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BASINS OF ATTRACTIONS OF NEW ITERATIVE METHODS FOR FINDING SIMPLE ZEROS

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Abstract. In this paper we present two new variants of Homeier's iterative method for finding simple, real or complex, solution of nonlinear equations. Increasing the order of convergence from three to four is achieved by one additional term. Through many numerical examples, by classical and criteria based on basins of attraction, it is shown that the new methods can be competitive to other fourth-order methods. **Keywords:** Homeier's iterative method, nonlinear equations, order of convergence, basins of attraction.

1. Introduction

Finding solutions of non-linear equations is one of the most important problems in numerical analysis. There is a vast literature about this topic , see [3], [5], [8]. Despite this, the development of new methods is still an actual problem with the aim of efficiently solving as wide a class of nonlinear equations as possible. In this paper we propose two new iterative methods for finding a simple real or complex solution of a non-linear equation $f(x) = 0$.

In Figure (1.1) we showed geometrical interpretations of three different iterative methods for finding the exact simple real zero of the function f.

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The first approximation y_n is obtained by the well-known Newton's method (order of convergence two) and represents the intersection of the tangent at the point $(x_n, f(x_n))$ with the x-axis

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}.$$

The second approximation x_{n+1} is obtained by Ostrowski's method (order of convergence four) as the intersection of the straight line y_{CD} through the points $C(\frac{x_n+y_n}{2}, \frac{1}{2}f(x_n))$ and $D(y_n, f(y_n))$ with the x-axis

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\Big(1 + \frac{f(x_n - \frac{f(x_n)}{f'(x_n)})}{f(x_n) - 2f(x_n - \frac{f(x_n)}{f'(x_n)})}\Big). \tag{1.1}$$

The third approximation is obtained by Homeier's third order method [3] and represents the intersection of the x-axis and the line $y_C = f'(y_n)(x - \frac{x_n+y_n}{2}) + \frac{1}{2}f(x_n)$ that passes through point C which is parallel to the tangent at the point D

$$x_{n+1} = x_n - \frac{f(x_n)}{2f'(x_n)}\Big(1 + \frac{f'(x_n)}{f'(x_n - \frac{f(x_n)}{f'(x_n)})}\Big) = x_n - \text{IM}. \tag{1.2}$$

To improve the local order of convergence of Homeier's method, we combine (1.2)

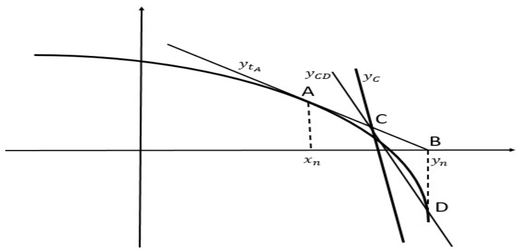

Fig. 1.1: Three different approximations of the exact zero

and two other methods. In Section 2. we showed that the order of convergence of the new methods is four. Their good features in difficult cases, as well as global convergence analysis using basin of attraction are given in Section 3..

2. The new methods and analysis of convergence

In the sequel we use some abbreviations: $e_{n+1} = x_{n+1} - \alpha, \quad e_n = x_n - \alpha,$

$$u(x) = \frac{f(x)}{f'(x)}, \qquad L(x) = \frac{f(x)f''(x)}{(f'(x))^2} \qquad C_k = \frac{f^{(k)}(\alpha)}{k!f'(\alpha)} \ \ (k = 2, 3, \ldots)$$

and definition of the order of convergence r, the asymptotic error constant AEC and error relation:

$$AEC = \lim_{n\to\infty} \frac{x_{n+1}-\alpha}{(x_n-\alpha)^r} \quad \Rightarrow \quad e_{n+1} = AEC\, e_n^r + O(e_n^{r+1})$$

We improve Homeier's iterative method (1.2) as a combination with an additional term IM(k), $k = 1, 2$, in order to obtain methods with a higher order of convergence

$$x_{n+1} = x_n - \theta \mathrm{IM} - (1-\theta)\mathrm{IM}(k), \qquad k = 1, 2. \tag{2.1}$$

In the last relation IM(1) represents the correction of Super-Halley's method [2]

$$x_{n+1} = x_n - u(x_n)\Big(1 + \frac{L(x_n)}{2(1-L(x_n))}\Big) = x_n - \mathrm{IM}(1) \tag{2.2}$$

and IM(2) is the correction of Chebyshev's method [8]

$$x_{n+1} = x_n - u(x_n)\Big(1 + \frac{1}{2}\, L(x_n) + \frac{1}{2}\,(L(x_n))^2\Big) = x_n - \mathrm{IM}(2). \tag{2.3}$$

Theorem 2.1. *Let $f(x)$ be sufficiently smooth in the neighborhood of the simple real or complex zero α. If* IM, IM(1) *and* IM(2) *are defined by* (1.2), (2.2) *and* (2.3) *then for $\theta = \frac{2}{3}$ the order of convergence of the methods* (2.1) *is four.*

Proof. In our estimations we use Taylor expansions of $f(x_n)$, $f'(x_n)$, $f''(x_n)$ and $f'(x_n - u(x_n))$ about α. According to (1.2), (2.2) and (2.3), using computer algebra system Mathematica, it is not difficult to find error relations of (2.1)

$$e_{n+1} = (\frac{3}{2}\theta - 1)C_3 e_n^3 + (\theta D + (1-\theta)D_k)e_n^4 + O(e_n^5), \quad k = 1, 2,$$

where we use notations $D = \frac{3}{2}C_2C_3$, $D_1 = C_2 - 3C_3$ and $D_2 = C_2^3 - 3C_4$. From previous error relation we conclude that if $\theta = \frac{2}{3}$ the order of convergency of iterative method (2.1) is four. □

In this way we found two new iterative methods of the fourth order:

Homeier-Super Halley method (HSH)

$$x_{n+1} = x_n - \frac{u(x_n)}{3}\Big(2 + \frac{f'(x_n)}{f'(x_n - u(x_n))} + \frac{L(x_n)}{2(1-L(x_n))}\Big) \tag{2.4}$$

and Homeier-Chebyshev method (HCh)

$$x_{n+1} = x_n - \frac{u(x_n)}{3}\Big(2 + \frac{f'(x_n)}{f'(x_n - u(x_n))} + \frac{1}{2}\, L(x_n) + \frac{1}{2}(L(x_n))^2\Big). \tag{2.5}$$

□

Table 3.1: The error norms e_n $(n = 1, 2, 3)$ where $A(-q)$ means $A \times 10^{-q}$

$f(x)$	e_n	(HSH)	(HCh)	(Ostr)	(Kiss)
$f_1(x)$	e_1	**1.00(-5)**	1.05(-5)	1.37(-5)	9.30(-5)
	e_2	**2.39(-24)**	1.64(-23)	1.58(-22)	1.80(-18)
	e_3	**7.76(-99)**	9.50(-95)	2.78(-90)	2.55(-73)
$f_2(x)$	e_1	**2.24(-4)**	8.35(-3)	4.44(-4)	2.71(-4)
	e_2	**9.28(-16)**	2.01(-8)	1.48(-14)	2.00(-15)
	e_3	**2.75(-61)**	6.71(-31)	1.84(-56)	5.99(-60)
$f_3(x)$	e_1	1.38(-6)	**3.56(-7)**	5.14(-7)	7.76(-7)
	e_2	5.11(-28)	**6.05(-31)**	3.69(-30)	2.78(-29)
	e_3	9.46(-114)	**5.02(-126)**	9.78(-123)	4.59(-119)

3. Numerical examples

For comparison purpose, besides the new methods (HSH) (2.4) and (HCh) (2.5) we tested Ostrowski's method (1.1) and Kiss' method [4]

$$x_{n+1} = x_n - u(x_n)\frac{1-\frac{1}{2}L(x_n)}{1-L(x_n)+\frac{f'''(x_n)}{6f'(x_n)}(u(x_n))^2}. \tag{3.1}$$

The choice of Ostovski's method is due to its similar geometric interpretation with Homeier's method (see Figure (1.1)), while Kiss' method was chosen due to the same number of functional calculations with the proposed methods.

Choosing nontrivial test functions with real and complex zeros, as Chun and Neta [1] (Example 1.) and test polynomials in [6] (Examle 2.), we have tested numerical examples using multi-precision arithmetic employing computer algebra system *Mathematica*.

Example 1. The goal of the first example is to demonstrate the convergence speed of the proposed methods (HSH) and (HCh). From Table (3.1), for a given test functions

$$\begin{array}{lll} f_1(x) = x^2 - e^x - 3x + 2 & x_0 = 0.5 & \alpha = 0.257530285439860760 \\ f_2(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5 & x_0 = -1 & \alpha = -1.207647827130918927 \\ f_3(x) = \ln x + \sqrt{x} - 5 & x_0 = 8 & \alpha = 8.309432694231571795 \end{array}$$

and many other tested functions, we conclude that the results obtained by the proposed methods coincide with the theoretical results given in Theorem 2.1. For $f_1(x) - f_3(x)$ the fastest convergence is achieved by one of the proposed methods.

Characteristics of our methods are discussed by basins of attractions, too [7].

Definition 3.1. *Let $S \subseteq \mathbb{C}$ be a complex domain and let f be a given sufficiently many times differentiable function in S having simple zeros $\alpha_1, \alpha_2, \ldots, \alpha_m \in S$. For* *a given root-finding iteration defined by* $x_{k+1} = g(x_k)$, *the basin of attraction for the zero* α_i *is the set (or the union of sets)*

$$\mathcal{B}_{f,g}(\alpha_i) = \{\xi \in S \mid \textit{the iteration } x_{k+1} = g(x_k) \textit{ with } x_0 = \xi \textit{ converges to } \alpha_i\}.$$

We drew basins respecting some rules: each basin have a different color and the shading is lighter when the number of iterations is smaller. If required accuracy is not achieved in less than 40 iterations, we paint the considered initial point black.

It is useful that the area of each basin of attraction is as large as possible and unvaried, with a straight lines as the boundaries. The number of blobs, fractals, divergent points and the required CPU time to achieve accuracy should be as small as possible.

As initial points for all tested methods we take equally spaced points within a rectangle $S = \{(x, y) \mid a \leqslant x \leqslant b,\, c \leqslant y \leqslant d\}$. The number of tested points is denoted with N. The stopping criterion was given by the inequality $|x_n - \alpha| < 10^{-6}$.

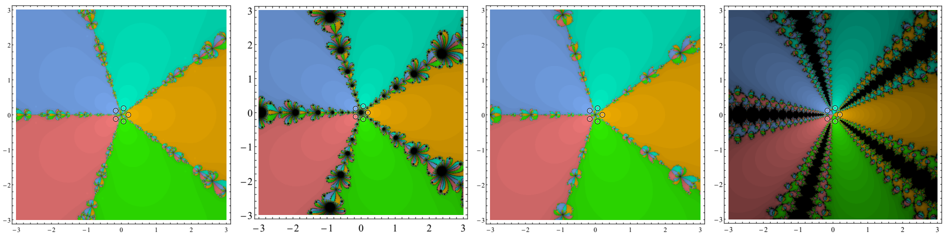

FIG. 3.1: $f_4(x)$, (HSH), (HCh), Ostrowski's, Kiss' methods, in that order

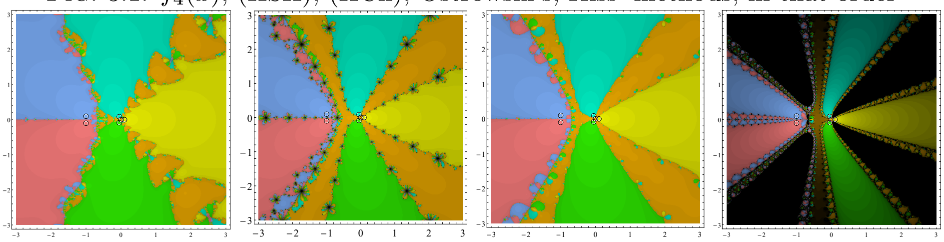

FIG. 3.2: $f_5(x)$, (HSH), (HCh), Ostrowski's, Kiss' methods, in that order

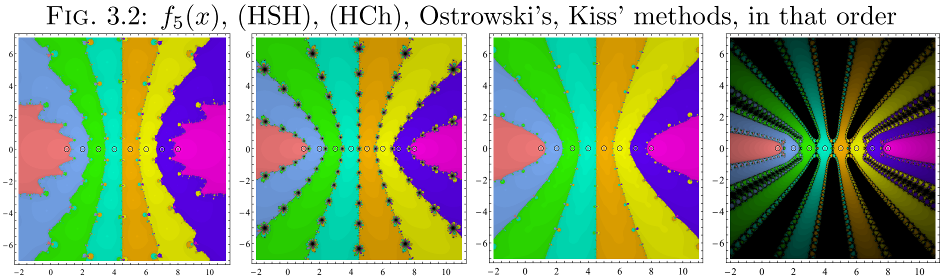

FIG. 3.3: $f_6(x)$, (HSH), (HCh), Ostrowski's, Kiss' methods, in that order

Example 2. In our experiments we chose rather challenging task and worked with test polynomials $f_4(x)$ and $f_5(x)$ containing clusters of zeros and Wilkinson's polinomial $f_6(x)$ that many methods have trouble with:

$$\begin{array}{ll} f_4(x) = x^5 - 0.00032, & 0.2e^{i2k\pi/5}, \quad k=\overline{0,4} \\ f_5(x) = (x^4 - 0.001x)(x^2+2x+1.01), & 0, \ -1 \pm 0.1i, \ 0.1e^{i2k\pi/3}, \quad k=\overline{0,2} \\ f_6(x) = \prod_{k=1}^{8}(x-k), & 1, \ 2, \ 3, \ 4, \ 5, \ 6, \ 7, \ 8 \end{array}$$

Due to many black points in case of Kiss' method (3.1), we have eliminated it

Table 3.2: (A) normalized CPU time for all points; (B) average number of iterations; (C) percentage of divergent points.

$f(x)$		(HSH)	(HCh)	Ostrowski	Kiss
	A	1.04	1.77	**1**	3.01
$f_4(x)$	B	**6.82**	10.12	7.51	22.45
$N = 360000$	C	**0**	2.297	0.003	15.594
	A	1.22	1.95	**1**	5.88
$f_5(x)$	B	**7.08**	9.56	7.76	22.45
$N = 360000$	C	**0**	0.163	**0**	44.458
	A	2.26	3.81	**1**	18.94
$f_6(x)$	B	**5.23**	7.40	5.98	22
$N = 455000$	C	**0**	0.321	**0**	38.054

from further discussion. There is a great similarity between the characteristics of the basin of attraction of the (HSH) method and the (1.1) method: there are no black points, in most of cases the CPU time is almost the same, as well as average number of iterations necessary to satisfy the termination criterion. Basins of (HCh) method possess negligible number of black points, but CPU time and average number of iterations are larger than for previous methods. From Figure (3.1)-(3.3), according to the size, hue and shapes of basins of attractions in all three cases, for the new methods we can conclude that most basins consist of a large unvaried area, which points to a good global convergence and simple structure of the new methods.

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