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CANONICAL BIHOLOMORPHICALLY PROJECTIVE MAPPINGS OF GENERALIZED RIEMANNIAN SPACE IN THE EISENHART SENSE

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Abstract. In this paper, canonical biholomorphically projective and equitorsion canonical biholomorphically projective mappings are defined. Some relations between corresponding curvature tensors of the generalized Riemannian spaces GR_N and $G\overline{R}_N$ are obtained. At the end, invariant geometric object of equitorsion canonical biholomorphically projective mapping is found.
Keywords: canonical biholomorphically projective mappings, curvature tensors, generalized Riemannian space.

1. Introduction and preliminaries

Differentiable manifolds GR_N with nonsymmetric metric tensor and GA_N with nonsymmetric affine connection have been studied in many papers, as well as their mappings [1–6, 8, 9, 11–15].

A generalized Riemannian space GR_N in the sense of Eisenhart's definition [3] is a differentiable N-dimensional manifold, equipped with a non-symmetric metric tensor g_{ij}. Connection coefficients are given by [10]

$$\Gamma^i_{jk} = g^{\underline{ip}}\Gamma_{p.jk}, \tag{1.1}$$

where $||g^{\underline{ij}}|| = ||g_{\underline{ij}}||^{-1}$, $g_{\underline{ij}} = \frac{1}{2}(g_{ij} + g_{ji})$, and $\Gamma_{i.jk} = \frac{1}{2}(g_{ji,k} - g_{jk,i} + g_{ik,j})$, where, for example, $g_{ij,k} = \frac{\partial g_{ij}}{\partial x^k}$. We suppose that $\det ||g_{ij}|| \neq 0$, $\det ||g_{\underline{ij}}|| \neq 0$. Generally,

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we have $\Gamma^i_{jk} \neq \Gamma^i_{kj}$, and the symmetric and antisymmetric part of Γ^i_{jk} are given by the formulas

$$\Gamma^i_{\underline{jk}} = \frac{1}{2}(\Gamma^i_{jk} + \Gamma^i_{kj}) = S^i_{jk}, \quad \Gamma^i_{\underset{\vee}{jk}} = \frac{1}{2}(\Gamma^i_{jk} - \Gamma^i_{kj}) = T^i_{jk}. \tag{1.2}$$

The magnitude T^i_{jk} is *the torsion tensor* of the space GR_N. Obviously,

$$\Gamma^i_{jk} = S^i_{jk} + T^i_{jk}. \tag{1.3}$$

In a generalized Riemannian space one can define four kinds of covariant derivatives [8]. For example, for a tensor a^i_j in GR_N we have

$$\begin{aligned} a^i_{j\underset{1}{|}m} &= a^i_{j,m} + \Gamma^i_{pm} a^p_j - \Gamma^p_{jm} a^i_p, \quad a^i_{j\underset{2}{|}m} = a^i_{j,m} + \Gamma^i_{mp} a^p_j - \Gamma^p_{mj} a^i_p, \\ a^i_{j\underset{3}{|}m} &= a^i_{j,m} + \Gamma^i_{pm} a^p_j - \Gamma^p_{mj} a^i_p, \quad a^i_{j\underset{4}{|}m} = a^i_{j,m} + \Gamma^i_{mp} a^p_j - \Gamma^p_{jm} a^i_p, \end{aligned} \tag{1.4}$$

where $\underset{\theta}{|}$ $(\theta = 1, 2, 3, 4)$ denotes a covariant derivative of the kind θ and $a^i_{j,m} = \dfrac{\partial a^i_j}{\partial x^m}$.

In the case of the space GR_N we have five independent curvature tensors [8]

$$\begin{aligned} \underset{1}{R}{}^i_{jmn} &= \Gamma^i_{jm,n} - \Gamma^i_{jn,m} + \Gamma^p_{jm}\Gamma^i_{pn} - \Gamma^p_{jn}\Gamma^i_{pm}, \\ \underset{2}{R}{}^i_{jmn} &= \Gamma^i_{mj,n} - \Gamma^i_{nj,m} + \Gamma^p_{mj}\Gamma^i_{np} - \Gamma^p_{nj}\Gamma^i_{mp}, \\ \underset{3}{R}{}^i_{jmn} &= \Gamma^i_{jm,n} - \Gamma^i_{nj,m} + \Gamma^p_{jm}\Gamma^i_{np} - \Gamma^p_{nj}\Gamma^i_{pm} + \Gamma^p_{mn}(\Gamma^i_{pj} - \Gamma^i_{jp}), \\ \underset{4}{R}{}^i_{jmn} &= \Gamma^i_{jm,n} - \Gamma^i_{nj,m} + \Gamma^p_{jm}\Gamma^i_{np} - \Gamma^p_{nj}\Gamma^i_{pm} + \Gamma^p_{nm}(\Gamma^i_{pj} - \Gamma^i_{jp}), \\ \underset{5}{R}{}^i_{jmn} &= \frac{1}{2}(\Gamma^i_{jm,n} + \Gamma^i_{mj,n} - \Gamma^i_{jn,m} - \Gamma^i_{nj,m} \\ &\quad + \Gamma^p_{jm}\Gamma^i_{pn} - \Gamma^p_{jn}\Gamma^i_{mp} + \Gamma^p_{mj}\Gamma^i_{pn} - \Gamma^p_{nj}\Gamma^i_{pm}). \end{aligned} \tag{1.5}$$

Let GR_N and $G\overline{R}_N$ be two generalized Riemannian spaces. We will observe these spaces in the common system of coordinates defined by the mapping $f : GR_N \to G\overline{R}_N$. If Γ^h_{ij} and $\overline{\Gamma}^h_{ij}$ are connection coefficients of the spaces GR_N and $G\overline{R}_N$, respectively, then $P^h_{ij} = \overline{\Gamma}^h_{ij} - \Gamma^h_{ij}$ is the deformation tensor of the connection for a mapping f.

The relations between corresponding curvature tensors of the spaces GR_N and $G\overline{R}_N$ are obtained in [10] as follows:

(1.6)
$$
\begin{aligned}
\underset{1}{\overline{R}}{}^i_{jmn} &= \underset{1}{R}{}^i_{jmn} + P^i_{jm\underset{1}{|}n} - P^i_{jn\underset{1}{|}m} + P^p_{jm}P^i_{pn} - P^p_{jn}P^i_{pm} + 2T^p_{mn}P^i_{jp},\\
\underset{2}{\overline{R}}{}^i_{jmn} &= \underset{2}{R}{}^i_{jmn} + P^i_{mj\underset{2}{|}n} - P^i_{nj\underset{2}{|}m} + P^p_{mj}P^i_{np} - P^p_{nj}P^i_{mp} + 2T^p_{nm}P^i_{pj},\\
\underset{3}{\overline{R}}{}^i_{jmn} &= \underset{3}{R}{}^i_{jmn} + P^i_{jm\underset{2}{|}n} - P^i_{nj\underset{1}{|}m} + P^p_{jm}P^i_{np} - P^p_{nj}P^i_{pm} + 2P^p_{nm}(T^i_{pj} + P^i_{\underset{\vee}{pj}}),\\
\underset{4}{\overline{R}}{}^i_{jmn} &= \underset{4}{R}{}^i_{jmn} + P^i_{jm\underset{2}{|}n} - P^i_{nj\underset{1}{|}m} + P^p_{jm}P^i_{np} - P^p_{nj}P^i_{pm} + 2P^p_{mn}(T^i_{pj} + P^i_{\underset{\vee}{pj}}),\\
\underset{5}{\overline{R}}{}^i_{jmn} &= \underset{5}{R}{}^i_{jmn} + \frac{1}{2}(P^i_{jm\underset{3}{|}n} - P^i_{jn\underset{4}{|}m} + P^i_{mj\underset{4}{|}n} - P^i_{nj\underset{3}{|}m}\\
&\quad + P^p_{jm}P^i_{pn} - P^p_{jn}P^i_{mp} + P^p_{mj}P^i_{np} - P^p_{nj}P^i_{pm}),
\end{aligned}
$$

where P^h_{ij} is a deformation tensor for a mapping f, $P^h_{\underset{\vee}{ij}}$ is its antisymmetric part, and T^h_{ij} is a torsion tensor.

2. Canonical biholomorphically projective mappings

In paper [7], we define biholomorphically projective mappings between two generalized Riemannian spaces with almost complex structures that are equal in a common system of coordinates. In that case,

$$\overline{\Gamma}^h_{ij} = \Gamma^h_{ij} + \psi_{(i}\delta^h_{j)} + \sigma_{(i}F^h_{j)} + \tau_{(i}\overset{2}{F}{}^h_{j)} + \xi^h_{ij}, \tag{2.1}$$

and the deformation tensor has the form

$$P^h_{ij} = \psi_{(i}\delta^h_{j)} + \sigma_{(i}F^h_{j)} + \tau_{(i}\overset{2}{F}{}^h_{j)} + \xi^h_{ij}, \tag{2.2}$$

where (ij) is a symmetrization without division by indices i and j, ψ_i, σ_i and τ_i are vectors, $\overset{2}{F}{}^h_p = F^h_q F^q_p$, and ξ^h_{ij} is an antisymmetric tensor.

Motivated by the form of deformation tensor (2.2), we will define new types of mappings. Let GR_N and $G\overline{R}_N$ be two generalized Riemannian spaces with almost complex structures F^h_i and $\overline{F}^h_i$, respectively, where $F^h_i = \overline{F}^h_i$ in the common system of coordinates defined by the mapping $f : GR_N \to G\overline{R}_N$, and assume that it holds $F^h_i \neq a\delta^h_i$, where a is scalar invariant.

The mapping $f : GR_N \to G\overline{R}_N$ is *canonical biholomorphically projective mapping* if in the common coordinate system connection coefficients Γ^h_{ij} and $\overline{\Gamma}^h_{ij}$ satisfy the relation

$$\overline{\Gamma}^h_{ij} = \Gamma^h_{ij} + \sigma_{(i}F^h_{j)} + \tau_{(i}\overset{2}{F}{}^h_{j)} + \xi^h_{ij}, \tag{2.3}$$

where (ij) is a symmetrization without division by indices i and j, σ_i and τ_i are vectors, $\overset{2}{F}{}^h_p = F^h_q F^q_p$, and ξ^h_{ij} is an antisymmetric tensor.

Let P^h_{ij} be deformation tensor with respect to the canonical biholomorphically projective mapping $f : GR_N \to G\overline{R}_N$. Then, we have

$$P^h_{ij} = \sigma_{(i}F^h_{j)} + \tau_{(i}\overset{2}{F}{}^h_{j)} + \xi^h_{ij}. \tag{2.4}$$

Below we will find the relations between corresponding curvature tensors of the spaces GR_N and $G\overline{R}_N$.

According to relations (1.5), (1.6) and (2.4), for the curvature tensor of the first kind we have

$$\begin{aligned}
\underset{1}{\overline{R}}{}^i_{jmn} = {} & \underset{1}{R}{}^i_{jmn} + \sigma_j(\sigma_{<n}\overset{2}{F}{}^i_{m>} + \tau_{<n}\overset{3}{F}{}^i_{m>}) + \sigma_j F^i_{<m\underset{1}{|}n>} + \sigma_{j\underset{1}{|}<n}F^i_{m>} \\
& + \sigma_{<m\underset{1}{|}n>}F^i_j + \sigma_m F^i_{j\underset{1}{|}n} - \sigma_n F^i_{j\underset{1}{|}m} + \tau_j(\sigma_{<n}\overset{3}{F}{}^i_{m>} + \tau_{<n}\overset{4}{F}{}^i_{m>}) \\
& + \tau_{j\underset{1}{|}<n}F^i_{m>} + \tau_m\overset{2}{F}{}^i_{j\underset{1}{|}n} - \tau_n\overset{2}{F}{}^i_{j\underset{1}{|}m} + \tau_j\overset{2}{F}{}^i_{<m\underset{1}{|}n>} + \tau_{<m\underset{1}{|}n>}\overset{2}{F}{}^i_j \\
& + \sigma_p\underset{1}{\mathcal{F}}{}^{pi}_{jmn} + \tau_p\underset{2}{\mathcal{F}}{}^{pi}_{jmn} + \mathcal{S}^i_{pn}\xi^p_{jm} + \mathcal{S}^p_{jm}\xi^i_{pn} - \mathcal{S}^p_{jn}\xi^i_{pm} - \mathcal{S}^i_{pm}\xi^p_{jn} \\
& + \xi^i_{j<m\underset{1}{|}n>} + \xi^p_{jm}\xi^i_{pn} - \xi^p_{jn}\xi^i_{pm} + 2T^p_{mn}(\mathcal{S}^i_{jp} + \xi^i_{jp}),
\end{aligned} \tag{2.5}$$

where (ij) is a symmetrization without division, $< ij >$ is an antisymmetrization without division by indices i, j, and

$$\begin{aligned}
& \overset{2}{F}{}^h_j = F^h_p F^p_j,\ \overset{3}{F}{}^h_j = F^h_p F^p_q F^q_j,\ \overset{4}{F}{}^h_j = F^h_p F^p_q F^q_r F^r_j,\ \mathcal{S}^i_{jp} = \sigma_{(j}F^i_{p)} + \tau_{(j}\overset{2}{F}{}^i_{p)} \\
& \underset{1}{\mathcal{F}}{}^{pi}_{jmn} = \sigma_j F^p_{<m}F^i_{n>} + \sigma_{<m}F^i_{n>}F^p_j + \tau_j\overset{2}{F}{}^p_{<m}F^i_{n>} + \tau_{<m}F^i_{n>}\overset{2}{F}{}^p_j, \\
& \underset{2}{\mathcal{F}}{}^{pi}_{jmn} = \sigma_j F^p_{<m}\overset{2}{F}{}^i_{n>} + \sigma_{<m}\overset{2}{F}{}^i_{n>}F^p_j + \tau_j\overset{2}{F}{}^p_{<m}\overset{2}{F}{}^i_{n>} + \tau_{<m}\overset{2}{F}{}^i_{n>}\overset{2}{F}{}^p_j.
\end{aligned} \tag{2.6}$$

Based on the facts given above, we have obtained the following statement.

Theorem 2.1. *A canonical biholomorphically projective relation between the curvature tensors of the first kind of the generalized Riemannian spaces GR_N and $G\overline{R}_N$ is given by the formula* (2.5)*, where T^h_{ij} is the torsion tensor and we denoted with respect to the* (2.6).

From relations (1.5), (1.6) and (2.4), for the curvature tensor of the second kind, we get:

$$
\begin{aligned}
\underset{2}{\overline{R}}{}^i_{jmn} &= \underset{2}{R}{}^i_{jmn} + \sigma_j(\sigma_{<n}\overset{2}{F}{}^i_{m>} + \tau_{<n}\overset{3}{F}{}^i_{m>}) + \sigma_j F^i_{<m\underset{2}{|}n>} + \sigma_{j\underset{2}{|}<n}F^i_{m>} \\
&+ \sigma_{<m\underset{2}{|}n>}F^i_j + \sigma_m F^i_{j\underset{2}{|}n} - \sigma_n F^i_{j\underset{2}{|}m} + \tau_j(\sigma_{<n}\overset{3}{F}{}^i_m + \tau_{<n}\overset{4}{F}{}^i_{m>}) \\
&+ \tau_m \overset{2}{F}{}^i_{j\underset{2}{|}n} - \tau_n\overset{2}{F}{}^i_{j\underset{2}{|}m} + \tau_{j\underset{2}{|}<n}\overset{2}{F}{}^i_{m>} + \tau_j\overset{2}{F}{}^i_{<m\underset{2}{|}n>} + \tau_{<m\underset{2}{|}n>}\overset{2}{F}{}^i_j \\
&+ \sigma_p \underset{1}{\mathcal{F}}{}^{pi}_{jmn} + \tau_p\underset{2}{\mathcal{F}}{}^{pi}_{jmn} + \mathcal{S}^i_{pn}\xi^p_{mj} + \mathcal{S}^p_{jm}\xi^i_{np} - \mathcal{S}^p_{jn}\xi^i_{mp} - \mathcal{S}^i_{pm}\xi^p_{nj} \\
&+ \xi^i_{mj\underset{2}{|}n} - \xi^i_{nj\underset{2}{|}m} + \xi^p_{mj}\xi^i_{np} - \xi^p_{nj}\xi^i_{mp} + 2T^p_{nm}(\mathcal{S}^i_{jp} + \xi^i_{pj}),
\end{aligned}
\tag{2.7}
$$

where $\overset{2}{F}{}^h_j$, $\overset{3}{F}{}^h_j$, $\overset{4}{F}{}^h_j$, $\underset{1}{\mathcal{F}}{}^{pi}_{jmn}$, $\underset{2}{\mathcal{F}}{}^{pi}_{jmn}$, $\mathcal{S}^i_{jp}$ are determined by the formula (2.6). Therefore, the following theorem is valid.

Theorem 2.2. *A canoninal biholomorphically projective relation between the curvature tensors of the second kind of the generalized Riemannian spaces GR_N and $G\overline{R}_N$ is given by the formula* (2.7), *where T^h_{ij} is the torsion tensor and we denoted with respect to the* (2.6).

Considering relations (1.5), (1.6) and (2.4), for the curvature tensor of the third kind we have the following:

$$
\begin{aligned}
\underset{3}{\overline{R}}{}^i_{jmn} &= \underset{3}{R}{}^i_{jmn} + \sigma_j(\sigma_{<n}\overset{2}{F}{}^i_{m>} + \tau_{<n}\overset{3}{F}{}^i_{m>}) + \sigma_j(F^i_{m\underset{2}{|}n} - F^i_{n\underset{1}{|}m}) \\
&+ \sigma_{j\underset{2}{|}n}F^i_m - \sigma_{j\underset{1}{|}m}F^i_n + (\sigma_{m\underset{2}{|}n} - \sigma_{n\underset{1}{|}m})F^i_j + \sigma_m F^i_{j\underset{2}{|}n} - \sigma_n F^i_{j\underset{1}{|}m} \\
&+ \tau_j(\sigma_{<n}\overset{3}{F}{}^i_{m>} + \tau_{<n}\overset{4}{F}{}^i_{m>}) + \tau_m\overset{2}{F}{}^i_{j\underset{2}{|}n} - \tau_n\overset{2}{F}{}^i_{j\underset{1}{|}m} + \tau_{j\underset{2}{|}n}\overset{2}{F}{}^i_m - \tau_{j\underset{1}{|}m}\overset{2}{F}{}^i_n \\
&+ \tau_j(\overset{2}{F}{}^i_{m\underset{2}{|}n} - \overset{2}{F}{}^i_{n\underset{1}{|}m}) + (\tau_{m\underset{2}{|}n} - \tau_{n\underset{1}{|}m})\overset{2}{F}{}^i_j + \xi^i_{jm\underset{2}{|}n} - \xi^i_{nj\underset{1}{|}m} \\
&+ \sigma_p\underset{1}{\mathcal{F}}{}^{pi}_{jmn} + \tau_p\underset{2}{\mathcal{F}}{}^{pi}_{jmn} + \mathcal{S}^i_{pn}\xi^p_{mj} + \mathcal{S}^p_{jm}\xi^i_{np} - \mathcal{S}^p_{jn}\xi^i_{pm} - \mathcal{S}^i_{pm}\xi^p_{nj} \\
&+ \xi^p_{jm}\xi^i_{np} - \xi^p_{nj}\xi^i_{mp} + 2(\mathcal{S}^p_{nm} + \xi^p_{nm})(T^i_{pj} + \xi^i_{pj}),
\end{aligned}
\tag{2.8}
$$

where we denoted with respect to (2.6). In this way, the following theorem is proven.

Theorem 2.3. *A canonical biholomorphically projective relation between the curvature tensors of the third kind of the generalized Riemannian spaces GR_N and $G\overline{R}_N$ is given by the formula* (2.8), *where T^h_{ij} is the torsion tensor and we denoted with respect to the* (2.6).

Using relations (1.5), (1.6) and (2.4), for the curvature tensor of the fourth kind we have the following:

$$
\begin{aligned}
\underset{4}{\overline{R}}{}^i_{jmn} &= \underset{4}{R}{}^i_{jmn} + \sigma_j(\sigma_{<n}\overset{2}{F}{}^i_{m>} + \tau_{<n}\overset{3}{F}{}^i_{m>}) + \sigma_j(F^i_{m\underset{2}{|}n} - F^i_{n\underset{1}{|}m}) \\
&+ \sigma_{j\underset{2}{|}n}F^i_m - \sigma_{j\underset{1}{|}m}F^i_n + (\sigma_{m\underset{2}{|}n} - \sigma_{n\underset{1}{|}m})F^i_j + \sigma_m F^i_{j\underset{2}{|}n} - \sigma_n F^i_{j\underset{1}{|}m} \\
&+ \tau_j(\sigma_{<n}\overset{3}{F}{}^i_{m>} + \tau_{<n}\overset{4}{F}{}^i_{m>}) + \tau_m\overset{2}{F}{}^i_{j\underset{2}{|}n} - \tau_n\overset{2}{F}{}^i_{j\underset{1}{|}m} + \tau_{j\underset{2}{|}n}\overset{2}{F}{}^i_m - \tau_{j\underset{1}{|}m}\overset{2}{F}{}^i_n \\
&+ \tau_j(\overset{2}{F}{}^i_{m\underset{2}{|}n} - \overset{2}{F}{}^i_{n\underset{1}{|}m}) + (\tau_{m\underset{2}{|}n} - \tau_{n\underset{1}{|}m})\overset{2}{F}{}^i_j + \sigma_p\underset{1}{\mathcal{F}}{}^{pi}_{jmn} + \tau_p\underset{2}{\mathcal{F}}{}^{pi}_{jmn} \\
&+ \mathcal{S}^i_{pn}\xi^p_{mj} + \mathcal{S}^p_{jm}\xi^i_{np} - \mathcal{S}^p_{jn}\xi^i_{pm} - \mathcal{S}^i_{pm}\xi^p_{nj} + \xi^i_{jm\underset{2}{|}n} - \xi^i_{nj\underset{1}{|}m} \\
&+ \xi^p_{jm}\xi^i_{np} - \xi^p_{nj}\xi^i_{mp} + 2(\mathcal{S}^p_{mn} + \xi^p_{mn})(T^i_{pj} + \xi^i_{pj}),
\end{aligned}
\tag{2.9}
$$

where we denoted with respect to (2.6). This proves the next statement.

Theorem 2.4. *A canonical biholomorphically projective relation between the curvature tensors of the fourth kind of the generalized Riemannian spaces GR_N and $G\overline{R}_N$ is given by the formula* (2.9)*, where T^h_{ij} is the torsion tensor and we denoted with respect to the* (2.6).

From relations (1.5), (1.6) and (2.4), for the curvature tensor of the fifth kind we get the following:

$$
\begin{aligned}
\underset{5}{\overline{R}}{}^i_{jmn} &= \underset{5}{R}{}^i_{jmn} + \frac{1}{2}\sigma_m(F^i_{j\underset{3}{|}n} + F^i_{j\underset{4}{|}n}) - \frac{1}{2}\sigma_n(F^i_{j\underset{3}{|}m} + F^i_{j\underset{4}{|}m}) \\
&+ \frac{1}{2}(\sigma_{<m\underset{3}{|}n>} + \sigma_{<m\underset{4}{|}n>})F^i_j + \frac{1}{2}(\sigma_{j\underset{3}{|}n} + \sigma_{j\underset{4}{|}n})F^i_m - \frac{1}{2}(\sigma_{j\underset{3}{|}m} + \sigma_{j\underset{4}{|}m})F^i_n \\
&+ \frac{1}{2}\sigma_j(F^i_{<m\underset{3}{|}n>} + F^i_{<m\underset{4}{|}n>}) + \frac{1}{2}(\tau_{<m\underset{3}{|}n>} + \tau_{<m\underset{4}{|}n>})\overset{2}{F}{}^i_j \\
&+ \frac{1}{2}(\tau_{j\underset{3}{|}n} + \tau_{j\underset{4}{|}n})\overset{2}{F}{}^i_m - \frac{1}{2}(\tau_{j\underset{3}{|}m} - \tau_{j\underset{4}{|}m})\overset{2}{F}{}^i_n + \frac{1}{2}\tau_m(\overset{2}{F}{}^i_{j\underset{3}{|}n} + \overset{2}{F}{}^i_{j\underset{4}{|}n}) \\
&+ \frac{1}{2}\tau_j(\overset{2}{F}{}^i_{<m\underset{3}{|}n>} + \overset{2}{F}{}^i_{<m\underset{4}{|}n>}) - \frac{1}{2}\tau_n(\overset{2}{F}{}^i_{j\underset{3}{|}m} + \overset{2}{F}{}^i_{j\underset{4}{|}m}) + \sigma_p\underset{1}{\mathcal{F}}{}^{pi}_{jmn} + \tau_p\underset{2}{\mathcal{F}}{}^{pi}_{jmn} \\
&+ \sigma_j(\sigma_{<n}\overset{2}{F}{}^i_{m>} + \tau_{<n}\overset{3}{F}{}^i_{m>}) + \tau_j(\sigma_{<n}\overset{3}{F}{}^i_{m>} + \tau_{<n}\overset{4}{F}{}^i_{m>}) \\
&+ \frac{1}{2}(\xi^i_{jm\underset{3}{|}n} - \xi^i_{nj\underset{3}{|}m} - \xi^i_{jn\underset{4}{|}m} + \xi^i_{mj\underset{4}{|}n}) + \xi^p_{jm}\xi^i_{pn} - \xi^p_{jn}\xi^i_{mp}.
\end{aligned}
\tag{2.10}
$$

where we denoted with respect to the (2.6).

Based on the facts given above, we have proved the next theorem related to curvature tensors of the fifth kind.

Theorem 2.5. *A canonical biholomorphically projective relation between the curvature tensors of the fifth kind of the generalized Riemannian spaces* GR_N *and* $G\overline{R}_N$ *is given by the formula* (2.10)*, where* T^h_{ij} *is the torsion tensor and we denoted with respect to the* (2.6)*.*

3. Equitorsion canonical biholomorphically projective mapping

The mapping $f : GR_N \to G\overline{R}_N$ is *equitorsion canonical biholomorphically projective mapping,* if the torsion tensors of the spaces GR_N and $G\overline{R}_N$ are equal in a common coordinate system after the mapping f. Then,

$$\xi^h_{ij} = 0. \tag{3.1}$$

In this case, the relation (2.4) becomes

$$P^h_{ij} = \sigma_{(i}F^h_{j)} + \tau_{(i}\overset{2}{F}{}^h_{j)}. \tag{3.2}$$

Considering (3.1), from (2.5) we get:

$$\begin{aligned}\underset{1}{\overline{R}}{}^i_{jmn} &= \underset{1}{R}{}^i_{jmn} + \sigma_j(\sigma_{<n}\overset{2}{F}{}^i_{m>} + \tau_{<n}\overset{3}{F}{}^i_{m>}) + \sigma_j F^i_{<m\underset{1}{|}n>} + \sigma_{j\underset{1}{|}<n}F^i_{m>}\\ &+ \sigma_{<m\underset{1}{|}n>}F^i_j + \sigma_m F^i_{j\underset{1}{|}n} - \sigma_n F^i_{j\underset{1}{|}m} + \tau_j(\sigma_{<n}\overset{3}{F}{}^i_{m>} + \tau_{<n}\overset{4}{F}{}^i_{m>})\\ &+ \tau_{j\underset{1}{|}<n}F^i_{m>} + \tau_m\overset{2}{F}{}^i_{j\underset{1}{|}n} - \tau_n\overset{2}{F}{}^i_{j\underset{1}{|}m} + \tau_j\overset{2}{F}{}^i_{<m\underset{1}{|}n>} + \tau_{<m\underset{1}{|}n>}\overset{2}{F}{}^i_j\\ &+ \sigma_p\underset{1}{\mathcal{F}}{}^{pi}_{jmn} + \tau_p\underset{2}{\mathcal{F}}{}^{pi}_{jmn} + 2T^p_{mn}\mathcal{S}^i_{jp}.\end{aligned} \tag{3.3}$$

Hence, the next theorem holds.

Theorem 3.1. *An equitorsion canonical biholomorphically projective relation between the curvature tensors of the first kind of the generalized Riemannian spaces* GR_N *and* $G\overline{R}_N$ *is given by the formula* (3.3)*, where* T^h_{ij} *is the torsion tensor and we denoted with respect to the* (2.6)*.*

The relation between the curvature tensors of the second kind (2.7), after applying the relation (3.1), becomes:

$$\begin{aligned}\underset{2}{\overline{R}}{}^i_{jmn} &= \underset{2}{R}{}^i_{jmn} + \sigma_j(\sigma_{<n}\overset{2}{F}{}^i_{m>} + \tau_{<n}\overset{3}{F}{}^i_{m>}) + \sigma_j F^i_{<m\underset{2}{|}n>} + \sigma_{j\underset{2}{|}<n}F^i_{m>}\\ &+ \sigma_{<m\underset{2}{|}n>}F^i_j + \sigma_m F^i_{j\underset{2}{|}n} - \sigma_n F^i_{j\underset{2}{|}m} + \tau_j(\sigma_{<n}\overset{3}{F}{}^i_{m} + \tau_{<n}\overset{4}{F}{}^i_{m>})\\ &+ \tau_m\overset{2}{F}{}^i_{j\underset{2}{|}n} - \tau_n\overset{2}{F}{}^i_{j\underset{2}{|}m} + \tau_{j\underset{2}{|}<n}\overset{2}{F}{}^i_{m>} + \tau_j\overset{2}{F}{}^i_{<m\underset{2}{|}n>} + \tau_{<m\underset{2}{|}n>}\overset{2}{F}{}^i_j\\ &+ \sigma_p\underset{1}{\mathcal{F}}{}^{pi}_{jmn} + \tau_p\underset{2}{\mathcal{F}}{}^{pi}_{jmn} + 2T^p_{nm}\mathcal{S}^i_{jp}.\end{aligned} \tag{3.4}$$

In this way, the following theorem is proven.

Theorem 3.2. *An equitorsion canonical biholomorphically projective relation between the curvature tensors of the second kind of the generalized Riemannian spaces* GR_N *and* $G\overline{R}_N$ *is given by the formula* (3.4)*, where* T^h_{ij} *is the torsion tensor and we denoted with respect to the* (2.6)*.*

The relation between the curvature tensors of the third kind (2.7), with respect to (3.1), becomes:

$$
\begin{aligned}
\underset{3}{\overline{R}}{}^i_{jmn} &= \underset{3}{R}{}^i_{jmn} + \sigma_j(\sigma_{<n}\overset{2}{F}{}^i_{m>} + \tau_{<n}\overset{3}{F}{}^i_{m>}) + \sigma_j(F^i_{m\underset{2}{|}n} - F^i_{n\underset{1}{|}m}) \\
&+ \sigma_{j\underset{2}{|}n}F^i_m - \sigma_{j\underset{1}{|}m}F^i_n + (\sigma_{m\underset{2}{|}n} - \sigma_{n\underset{1}{|}m})F^i_j + \sigma_m F^i_{j\underset{2}{|}n} - \sigma_n F^i_{j\underset{1}{|}m} \\
&+ \tau_j(\sigma_{<n}\overset{3}{F}{}^i_{m>} + \tau_{<n}\overset{4}{F}{}^i_{m>}) + \tau_m\overset{2}{F}{}^i_{j\underset{2}{|}n} - \tau_n\overset{2}{F}{}^i_{j\underset{1}{|}m} + \tau_{j\underset{2}{|}n}\overset{2}{F}{}^i_m - \tau_{j\underset{1}{|}m}\overset{2}{F}{}^i_n \\
&+ \tau_j(\overset{2}{F}{}^i_{m\underset{2}{|}n} - \overset{2}{F}{}^i_{n\underset{1}{|}m}) + (\tau_{m\underset{2}{|}n} - \tau_{n\underset{1}{|}m})\overset{2}{F}{}^i_j + \sigma_p\underset{1}{\mathcal{F}}{}^{pi}_{jmn} + \tau_p\underset{2}{\mathcal{F}}{}^{pi}_{jmn} + 2\mathcal{S}^p_{nm}T^i_{pj},
\end{aligned}
\tag{3.5}
$$

and we may formulate the following theorem.

Theorem 3.3. *An equitorsion canonical biholomorphically projective relation between the curvature tensors of the third kind of the generalized Riemannian spaces* GR_N *and* $G\overline{R}_N$ *is given by the formula* (3.5)*, where* T^h_{ij} *is the torsion tensor and we denoted with respect to the* (2.6)*.*

In particular, from the relations (2.9) and (3.1) we have

$$
\begin{aligned}
\underset{4}{\overline{R}}{}^i_{jmn} &= \underset{4}{R}{}^i_{jmn} + \sigma_j(\sigma_{<n}\overset{2}{F}{}^i_{m>} + \tau_{<n}\overset{3}{F}{}^i_{m>}) + \sigma_j(F^i_{m\underset{2}{|}n} - F^i_{n\underset{1}{|}m}) \\
&+ \sigma_{j\underset{2}{|}n}F^i_m - \sigma_{j\underset{1}{|}m}F^i_n + (\sigma_{m\underset{2}{|}n} - \sigma_{n\underset{1}{|}m})F^i_j + \sigma_m F^i_{j\underset{2}{|}n} - \sigma_n F^i_{j\underset{1}{|}m} \\
&+ \tau_j(\sigma_{<n}\overset{3}{F}{}^i_{m>} + \tau_{<n}\overset{4}{F}{}^i_{m>}) + \tau_m\overset{2}{F}{}^i_{j\underset{2}{|}n} - \tau_n\overset{2}{F}{}^i_{j\underset{1}{|}m} + \tau_{j\underset{2}{|}n}\overset{2}{F}{}^i_m - \tau_{j\underset{1}{|}m}\overset{2}{F}{}^i_n \\
&+ \tau_j(\overset{2}{F}{}^i_{m\underset{2}{|}n} - \overset{2}{F}{}^i_{n\underset{1}{|}m}) + (\tau_{m\underset{2}{|}n} - \tau_{n\underset{1}{|}m})\overset{2}{F}{}^i_j + \sigma_p\underset{1}{\mathcal{F}}{}^{pi}_{jmn} + \tau_p\underset{2}{\mathcal{F}}{}^{pi}_{jmn} + 2\mathcal{S}^p_{mn}T^i_{pj}.
\end{aligned}
\tag{3.6}
$$

Therefore, the next theorem holds.

Theorem 3.4. *An equitorsion canonical biholomorphically projective relation between the curvature tensors of the fourth kind of the generalized Riemannian spaces* GR_N *and* $G\overline{R}_N$ *is given by the formula* (3.6)*, where* T^h_{ij} *is the torsion tensor and we denoted with respect to the* (2.6)*.*

Analogously, from (2.10), with respect to the (3.1), we get:

$$
\begin{aligned}
\overline{\underset{5}{R}}{}^i_{jmn} &= \underset{5}{R}{}^i_{jmn} + \frac{1}{2}\sigma_m(F^i_{j\underset{3}{|}n} + F^i_{j\underset{4}{|}n}) - \frac{1}{2}\sigma_n(F^i_{j\underset{3}{|}m} + F^i_{j\underset{4}{|}m}) \\
&+ \frac{1}{2}(\sigma_{<m\underset{3}{|}n>} + \sigma_{<m\underset{4}{|}n>})F^i_j + \frac{1}{2}(\sigma_{j\underset{3}{|}n} + \sigma_{j\underset{4}{|}n})F^i_m - \frac{1}{2}(\sigma_{j\underset{3}{|}m} + \sigma_{j\underset{4}{|}m})F^i_n \\
&+ \frac{1}{2}\sigma_j(F^i_{<m\underset{3}{|}n>} + F^i_{<m\underset{4}{|}n>}) + \frac{1}{2}(\tau_{<m\underset{3}{|}n>} + \tau_{<m\underset{4}{|}n>})\overset{2}{F}{}^i_j \\
&+ \frac{1}{2}(\tau_{j\underset{3}{|}n} + \tau_{j\underset{4}{|}n})\overset{2}{F}{}^i_m - \frac{1}{2}(\tau_{j\underset{3}{|}m} - \tau_{j\underset{4}{|}m})\overset{2}{F}{}^i_n + \frac{1}{2}\tau_m(\overset{2}{F}{}^i_{j\underset{3}{|}n} + \overset{2}{F}{}^i_{j\underset{4}{|}n}) \\
&+ \frac{1}{2}\tau_j(\overset{2}{F}{}^i_{<m\underset{3}{|}n>} + \overset{2}{F}{}^i_{<m\underset{4}{|}n>}) - \frac{1}{2}\tau_n(\overset{2}{F}{}^i_{j\underset{3}{|}m} + \overset{2}{F}{}^i_{j\underset{4}{|}m}) + \sigma_p\underset{1}{\mathcal{F}}{}^{pi}_{jmn} + \tau_p\underset{2}{\mathcal{F}}{}^{pi}_{jmn} \\
&+ \sigma_j(\sigma_{<n}\overset{2}{F}{}^i_{m>} + \tau_{<n}\overset{3}{F}{}^i_{m>}) + \tau_j(\sigma_{<n}\overset{3}{F}{}^i_{m>} + \tau_{<n}\overset{4}{F}{}^i_{m>}).
\end{aligned} \tag{3.7}
$$

i.e. the following theorem is valid:

Theorem 3.5. *An equitorsion canonical biholomorphically projective relation between the curvature tensors of the fifth kind of the generalized Riemannian spaces GR_N and $G\overline{R}_N$ is given by the formula* (3.7)*, where T^h_{ij} is the torsion tensor and we denoted with respect to the* (2.6).

4. Invariant geometric objects

In this section, we will obtain an invariant geometric object of equitorsion canonical biholomorphically projective mapping. In relation to that, in relation (3.2) let us put

$$\sigma_i = -\tau_p F^p_i .$$

Then, we have

$$\overline{\Gamma}^h_{ij} - \Gamma^h_{ij} = -\tau_p F^p_{(i}F^h_{j)} + \tau_{(i}\overset{2}{F}{}^h_{j)}. \tag{4.1}$$

Contracting by indicies h and i in (4.1), assuming that it is valid

$$Tr(F^2) = 0, \text{ i.e. } \overset{2}{F}{}^p_p = F^p_q F^q_p = 0, \text{ and } \quad F^p_j F^k_k = e\delta^h_j \ \ (e = \pm 1), \tag{4.2}$$

we get

$$\tau_j = -\frac{1}{e}(\overline{\Gamma}^p_{\underline{p}j} - \Gamma^p_{\underline{p}j}). \tag{4.3}$$

Substituting (4.3) in (4.1) we have

$$
\begin{aligned}
&\overline{\Gamma}^h_{ij} - \frac{1}{e}\left(\overline{\Gamma}^k_{\underline{k}p}F^p_{(i}F^h_{j)} - \overline{\Gamma}^k_{\underline{k}i}\overset{2}{F}{}^h_j - \overline{\Gamma}^k_{\underline{k}j}\overset{2}{F}{}^h_i\right) \\
&= \Gamma^h_{ij} - \frac{1}{e}\left(\Gamma^k_{\underline{k}p}F^p_{(i}F^h_{j)} - \Gamma^k_{\underline{k}i}\overset{2}{F}{}^h_j - \Gamma^k_{\underline{k}j}\overset{2}{F}{}^h_i\right).
\end{aligned} \tag{4.4}
$$

If we denote

$$\mathcal{CHT}^h_{ij} = \Gamma^h_{ij} - \frac{1}{e}\left(\Gamma^k_{\underline{k}p} F^p_{(i} F^h_{j)} - \Gamma^k_{\underline{k}i} \overset{2}{F}{}^h_j - \Gamma^k_{\underline{k}j} \overset{2}{F}{}^h_i\right), \tag{4.5}$$

the relation (4.4) can be presented in the form

$$\overline{\mathcal{CHT}}{}^h_{ij} = \mathcal{CHT}^h_{ij}, \tag{4.6}$$

where $\overline{\mathcal{CHT}}{}^h_{ij}$ is an object of the space $G\overline{R}_N$. The magnitude $\mathcal{CHT}^h_{ij}$ is not tensor and it is called *Thomas equitorsion canonical biholomorphically projective parameter.*

Accordingly, we conclude that the following assertion is valid.

Theorem 4.1. *The geometic object* $\mathcal{CHT}^h_{ij}$ *given by equation* (4.5) *is an invariant of the equitorsion canonical biholomorphically projective mapping* $f : GR_N \to G\overline{R}_N$*, provided that the relations* (4.2) *are valid.*

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