$ is a weak β-Kenmotsu manifold with $\beta = (\partial_t\sigma)/\sigma$. Indeed, define tensors,

$$(8.2)\qquad f = \begin{pmatrix} 0 & 0 \\ 0 & \bar f \end{pmatrix},\ Q = \begin{pmatrix} 0 & 0 \\ 0 & -\bar f^{\,2} \end{pmatrix},\ \xi = \begin{pmatrix} \partial_t \\ 0 \end{pmatrix},\ \ \eta = (dt, 0),\ g = \begin{pmatrix} dt^2 & 0 \\ 0 & \sigma^2 \bar g \end{pmatrix}$$

on $M = \mathbb{R}\times \bar M$. Note that $X(\beta) = 0$ $(X \perp \xi)$.
If $X, Y \in \mathcal{D}$, then $\pi_{2*}((\nabla_X f)Y) = (\bar\nabla_X \bar f)Y = 0$ and $\pi_{1*}((\nabla_X f)Y) = -\beta\, g(X, fY)$. If $X \in \mathcal{D}$ and $Y = \xi$, then $(\nabla_X f)\xi = -f\nabla_X\,\xi = -\beta\, fX$. If $X = \xi$ and $Y \in \mathcal{D}$, then $(\nabla_\xi f)Y = \beta(fY) - f(\beta Y) = 0$. Also, we get $(\nabla_\xi f)\xi = 0$. By the above, (1.11) is true.

Theorem 8.1. *Every point of a weak β-Kenmotsu manifold $M(f, Q, \xi, \eta, g)$ has a neighborhood U that is a warped product $(-\varepsilon, \varepsilon)\times_\sigma \bar M$, where $(\partial_t\sigma)/\sigma = \beta$, $(\bar M, \bar g, \bar f)$ is a weak Kählerian manifold, and the structure (ξ, η, f, Q, g) is given on U as* (8.2)*. Thus, the equality $X(\beta) = 0$ $(X \perp \xi)$ is valid.*

From Example 8.1 and Theorem 8.1, we obtain the following generalization of [19, Theorem 2.3]. A weak almost contact metric manifold $M(f, Q, \xi, \eta, g)$ is a weak β-Kenmotsu manifold if and only if the following conditions are valid:
– the ξ-trajectories are geodesics and f is ξ-invariant, i.e., $\pounds_\xi f = 0$,
– the distribution $\ker\eta$ is integrable and defines a totally umbilical foliation $\mathcal{F}$ of codimension one, whose leaves have constant mean curvature,
– a weak Hermitian structure $(\bar g, \bar f)$ induced on a leaf $\bar M \in \mathcal{F}$ is weak Kählerian.

Example 8.2. Let $(\bar M, \bar g, \bar f)$ be a weak Kählerian manifold and $\sigma(t) = c\,e^t$ ($c = const \ne 0$) a function on a line $\mathbb{R}$. Then the warped product manifold $M = \mathbb{R}\times_\sigma \bar M$ has a weak almost contact metric structure which satisfies (1.11) with $\beta \equiv 1$.

Lemma 8.1. *The following formulas hold on weak β-Kenmotsu manifolds:*

$$\begin{aligned}
&R_{X,Y}\,\xi = (\xi(\beta) + \beta^2)(\eta(X)Y - \eta(Y)X) \quad (X, Y \in \mathfrak{X}_M),\\
&\mathrm{Ric}^\sharp\,\xi = -2n(\xi(\beta) + \beta^2)\xi,\\
&(\nabla_\xi\,\mathrm{Ric}^\sharp)X = -2\,\beta\,\mathrm{Ric}^\sharp X - 2\,(\xi(\beta^2) + 2\,n\,\beta^3)X + 2\,(1-n)\,\xi(\beta^2)\,\eta(X)\,\xi,\\
&\xi(r) = -2\,\beta\,(r + 2\,n(2\,n+1)\beta^2) - 6\,n\,\xi(\beta^2), \quad X \in \mathfrak{X}_M;
\end{aligned}$$

in particular, if $\beta = const$, then $(\nabla_\xi\,\mathrm{Ric}^\sharp)X = -2\,\beta\,\mathrm{Ric}^\sharp X - 4n\,\beta^3 X$.

Theorem 8.2. *If a weak β-Kenmotsu manifold $M(f, Q, \xi, \eta, g)$ with $\beta = const \ne 0$ satisfies $\nabla_\xi\,\mathrm{Ric}^\sharp = 0$, then (M, g) is an Einstein manifold with $r = -2n(2n+1)\beta^2$.*

A 3-dimensional Einstein manifold has constant sectional curvature. Thus, we get

Corollary 8.1. *Let $M^3(f, Q, \xi, \eta, g)$ be a weak β-Kenmotsu manifold with $\beta = const \ne 0$. If $\nabla_\xi\,\mathrm{Ric}^\sharp = 0$, then M has sectional curvature $-\beta^2$.*

8.2. η-Ricci solitons on weak β-Kenmotsu manifolds

Here, we study η-Ricci solitons on weak β-Kenmotsu manifolds. We consider an η-Einstein weak β-Kenmotsu metric as an η-Ricci soliton and characterize the Einstein metrics in such a wider class of metrics.

Lemma 8.2. *Let $M^{2n+1}(f,Q,\xi,\eta,g)$ be a weak β-Kenmotsu manifold. If g represents an η-Ricci soliton with the potential vector field V, then for all $X\in\mathfrak{X}_M$:*

$$(\pounds_V R)_{X,\,\xi}\,\xi = -2\,\xi(\beta)\,\mathrm{Ric}^\sharp X + \big(4\,n\,\xi(\beta^3) + 2\,\xi(\xi(\beta^2))\big)X \\ -\big(8\,n\,\xi(\beta^3) + 2(n+1)\,\xi(\xi(\beta^2))\big)\,\eta(X)\,\xi;$$

moreover, if $\beta = const \neq 0$, then $(\pounds_V R)_{X,\xi}\xi = 0$.

Lemma 8.3. *On an η-Einstein weak β-Kenmotsu manifold $M^{2n+1}(f,Q,\xi,\eta,g)$, the expression of $\mathrm{Ric}^\sharp$ is the following:*

$$\mathrm{Ric}^\sharp X = \big(\xi(\beta)+\beta^2+\frac{r}{2\,n}\big)\,X - \big((2n+1)(\xi(\beta)+\beta^2)+\frac{r}{2\,n}\big)\,\eta(X)\,\xi \quad (X\in\mathfrak{X}_M).$$

Theorem 8.3. *Let $M^{2n+1}(f,Q,\xi,\eta,g)$, $n>1$, be an η-Einstein weak β-Kenmotsu manifold with $\beta = const \neq 0$. If g represents an η-Ricci soliton with the potential vector field V, then (M,g) is an Einstein manifold with $r=-2\,n(2\,n+1)\,\beta^2$.*

Definition 8.1. A vector field X on a weak contact metric manifold is called a *weak contact vector field*, if there exists a smooth function $\rho: M\to\mathbb{R}$ such that $\pounds_X\,\eta = \rho\,\eta$, and if $\rho=0$, then X is said to be *strict weak contact vector field*.

We consider a weak β-Kenmotsu metric as an η-Ricci soliton, whose potential vector field V is weak contact, or V is collinear to ξ. First, we derive the following.

Lemma 8.4. *Let $M^{2n+1}(f,Q,\xi,\eta,g)$ be a weak β-Kenmotsu manifold. If g represents an η-Ricci soliton with a potential vector field V, then $\lambda+\mu = 2n(\xi(\beta)+\beta^2)$.*

Theorem 8.4. *Let $M^{2n+1}(f,Q,\xi,\eta,g)$, $n>1$, be a weak β-Kenmotsu manifold with $\beta = const \neq 0$ and $\xi(r)=0$. If g represents an η-Ricci soliton with a weak contact potential vector field V, then V is strict and (M,g) is an Einstein manifold with scalar curvature $r=-2\,n(2n+1)\beta^2$.*

Theorem 8.5. *Let $M^{2n+1}(f,Q,\xi,\eta,g)$ be a weak β-Kenmotsu manifold with $\beta = const \neq 0$ and $\xi(r)=0$. Suppose that g represents an η-Ricci soliton with a non-zero vector field V collinear to ξ: $V=\delta\,\xi$ for a smooth function $\delta\neq 0$ on M. Then $\delta = const$ and (M,g) is an Einstein manifold with scalar curvature $-2n(2\,n+\delta(\beta-1))$.*

In [15], K. Kenmotsu found the following necessary and sufficient condition for a Kenmotsu manifold M to have constant f-holomorphic sectional curvature H:

$$\begin{aligned} 4R_{X,Y}Z &= (H-3)\{g(Y,Z)X - g(X,Z)Y\} \\ &+(H+1)\{\eta(X)\,\eta(Z)Y - \eta(Y)\,\eta(Z)X + \eta(Y)g(X,Z)\xi - \eta(X)g(Y,Z)\xi \\ &+ g(X,fZ)fY - g(Y,fZ)fX + 2g(X,fY)fZ\}, \quad X,Y,Z\in\mathfrak{X}_M. \end{aligned}$$

The following corollary of Theorem 8.3 illustrates Lemma 8.4 and gives an example of a weak β-Kenmotsu manifold that admits an η-Ricci soliton.

Corollary 8.2. *Let a metric g of a warped product $\mathbb{R}\times_\sigma N^{2n}$, where $n>1$, $(\partial_t\sigma)/\sigma=\beta=const\neq 0$ and N is a weak Kählerian manifold, represent an η-Ricci soliton with the potential vector field V. Then g has constant f-holomorphic sectional curvature $H=-\beta^2$.*

We complete Theorem 8.3 by studying the 3-dimensional case.

Lemma 8.5. *A weak β-Kenmutsu manifold $M^3(f,Q,\xi,\eta,g)$ with $\beta=const\neq 0$ represents an η-Einstein manifold.*

Using Lemma 8.5 and Theorem 8.3, we obtain the following.

Corollary 8.3. *If a weak β-Kenmutsu manifold $M^3(f,Q,\xi,\eta,g)$ with $\beta=const\neq 0$ and $X(r)=0$ $(X\perp\xi)$ represents an η-Ricci soliton* (1.6)*, then (M,g) has constant sectional curvature $-\beta^2$.*

Lemma 8.6. *If a weak β-Kenmutsu manifold $M(f,Q,\xi,\eta,g)$ with $\beta=const\neq 0$ represents an η-Ricci soliton* (1.6) *with the potential vector field V, then*

$$(i)\ 2\,(\pounds_V\nabla)(X,\xi) = -\xi(r)\,(X-\eta(X)\xi), \quad \text{where} \quad \xi(r) = -2\,\beta\,(r+6\,\beta^2),$$

$$(ii)\ (r+6\,\beta^2)\,\big\{g(X,\pounds_V\xi)-\eta(X)\,\eta(\pounds_V\xi)\big\} + X(\xi(r)) + \xi(\xi(r))\,\eta(X) - 4\big\{\beta\,\xi(r) - 2\,\beta^2\,(\lambda-2\,\beta^2)\big\}\,\eta(X) = 0, \quad X\in\mathfrak{X}_M.$$

Theorem 8.6. *If a weak β-Kenmutsu manifold $M^3(f,Q,\xi,\eta,g)$ with $\beta=const\neq 0$ represents an η-Ricci soliton* (1.6) *with the potential vector field V, then (M,g) has constant sectional curvature $-\beta^2$.*

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