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GEOMETRIC INEQUALITIES FOR CR-SUBMANIFOLDS

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Dedicated to the memory of Professor Stana Nikčević.

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Abstract. We study two kinds of curvature invariants of Riemannian manifold equipped with a complex distribution D (for example, a CR-submanifold of an almost Hermitian manifold) related to sets of pairwise orthogonal subspaces of the distribution. One kind of invariant is based on the mutual curvature of the subspaces and another is similar to Chen's δ-invariants. We compare the mutual curvature invariants with Chen-type invariants and prove geometric inequalities with intermediate mean curvature squared for CR-submanifolds in almost Hermitian spaces. In the case of a set of complex planes, we introduce and study curvature invariants based on the concept of holomorphic bisectional curvature. As applications, we give consequences of the absence of some D-minimal CR-submanifolds in almost Hermitian manifolds.
Keywords: almost Hermitian manifold, CR-submanifold, distribution, mutual curvature, mean curvature.

1. Introduction

In 1978, A. Bejancu introduced the notion of a CR-submanifold as a generalization of holomorphic and totally real submanifolds of almost Hermitian manifolds. Since then, CR-submanifolds in various ambient spaces have been actively studied, for example, [4, 5, 9]. The development of the extrinsic geometry of submanifolds led

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to the following problem (for example, [3]): *find a simple optimal connection between the intrinsic and extrinsic invariants of a submanifold in a Riemannian manifold; in particular, in space forms.* B.Y. Chen introduced the concept of δ-curvature invariants for Riemannian manifolds in the 1990s and proved an optimal inequality for a submanifold, including δ-curvature invariants and the square of mean curvature. Chen invariants are obtained from the scalar curvature by discarding some of the sectional curvatures. The case of equality led to the notion of "ideal immersions" in Euclidean space, that is, isometric immersions with the smallest possible tension. Chen's theory was extended by geometers to Kähler, (para-)contact, Lagrangian and affine submanifolds, warped products and submersions, see [3, 4].

In [8, 7], we introduced invariants of a Riemannian manifold, which are related to the mutual curvature of noncomplementary pairwise orthogonal subspaces of the tangent bundle, and proved geometrical inequalities for Riemannian submanifolds with applications to foliations.

In this paper, as indicated in the Abstract, two types of mutual invariants of the curvature of a Riemannian manifold equipped with a complex distribution D (in particular, a CR-submanifold of an almost Hermitian manifold) are studied.

The paper is organized as follows. In Section 2, we report some basic information about the curvature invariants of a manifold with a distribution. In Section 3, we study geometric inequalities for CR-submanifolds in almost Hermitian manifolds.

2. Curvature invariants of a manifold with a distribution

In this section, we recall two kinds of curvature invariants of a manifold with a distribution (Chen-type invariants and invariants based on the mutual curvature, see [8, 7]), and for the complex distribution we define invariants based on the holomorphic bisectional curvature. Let (M^{d+l}, g) $(d, m > 0)$ be a Riemannian manifold with a d-dimensional distribution D. Denote by ∇ the Levi-Civita connection of g and $R_{X,Y} = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$ the curvature tensor, where X, Y are any vector fields on the tangent bundle TM. The scalar curvature τ (function on M) is the trace of the Ricci tensor $\mathrm{Ric}_{X,Y} = \mathrm{trace}(Z \mapsto R_{Z,X}\,Y)$. Some authors, for example, [3], define the scalar curvature as half of "trace Ricci".

Example 2.1. Let g be an admissible metric for an almost contact structure (φ, ξ, η) on a manifold M^{2n+1},

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\,\eta(Y), \quad \eta(\xi) = 1, \quad X, Y \in \mathfrak{X}_M,$$

see [2], where φ is a $(1,1)$-tensor, ξ is a unit vector field (called Reeb vector field) and η is a 1-form. Then $d = 2n$ and $D = \ker\eta$ is a $2n$-dimensional contact distribution on M^{2n+1}.

Next, we define curvature invariants related with D, see [7, Section 4, page 7], which for $D = TM$ are reduced to Chen's δ-invariants, for example, [3, Section 13.2].

Definition 2.1. Define Chen's type curvature invariants δ_D and $\hat\delta_D$ by

$$
\begin{aligned}
&2\,\delta_D(n_1,\dots,n_k)(x) = \tau_D(x) - \min\{\tau(V_1)+\ldots+\tau(V_k)\},\\
&2\,\hat\delta_D(n_1,\dots,n_k)(x) = \tau_D(x) - \max\{\tau(V_1)+\ldots+\tau(V_k)\},
\end{aligned}
\tag{2.1}
$$

where $\tau(V_i) = \operatorname{trace}_g \operatorname{Ric}|_{V_i}$, and $V_1,\dots,V_k$ run over all $k \geqslant 0$ mutually orthogonal subspaces of D_x at $x\in M$ such that $\dim V_i = n_i$ $(0\leqslant i\leqslant k)$.

For example, $2\,\delta_D = \tau_D$ if $k=0$, and $2\,\delta_D(n_1)(x) = \tau_D(x) - \min\tau(V_1)$ if $k=1$.

Remark 2.1. In (2.1), we use max and min (instead of sup and inf, see [3]) since the set "all mutually orthogonal subspaces $V_1,\dots,V_k$ at a point $x\in M$ such that ..." is compact.

Let $\{e_i\}$ be an orthonormal frame of a subspace $V=\bigoplus_{i=1}^k V_i$ of T_xM such that $\{e_1,\dots,e_{n_1}\}\subset V_1,\ \dots\ \{e_{n_{k-1}+1},\dots,e_{n_k}\}\subset V_k$. For $k\geqslant 2$, the *mutual curvature of a set* $\{V_1,\dots,V_k\}$ is defined by

$$
\mathrm{S_m}(V_1,\dots,V_k) = \sum\nolimits_{i<j}\mathrm{S_m}(V_i,V_j),
\tag{2.2}
$$

where $\mathrm{S_m}(V_i,V_j) = \sum\limits_{n_{i-1}<a\leqslant n_i,\ n_{j-1}<b\leqslant n_j} g(R_{e_a,e_b}\,e_b,e_a)$ is the *mutual curvature* of (V_i,V_j). Note that $\mathrm{S_m}(V_1,\dots,V_k)$ does not depend on the choice of frames. We get

$$
\tau(V) = 2\,\mathrm{S_m}(V_1,\dots,V_k) + \sum\nolimits_{i=1}^k \tau(V_i),
\tag{2.3}
$$

where $\tau(V) = \operatorname{trace}_g\operatorname{Ric}|_V$ is the trace of the Ricci tensor on $V=\bigoplus_{i=1}^k V_i$. For example, if all subspaces V_i are one-dimensional, then $2\,\mathrm{S_m}(V_1,\dots,V_k)=\tau(V)$.

We introduce the curvature invariants based on the concept of mutual curvature.

Definition 2.2. Define the *mutual curvature invariants* of a Riemannian manifold (M^{d+l},g) equipped with a d-dimensional distribution D by, see [7, page 7],

$$
\begin{aligned}
\delta^+_{\mathrm{m},D}(n_1,\dots,n_k)(x) &= \max \mathrm{S_m}(V_1,\dots,V_k),\\
\delta^-_{\mathrm{m},D}(n_1,\dots,n_k)(x) &= \min \mathrm{S_m}(V_1,\dots,V_k),
\end{aligned}
\tag{2.4}
$$

where $x\in M$ and $V_1,\dots,V_k$ run over all $k\geqslant 2$ mutually orthogonal subspaces of D_x such that $\dim V_i = n_i$ $(2\leqslant i\leqslant k)$. For $D=TM$, we get the invariants $\delta^\pm_{\mathrm{m}} = \delta^\pm_{\mathrm{m},TM}$.

The invariants in (2.1) and (2.4) are related by the following inequalities.

Proposition 2.1. *For $k\geqslant 2$ and $n_1+\ldots+n_k<d$, the following inequalities hold:*

$$
\begin{aligned}
\delta^+_{\mathrm{m},D}(n_1,\dots,n_k) &\geqslant \delta_D(n_1,\dots,n_k) - \delta_D(n_1+\ldots+n_k)\,,\\
\delta^-_{\mathrm{m},D}(n_1,\dots,n_k) &\leqslant \hat\delta_D(n_1,\dots,n_k) - \hat\delta_D(n_1+\ldots+n_k)\,,
\end{aligned}
$$

and if $n_1+\ldots+n_k=d$*, then*

$$\hat\delta_D(n_1,\ldots,n_k)=\delta^-_{\mathrm{m},D}(n_1,\ldots,n_k)\leqslant\delta^+_{\mathrm{m},D}(n_1,\ldots,n_k)=\delta_D(n_1,\ldots,n_k).$$

If the sectional curvature K *along* D *satisfies* $c\leqslant K\leqslant C$ *and* $\sum_{i=1}^k n_i=s\leqslant d$*, then*

$$\begin{aligned}&\tfrac{c}{2}\,(s^2-\textstyle\sum_i n_i^2)=c\textstyle\sum_{i<j}n_i\,n_j\leqslant\delta^-_{\mathrm{m},D}(n_1,\ldots n_k)\\ &\leqslant\delta^+_{\mathrm{m},D}(n_1,\ldots n_k)\leqslant C\textstyle\sum_{i<j}n_i\,n_j=\tfrac{C}{2}(s^2-\textstyle\sum_i n_i^2)\,.\end{aligned}\tag{2.5}$$

Proof. This is similar to the proof of [7, Proposition 1]. □

Corollary 2.1. *If* (M^{d+l},D,g) *has non-negative sectional curvature of planes tangent to* D*, then*

$$\hat\delta_D(n_1,\ldots,n_k)\leqslant\delta^-_{\mathrm{m},D}(n_1,\ldots,n_k)\leqslant\delta^+_{\mathrm{m},D}(n_1,\ldots,n_k)\leqslant\delta_D(n_1,\ldots,n_k),$$

and if this sectional curvature is nonpositive, then the above inequalities are opposite.

Given two J-invariant planes σ and σ' (2-dimensional subspaces) in T_xM of an almost Hermitian manifold (M,J,g), and unit vectors $X\in\sigma$ and $Y\in\sigma'$, Goldberg and Kobayashi [6] defined the *holomorphic bisectional curvature* $K_{\mathrm{h}}(\sigma,\sigma')$ by

$$K_{\mathrm{h}}(\sigma,\sigma')=R(X,JX,Y,JY).\tag{2.6}$$

This depends on σ and σ' only, and for $\sigma=\sigma'$ gives the *holomorphic sectional curvature.* For a set of J-invariant planes in a complex distribution of real dimension $d\geqslant 4$, we introduce invariants based on the holomorphic bisectional curvature.

Definition 2.3. Let D be a d-dimensional complex distribution of a Riemannian manifold (M,g), i.e., there is a skew-symmetric (1,1)-tensor $J:D\to D$ such that $J^2X=-X$ and $g(JX,JY)=g(X,Y)$ for $X,Y\in D$. The *holomorphic mutual curvature invariants* $\delta^{\pm}_{\mathrm{h},D}(k)$ $(1<k\leqslant d/2)$ are defined by

$$\delta^+_{\mathrm{h},D}(k)(x)=\max\mathrm{S}_{\mathrm{h}}(\sigma_1,\ldots,\sigma_k),\quad \delta^-_{\mathrm{h},D}(k)(x)=\min\mathrm{S}_{\mathrm{h}}(\sigma_1,\ldots,\sigma_k),$$

where $\sigma_1,\ldots,\sigma_k$ run over all k mutually orthogonal J-invariant planes of D_x at a point $x\in M$, and $\mathrm{S}_{\mathrm{h}}(\sigma_1,\ldots,\sigma_k)$ is defined using (2.6) by

$$\mathrm{S}_{\mathrm{h}}(\sigma_1,\ldots,\sigma_k)=\sum\nolimits_{i<j}K_{\mathrm{h}}(\sigma_i,\sigma_j).\tag{2.7}$$

For $D=TM$, i.e., for an almost Hermitian manifold, we get the *holomorphic mutual curvature invariants* $\delta^{\pm}_{\mathrm{h}}(k):=\delta^{\pm}_{\mathrm{h},TM}(k)$.

Lemma 2.1. *Let* $\{\sigma_1,\ldots,\sigma_k\}$ $(2\leqslant k\leqslant d/2)$ *be mutually orthogonal* J*-invariant planes of a complex distribution* D_x *at a point* $x\in M$*. Then*

$$2\,\mathrm{S}_{\mathrm{h}}(\sigma_1,\ldots,\sigma_k)=\mathrm{S}_{\mathrm{m}}(\sigma_1,\ldots,\sigma_k).\tag{2.8}$$

Proof. By the Bianchi identity, we get

$$K_{\rm h}(\sigma,\sigma') = R(X,Y,X,Y) + R(X,JY,X,JY). \tag{2.9}$$

Replacing X with JX, we get $K_{\rm h}(\sigma,\sigma') = R(JX,Y,JX,Y) + R(JX,JY,JX,JY)$. Thus, $\mathrm{S_m}(\sigma,\sigma') = 2\,K_{\rm h}(\sigma,\sigma')$. From this, (2.2) and (2.7), we get (2.8). □

Corollary 2.2. *The following inequalities are true for $2 \leqslant k \leqslant d/2$:*

$$2\,\delta^+_{{\rm h},D}(k) \leqslant \delta^+_{{\rm m},D}(\underbrace{2,\dots,2}_{k}), \quad 2\,\delta^-_{{\rm h},D}(k) \geqslant \delta^-_{{\rm m},D}(\underbrace{2,\dots,2}_{k}).$$

3. CR-submanifolds in almost Hermitian manifolds

In this section, using mutual curvature invariants, Chen-type invariants and holomorphic mutual curvature invariants, we prove several geometric inequalities for CR-submanifolds in almost Hermitian manifolds.

An even-dimensional Riemannian manifold $(\bar M,\bar g)$ equipped with a skew-symmetric (1,1)-tensor $\bar J$ such that $\bar J^{\,2}X = -X$ and $\bar g(\bar JX,\bar JY) = \bar g(X,Y)$ for all $X,Y\in T\bar M$ is called an almost Hermitian manifold. We will put a top "bar" for objects related to $\bar M$. A submanifold M^{d+l} $(d,l>0)$ of an almost Hermitian manifold $(\bar M,\bar J,\bar g)$ is called a *CR-submanifold* if $D = \bar J(TM)\cap TM$ is a complex distribution (the maximal $\bar J$-invariant subbundle) of constant real dimension d, see [5, Definition 7.2]. A different definition is given in [1, 9]: a real submanifold M^{d+l} $(d,l>0)$ of an almost Hermitian manifold $(\bar M,\bar J,\bar g)$ is called a CR-submanifold if there exists on M a totally real distribution $D^\perp$ (i.e., $\bar J(D^\perp)\subset T^\perp M$) whose orthogonal complement D (i.e., $TM = D\oplus D^\perp$) is a complex distribution (i.e., $\bar J(D) = D$) of constant real dimension d. Both definitions above give the same thing when the dimension of D is maximum, that is, $D^\perp$ is one-dimensional. The main examples are real hypersurfaces, other examples are explained in [5].

Example 3.1. Let (M^{d+1},g) be a CR-submanifold of an almost Hermitian manifold $(\bar M,\bar J,\bar g)$ with a d-dimensional complex distribution $D = \bar J(TM)\cap TM$. Then M^{d+1} admits an almost contact metric structure (φ,ξ,η,g), where $\varphi{=}\bar J|_D$ and ξ is the unit tangent vector field orthogonal to D.

Let $h: TM\times TM\to TM^\perp$ be the 2nd fundamental form of the submanifold (M,g) of the Riemannian manifold $(\bar M,\bar g)$. Recall the Gauss equation [3, page 34]:

$$\bar g(\bar R_{Y,Z}\,U,X) = g(R_{Y,Z}\,U,X) + g(h(Y,U),h(Z,X)) - g(h(Z,U),h(Y,X)), \tag{3.1}$$

where $U,X,Y,Z\in TM$ and $\bar R$ and R are the curvature tensors of $(\bar M,\bar g)$ and (M,g), respectively. The mean curvature vector field of a subspace $V\subset T_xM$ is given by $H_V = \sum_i h(e_i,e_i)$, where e_i is an orthonormal basis of V. In short form, we will write H_i instead of H_{V_i}, and H if $V = T_xM$. For a CR-submanifold (M^{d+l},g), set

$$\mathcal{H}_{D_x}(s) = \max\{\,\|H_V\|:\ V\subset D_x,\ \dim V = s>0\}.$$

If $s=d$, then $\mathcal{H}_{D_x}(d)=\|H_{D_x}\|$, where H_{D_x} is the mean curvature vector of D_x. Note that for $s<d$, the equality $\mathcal{H}_{D_x}(s)=0$ implies $h\,|_{D_x}=0$.

A CR-submanifold (M,g) in an almost Hermitian space $(\bar M,\bar J,\bar g)$ is called *D-minimal* (where $D=\bar J(TM)\cap TM$) if $H_D\equiv 0$. A CR-submanifold (M^{d+l},g) is called *mixed totally geodesic* on $V=\bigoplus_{i=1}^k V_i\subset D$ if $h(X,Y)=0$ for all $X\in V_i$, $Y\in V_j$ and $i\neq j$. Note that $\delta^+_{\mathrm{m},D}(n_1,\dots,n_k)\leqslant\delta^+_{\mathrm{m}}(n_1,\dots,n_k)$, and, for $s<d$, by $\mathcal{H}_D(s)=0$, M is totally geodesic on D, i.e., $h\,|_{D_x}=0$.

Theorem 3.1. *Let (M^{d+l},g) be a CR-submanifold of an almost Hermitian manifold $(\bar M,\bar J,\bar g)$, and $D=\bar J(TM)\cap TM$. For any natural numbers $n_1,\dots,n_k$ such that $\sum_i n_i=s\leqslant d$, we obtain*

$$\delta^+_{\mathrm{m},D}(n_1,\dots,n_k)\leqslant\bar\delta^+_{\mathrm{m}}(n_1,\dots,n_k)+\frac{k-1}{2\,k}\begin{cases}\mathcal{H}_D(s)^2, & \text{if } s<d,\\ \|\,H_D\,\|^2, & \text{if } s=d,\end{cases}\tag{3.2}$$

where $\bar\delta^+_{\mathrm{m}}(n_1,\dots,n_k)$ are defined for $(\bar M,\bar g)$ similarly to $\delta^+_{\mathrm{m}}(n_1,\dots,n_k)$ for (M,g), see Definition 2.2. The equality in (3.2) holds at a point $x\in M^{d+l}$ if and only if there exist mutually orthogonal subspaces $V_1,\dots,V_k$ of D_x with $\sum_i n_i=s$ such that M^{d+l} is mixed totally geodesic on $V=\bigoplus_{i=1}^k V_i$, $H_1=\ldots=H_k$, $\|H_V\|=\mathcal{H}_{D_x}(s)$ and $\bar{\mathrm{S}}_{\mathrm{m}}(V_1,\dots,V_k)=\bar\delta^+_{\mathrm{m}}(n_1,\dots,n_k)(x)$.

Proof. Taking a trace of (3.1) for the submanifold M^{d+l} on V and V_i yields

$$\bar\tau(V)-\tau(V)=\|h_V\|^2-\|H_V\|^2,\quad \bar\tau(V_i)-\tau(V_i)=\|h_i\|^2-\|H_i\|^2,\tag{3.3}$$

where $\bar\tau(V)$, $\bar\tau(V_i)$ and $\tau(V)$, $\tau(V_i)$ are the scalar curvatures of subspaces $V=\bigoplus_{i=1}^k V_i$ and V_i for the curvature tensors $\bar R$ and R, respectively, at the point $x\in M$.

Let $H_V\neq 0$ hold on an open set $U\subset M$. We complement an adapted local orthonormal frame $\{e_1,\dots,e_d\}$ of D over U with a vector field e_{d+1} parallel to H_V. By $H_V=\sum_{i=1}^k H_i$ and $a_1^2+\ldots+a_k^2\geqslant\frac{1}{k}\,(a_1+\ldots+a_k)^2$ for $a_i=\bar g(H_i,e_{d+1})$, we get

$$\sum\nolimits_i\|H_i\|^2\geqslant\sum\nolimits_i\bar g(H_i,e_{d+1})^2\geqslant\frac{1}{k}\,\|H_V\|^2,\tag{3.4}$$

and the equality holds if and only if $H_1=\ldots=H_k$. The (3.4) is true for $H_V=0$; thus, it is valid on M. Set $\|h^{\mathrm{mix}}_{ij}\|^2=\sum_{e_a\in V_i,\,e_b\in V_j}\|h(e_a,e_b)\|^2$ for $i\neq j$. Note that

$$\|h_V\|^2=\sum\nolimits_i\|h_i\|^2+\sum\nolimits_{i<j}\|h^{\mathrm{mix}}_{ij}\|^2\geqslant\sum\nolimits_i\|h_i\|^2,\tag{3.5}$$

and the equality holds if and only if $\|h^{\mathrm{mix}}_{ij}\|^2=0$ $(\forall\,i<j)$, i.e., M^{d+l} is mixed totally geodesic along V. By (3.3)–(3.5) and the following equalities, see (2.3):

$$\bar\tau(V)=2\,\bar{\mathrm{S}}_{\mathrm{m}}(V_1,\dots,V_k)+\sum\nolimits_i\bar\tau(V_i),\quad \tau(V)=2\,\mathrm{S}_{\mathrm{m}}(V_1,\dots,V_k)+\sum\nolimits_i\tau(V_i),$$

we obtain

$$
\begin{aligned}
&2\,\mathrm{S}_{\mathrm{m}}(V_1,\dots,V_k) = 2\,\bar{\mathrm{S}}_{\mathrm{m}}(V_1,\dots,V_k) + \sum\nolimits_i \big(\bar\tau(V_i)-\tau(V_i)\big) + \|H_V\|^2 - \|h_V\|^2\\
&\leqslant 2\,\bar\delta^+_{\mathrm{m},D}(n_1,\dots,n_k) - (\|h_V\|^2 - \sum\nolimits_i \|h_i\|^2) + (\|H_V\|^2 - \sum\nolimits_i \|H_i\|^2)\\
&\leqslant 2\,\bar\delta^+_{\mathrm{m},D}(n_1,\dots,n_k) + \frac{k-1}{k}\,\mathcal{H}_D(s)^2,
\end{aligned}
$$

and the equality holds in the 2nd line if and only if $\bar{\mathrm{S}}_{\mathrm{m}}(V_1,\dots,V_k)=\bar\delta^+_{\mathrm{m},D}(n_1,\dots,n_k)$ and $\|H_V\| = \mathcal{H}_x(s)$ at each point $x\in M$. This proves (3.2) for $s<d$. The case $\sum_i n_i = d$ of (3.2) can be proved similarly. □

Remark 3.1. For a CR-submanifold (M^{d+l},g) in an almost Hermitian space $(\bar M,\bar J,\bar g)$ with sectional curvature bounded above by c, for $\sum_i n_i = s\leqslant d$ from (2.5) and (3.2), we get

$$
\delta^+_{\mathrm{m},D}(n_1,\dots,n_k) \leqslant \begin{cases} \frac{c}{2}\,(s^2-\sum_i n_i^2) + \frac{k-1}{2\,k}\,\mathcal{H}_D(s)^2, & \text{if } s<d,\\ \frac{c}{2}\,(d^2-\sum_i n_i^2) + \frac{k-1}{2\,k}\,\|H_D\|^2, & \text{if } s=d.\end{cases} \tag{3.6}
$$

For $s=d$, the RHS of (3.6) coincides with the RHS of [3, Eqn. (13.43)] for $\sum_i n_i = d$.

As real hypersurfaces of almost Hermitian manifolds are the main examples of CR-submanifolds and important objects in the study of geometrical inequalities, we reformulate Theorem 3.1 especially for this case.

Corollary 3.1. *Let (M^{2n+1},g) be a real hypersurface with a complex distribution $D=\bar J(TM)\cap TM$ of an almost Hermitian manifold $(\bar M,\bar J,\bar g)$. For any natural numbers $n_1,\dots,n_k$ such that $\sum_i n_i = s\leqslant 2n$, we obtain the inequality*

$$
\delta^+_{\mathrm{m},D}(n_1,\dots,n_k) \leqslant \bar\delta^+_{\mathrm{m}}(n_1,\dots,n_k) + \frac{k-1}{2\,k}\begin{cases}\mathcal{H}_D(s)^2, & \text{if } s<2n,\\ \|H_D\|^2, & \text{if } s=2n.\end{cases} \tag{3.7}
$$

The equality in (3.7) holds at a point $x\in M^{2n+1}$ if and only if there exist mutually orthogonal subspaces $V_1,\dots,V_k$ of D_x with $\sum_i n_i = s$ such that M^{2n+1} is mixed totally geodesic on $V=\bigoplus_{i=1}^k V_i$, $H_1=\ldots=H_k$, $\|H_V\|=\mathcal{H}_{D_x}(s)$ and $\bar{\mathrm{S}}_{\mathrm{m}}(V_1,\dots,V_k)=\bar\delta^+_{\mathrm{m}}(n_1,\dots,n_k)(x)$.

For any k-tuple $(n_1,\dots,n_k)$ with $\sum_i n_i = s\leqslant d$, define the *normalized* $\delta_{\mathrm{m},D}$-*curvature* by $\Delta_{\mathrm{m},D}(n_1,\dots,n_k) = \frac{2\,k}{k-1}\,\delta^+_{\mathrm{m},D}(n_1,\dots,n_k)$, and for $\sum_i n_i = s$ put $\bar\Delta_{\mathrm{m},D} := \max\Delta_{\mathrm{m},D}(n_1,\dots,n_k)$.
Theorem 3.1 gives the following (compare with the maximum principle [3, page 268]).

Proposition 3.1. *If the equality $\mathcal{H}_D(s)^2=\Delta_{\mathrm{m},D}(n_1,\dots,n_k)$ holds for a CR-submanifold (M^{d+l},g) of $\mathbb{C}^q$ for some k-tuple $(n_1,\dots,n_k)$ with $\sum_i n_i = s\leqslant d$, then for all $(m_1,\dots,m_k)$ with $\sum_i m_i = s$, we get $\Delta_{\mathrm{m},D}(n_1,\dots,n_k)\geqslant\Delta_{\mathrm{m},D}(m_1,\dots,m_k)$.*

Proof. By the conditions, $\Delta_{\mathrm{m},D}(n_1,\dots,n_k)=\bar\Delta_{\mathrm{m},D}(s)$. Since $\Delta_{\mathrm{m},D}(m_1,\dots,m_k)\leqslant\mathcal{H}_D(s)^2$, we obtain the inequality $\Delta_{\mathrm{m},D}(m_1,\dots,m_k)\leqslant\Delta_{\mathrm{m},D}(n_1,\dots,n_k)$. □

Corollary 3.2. *For every CR-submanifold (M^{d+l}, g) in $\mathbb{C}^q$, we have $\mathcal{H}_D(s)^2 \geqslant \bar\Delta_{\mathrm{m},D}(s)$ for any $s < d$, and $\|H_D\|^2 \geqslant \bar\Delta_{\mathrm{m},D}$.*

The case of equality in Corollary 3.2 is of special interest. Such extremal CR-immersions in $\mathbb{C}^q$ can be compared to "ideal immersions" introduced by Chen's for real space forms in terms of δ-invariants, for example, [3, Definition 13.3].

The theory of δ_D-invariants (2.1) of CR-submanifolds can be developed similarly to the theory of Chen's δ-invariants of a Riemannian submanifold.

Theorem 3.2. *Let (M^{d+l}, g) be a CR-submanifold of an almost Hermitian manifold $(\bar M, \bar J, \bar g)$ with sectional curvature bounded above by $c \in \mathbb{R}$. For each k-tuple $(n_1, \dots, n_k)$ such that $\sum_i n_i \leqslant d$, we obtain (similarly to [3, Theorem 13.5])*

$$(3.8)\quad \delta_D(n_1, \dots, n_k) \leqslant \frac{d^2(d+k-1-\sum_i n_i)}{2(d+k-\sum_i n_i)}\|H_D\|^2 + \frac{c}{2}\,[d(d-1) - \sum\nolimits_i n_i(n_i-1)].$$

The case of equality in (3.8) is of special interest: extremal CR-submanifolds in terms of δ_D-invariants are an analogue of Chen's "ideal immersions".

Set $\delta^+_{\mathrm{m},D}(k) = \max \delta^+_{\mathrm{m},D}(n_1, \dots, n_k)$ and $\delta^-_{\mathrm{m},D}(k) = \min \delta^-_{\mathrm{m},D}(n_1, \dots, n_k)$, where $\sum_i n_i \leqslant d$. The $\bar\delta^+_{\mathrm{m}}(k+1)$ are defined for $(\bar M, \bar g)$ similarly to $\delta^+_{\mathrm{m}}(k+1)$ for (M, g).

Theorem 3.3. *Let (M^{d+l}, g) be a CR-submanifold of an almost Hermitian manifold $(\bar M, \bar J, \bar g)$. For any $k \geqslant 2$, we obtain the inequality that supplements* (3.7)*:*

$$(3.9)\qquad \delta^-_{\mathrm{m},D}(k) \leqslant \frac{k-1}{2\,k(k+1)}\,\|H_D\|^2 + \bar\delta^+_{\mathrm{m}}(k+1)\,.$$

The equality in (3.9) *holds at a point $x \in M^{d+l}$ if and only if there exist mutually orthogonal subspaces $V_1, \dots, V_{k+1}$ of D_x with $\sum_{i=1}^{k+1} n_i = d$ such that M^{d+l} is mixed totally geodesic, $H_1 = \ldots = H_{k+1}$, $\bar{\mathrm{S}}_{\mathrm{m}}(V_1, \dots, V_{k+1}) = \bar\delta^+_{\mathrm{m}}(n_1, \dots, n_{k+1})$ and $\mathrm{S}_{\mathrm{m}}(V_1, \dots, \widehat{V}_i, \dots, V_{k+1}) = \delta^-_{\mathrm{m},D}(k)$ for any $i = 1, \dots, k+1$, where $\widehat{V}_i$ means removing the space V_i from the set $\{V_1, \dots, V_{k+1}\}$.*

Proof. Let V_{k+1} be the orthogonal complement to $V = \bigoplus_{i=1}^k V_i$ in D_x. Note that $\sum_i \mathrm{S}_{\mathrm{m}}(V_1, \dots, \widehat{V}_i, \dots, V_{k+1}) = (k+1)\,\mathrm{S}_{\mathrm{m}}(V_1, \dots, V_{k+1})$. We also obtain $\delta^-_{\mathrm{m},D}(k) \leqslant \delta^-_{\mathrm{m},D}(n_1, \dots, \widehat{n}_i, \dots, n_{k+1}) \leqslant \mathrm{S}_{\mathrm{m},D}(V_1, \dots, \widehat{V}_i, \dots, V_{k+1})$ for any $i = 1, \dots, k+1$. Thus, $\delta^-_{\mathrm{m},D}(k) \leqslant \mathrm{S}_{\mathrm{m}}(V_1, \dots, V_{k+1})$, and using (3.7) for $\sum_i n_i = d$ gives (3.9). □

Theorems 3.1 and 3.3 give the assertions on the absence of some CR-submanifolds.

Corollary 3.3. *There are no D-minimal CR-submanifolds (M^{d+l}, g) in $\mathbb{C}^q$ with any of the following properties: $\delta^+_{\mathrm{m},D}(n_1, \dots, n_k) > 0$ for some $(n_1, \dots, n_k)$ with $\sum_i n_i = d$, and $\delta^-_{\mathrm{m},D}(k) > 0$ for some $k \geqslant 2$.*

Next, we use the fact that the tangent distribution TM is the sum $TM = D \oplus D^\perp$ of two mutually orthogonal distributions D and $D^\perp$ of ranks d and l. Let $x \in M$ and $\{e_i\}$ on (M, g) be an adapted orthonormal frame, i.e., $\{e_1, \ldots, e_d\} \subset D(x)$, $\{e_{d+1}, \ldots, e_{d+l}\} \subset D^\perp(x)$. The *mutual curvature* of $(D, D^\perp)$ is a function $\mathrm{S_m}(D, D^\perp)$, given at $x \in M$ by $\mathrm{S_m}(D(x), D^\perp(x)) = \sum_{1\leqslant a\leqslant d,\ d<b\leqslant d+l} K(e_a, e_b)$. In this case, $\mathrm{S_m}(D, D^\perp)$ is the *mixed scalar curvature*; see [8, page 2]. A CR-submanifold (M, g) is called *mixed totally geodesic* on $(D, D^\perp)$ if $h(X, Y) = 0$ $(X \in D,\ Y \in D^\perp)$.

Theorem 3.4. *Let $(M^{d+l}, g; D, D^\perp)$, where $D = \bar J(TM) \cap TM$, be a CR-submanifold of an almost Hermitian manifold $(\bar M, \bar J, \bar g)$. Then the following inequality holds:*

$$\mathrm{S_m}(D, D^\perp) \leqslant (1/4)\, \|H_D\|^2 + \bar\delta^+_{\mathrm m}(d, l)\,. \tag{3.10}$$

The equality in (3.10) *holds at a point $x \in M$ if and only if M^{d+l} is mixed totally geodesic, $H_D(x) = H_{D^\perp}(x)$ and $\bar{\mathrm S}_{\mathrm m}(D(x), D^\perp(x)) = \bar\delta^+_{\mathrm m}(d, l)(x)$.*

Proof. The proof of the first statement is similar to the proof of Theorem 3.1. The second assertion follows directly from the cases of equality, as for Theorem 3.1. □

Corollary 3.4. *A CR-submanifold in $\mathbb{C}^q$ with $\mathrm{S_m}(D, D^\perp) > 0$ cannot be D-minimal.*

Example 3.2. Consider distributions $D, D^\perp$ on a domain M on a unit sphere $S^{d+l}(1)$ in a complex Euclidean space; thus, $\bar\delta^+_{\mathrm m}(d, l) = 0$. Using coordinate charts, we can take integrable distributions $D, D^\perp$, and M is diffeomorphic to the product of two manifolds.

Let $d = 2$ and $l = 1$; then, $\|H\|^2 = 9$ and $\mathrm{S_m}(D, D^\perp) = 2$. Hence, (3.10) reduces to the inequality $2 < 9/4$. Note that $H_D = \frac{1}{3} H \neq \frac{2}{3} H = H_{D^\perp}$.

Let $d = l = 2$ and locally $M \subset S^4(1)$ be diffeomorphic to $\mathbb{C} \times \mathbb{C}$. Then, $\|H\|^2 = 16$, $H_D = H_{D^\perp}$, $\mathrm{S_m}(D, D^\perp) = 4$ and (3.10) reduces to the equality $4 = 16/4$.

The following theorem deals with holomorphic bisectional curvature invariants.

Theorem 3.5. *Let (M^{d+l}, g) be a CR-submanifold of an almost Hermitian manifold $(\bar M, \bar J, \bar g)$. For any natural number $k \in [2, d/2]$, we obtain*

$$\delta^+_{\mathrm h, D}(k) \leqslant \bar\delta^+_{\mathrm h}(k) + \frac{k-1}{4\,k} \begin{cases} \mathcal{H}_D(2k)^2, & \text{if } 2k < d, \\ \|\, H_D\,\|^2, & \text{if } 2k = d, \end{cases} \tag{3.11}$$

where $\bar\delta^+_{\mathrm h}(k)$ are defined for $(\bar M, \bar g)$ similarly to $\delta^+_{\mathrm h}(k)$ for (M, g), see Definition 2.3. The equality in (3.11) *holds at $x \in M^{d+l}$ if and only if there exist mutually orthogonal J-invariant planes $\{\sigma_1, \ldots, \sigma_k\}$ of D_x such that M^{d+l} is mixed totally geodesic on $V = \bigoplus_{i=1}^k \sigma_i$, $H_1 = \ldots = H_k$, $\|H_V\| = \mathcal{H}_{D_x}(2k)$ and $\bar{\mathrm S}_{\mathrm h}(\sigma_1, \ldots, \sigma_k) = \bar\delta^+_{\mathrm h}(k)(x)$.*

Proof. This is similar to the proof of Theorem 3.1. □

Corollary 3.5. *Let (M^{d+1}, g) be a real hypersurface of an almost Hermitian manifold $(\bar M, \bar J, \bar g)$. Then* (3.11) *is true for any natural number $k \in [2, d/2]$.*

Using $\delta(2,\dots,2)$-invariants, Chen classified in [3, Section 15.7] extremal real hypersurfaces of Kählerian space forms. Similarly, we would like to study the extreme case of Corollary 3.5 when $(\bar M,\bar J,\bar g)$ is a Kählerian space form.

From Theorem 3.5 we get the assertion on the absence of some CR-submanifolds.

Corollary 3.6. *A CR-submanifold (M^{d+l},g) in $\mathbb{C}^q$ satisfying $\delta^{+}_{\mathrm{h},D}(d/2)>0$ cannot be D-minimal.*

4. Conclusions

We studied the question of finding a simple optimal connection between the intrinsic and extrinsic invariants of a manifold equipped with a complex distribution. The main contribution of the paper is the concept of curvature invariants $\delta^{\pm}_{\mathrm{m},D}$ of CR-submanifolds of almost Hermitian manifolds, based on the mutual curvature of several pairwise orthogonal subspaces of a contact distribution D. We used these curvature invariants and Chen-type curvature invariants $\delta^{\pm}_{D}$ to prove new geometric inequalities involving the squared intermediate mean curvature for CR-submanifolds of almost Hermitian manifolds. In the case of complex planes, we study curvature invariants $\delta^{\pm}_{\mathrm{h},D}$ based on the concept of holomorphic bisectional curvature. Consequences of the absence of some D-minimal CR-submanifolds were provided.

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